Some new results on bi-skew braces

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Joint work with Senne Trappeniers

The algebra of the Yang-Baxter equation, 14 July 2022

Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A *skew brace* is a triple $(A, \cdot, \circ)$, where $(A, \cdot)$, $(A, \circ)$ are groups and

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c).$$

Here a^{-1} denotes the inverse of a in $(A, \cdot)$.

Skew braces are connected with

- radical rings;
- solutions of the set-theoretic Yang–Baxter equation;
- regular subgroups of holomorphs of groups;
- Hopf–Galois structures.

Bi-skew braces

Definition ([Childs, 2019])

A *bi-skew brace* is a skew brace $(A, \cdot, \circ)$ such that also $(A, \circ, \cdot)$ is a skew brace.

Example

Let $(A, \circ)$ be a group.

- A *trivial* skew brace $(A, \circ, \circ)$ is a bi-skew brace.
- An *almost trivial* skew brace $(A, \circ_{\mathrm{op}}, \circ)$, where $a \circ_{\mathrm{op}} b = b \circ a$, is a bi-skew brace.

Example

$(\mathbb{Z}, +, \tilde{\circ})$ is a bi-skew brace, where $a \,\tilde{\circ}\, b = a + (-1)^a b$.

Byott's conjecture

Conjecture (Byott's conjecture)

Let $(A, \cdot, \circ)$ *be a finite skew brace. If* $(A, \cdot)$ *is soluble, then* $(A, \circ)$ *is soluble.*

Theorem ([LS and Trappeniers, 2022])

Let $(A, \cdot, \circ)$ *be a bi-skew brace. Then* $(A, \cdot)$ *is soluble if and only if* $(A, \circ)$ *is soluble.*

A key player

Recall that *ideals* of skew braces are the substructures to consider in order to define quotient skew braces.

Fact

Let $(A, \cdot, \circ)$ *be a bi-skew brace. Then there exists an ideal* I *of* $(A, \cdot, \circ)$ *such that*

- $(I, \cdot, \circ)$ *is a trivial skew brace;*
- $(A/I, \cdot, \circ)$ *is an almost trivial skew brace.*

(Here $I = A^2_{\text{op}}$*.)*

Proof.

Suppose that $(A, \cdot)$ is soluble. Then $(I, \cdot)$ and $(A/I, \cdot)$ are soluble. As $(I, \cdot) = (I, \circ)$ and $(A/I, \cdot) \cong (A/I, \circ)$, also $(I, \circ)$ and $(A/I, \circ)$ are soluble. We conclude that $(A, \circ)$ is soluble. □

Fact

This argument works also for skew braces $(A, \cdot, \circ)$ *admitting series* $1 = A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1 = A$ *where* A_{i+1} *is an ideal of* A_i *with* $(A_i/A_{i+1}, \cdot) \cong (A_i/A_{i+1}, \circ)$ *for all* i*.*

In particular, Byott's conjecture holds for soluble skew braces, so also for skew braces that are right nilpotent, left nilpotent, strongly, central nilpotent, metatrivial, γ*-homomorphic, ...*

A classification result

A classification problem

Recall that $(\mathbb{Z}, +, \tilde{\circ})$ is a bi-skew brace, where $a \,\tilde{\circ}\, b = a + (-1)^a b$.

Proposition ([Rump, 2007])

Let $(\mathbb{Z}, +, \circ)$ be a skew brace. If $(\mathbb{Z}, +, \circ)$ is not trivial, then $(\mathbb{Z}, \circ) = (\mathbb{Z}, \tilde{\circ})$.

Problem ([Vendramin, 2019])

Determine the skew braces of the form $(\mathbb{Z}, \cdot, +)$.

Theorem ([Cedó et al., 2019])

Let $(\mathbb{Z}, \cdot, +)$ be a skew brace such that $(\mathbb{Z}, \cdot)$ is abelian. Then $(\mathbb{Z}, \cdot, +)$ is trivial.

The resolution

We remark that we always have the skew braces $(\mathbb{Z}, +, +)$, $(\mathbb{Z}, \tilde{\circ}, +)$, and $(\mathbb{Z}, \tilde{\circ}_{\mathsf{op}}, +)$.

Theorem ([LS and Trappeniers, 2022])

Let $(\mathbb{Z}, \cdot, +)$ be a skew brace. If $(\mathbb{Z}, \cdot, +)$ is not trivial, then $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ})$ or $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ}_{\mathsf{op}})$.

Sketch of the proof.

- If $(\mathbb{Z}, \cdot)$ is abelian, then $(\mathbb{Z}, \cdot, +)$ is trivial.
- If $(\mathbb{Z}, \cdot)$ is not abelian, then we can show that either $(\mathbb{Z}, \cdot, +)$ or $(\mathbb{Z}, \cdot_{\mathsf{op}}, +)$ is a bi-skew brace.

  In particular, either $(\mathbb{Z}, +, \cdot)$ or $(\mathbb{Z}, +, \cdot_{\mathsf{op}})$ is a skew brace.

  We obtain that either $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ})$ or $(\mathbb{Z}, \cdot_{\mathsf{op}}) = (\mathbb{Z}, \tilde{\circ})$. $\square$

The Yang–Baxter equation

The Yang–Baxter equation

Let (X, r) be a *(nondegenerate bijective) solution (of the set-theoretic Yang–Baxter equation)*. Write

- $r(x, y) = (\sigma_x(y), \tau_y(x))$;
- $r^{-1}(x, y) = (\hat{\sigma}_x(y), \hat{\tau}_y(x))$.

Define the *structure group* of (X, r) to be

$$G(X, r) = \langle X \mid x \circ y = \sigma_x(y) \circ \tau_y(x) \rangle;$$

then (X, r) is *injective* if the natural map $X \to G(X, r)$ is injective.

For example, this is the case when (X, r) is *involutive*, that is, when $r^2 = \mathrm{id}$.

We remark that $G(X, r)$ has a natural structure of a skew brace.

Question

When is $G(X, r)$ a bi-skew brace?

Theorem ([LS and Trappeniers, 2022])

Let (X, r) be a solution.

- *If for all $x, y \in X$,*
$$\sigma_{\hat{\sigma}_x(y)} = \sigma_y,$$
then $G(X, r)$ is a bi-skew brace.
- *If (X, r) is injective, then also the opposite implication holds.*

Fact

When (X, r) is involutive, we find solutions studied in [Jedlička et al., 2020].

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