Lecture 10
The Hausdorff metric.
Hutchinson's theorem.
Fractal images
Similarity dimension and Hausdorff dimension.

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Compact sets and their collars

Let X be a complete metric space. Let $\mathcal{H}(X)$ denote the space of non-empty compact subsets of X. For any $A \in \mathcal{H}(X)$ and any positive number ϵ, let

$$A_\epsilon = \{x \in X | d(x,y) \leq \epsilon, \text{for some } y \in A\}.$$

We call A_ϵ the *ϵ-collar* of A. Recall that we defined

$$d(x,A) = \inf_{y \in A} d(x,y)$$

to be the distance from any $x \in X$ to A. So we can write the definition of the ϵ-collar as

$$A_\epsilon = \{x | d(x,A) \leq \epsilon\}.$$

$$d(x, A) = \inf_{y \in A} d(x, y)$$

Notice that the infimum in the definition of $d(x, A)$ is actually achieved, that is, there is some point $y \in A$ such that

$$d(x, A) = d(x, y).$$

This is because A is compact.

$d(A, B)$.

For a pair of non-empty compact sets, A and B, define

$$d(A, B) = \max_{x \in A} d(x, B).$$

So

$$d(A, B) \leq \epsilon \iff A \subset B_\epsilon.$$

Notice that this condition is not symmetric in A and B.

The Hausdorff metric.

So Hausdorff introduced

$$\begin{aligned} h(A,B) &= \max\{d(A,B), d(B,A)\} && (1)\\ &= \inf\{\epsilon \mid A \subset B_\epsilon \ \text{ and } \ B \subset A_\epsilon\}. && (2) \end{aligned}$$

as a distance on $\mathcal{H}(X)$.

h really is a metric.

He proved

Theorem

The function h on $\mathcal{H}(X) \times \mathcal{H}(X)$ satsifies the axioms for a metric and makes $\mathcal{H}(X)$ into a complete metric space. Furthermore, if

$$A, B, C, D \in \mathcal{H}(X)$$

then

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}. \tag{3}$$

Proof of $h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$. (3)

If ϵ is such that $A \subset C_\epsilon$ and $B \subset D_\epsilon$ then clearly $A \cup B \subset C_\epsilon \cup D_\epsilon = (C \cup D)_\epsilon$. Repeating this argument with the roles of A, C and B, D interchanged proves (3).

We prove that h is a metric: h is symmetric, by definition. Also, $h(A, A) = 0$, and if $h(A, B) = 0$, then every point of A is within zero distance of B,and hence must belong to B since B is compact, so $A \subset B$ and similarly $B \subset A$. So $h(A, B) = 0$ implies that $A = B$.

The triangle inequality.

For this it is enough to prove that
$d(A,B) \leq d(A,C) + d(C,B),$ because interchanging the role of A and B gives the desired result. Now for any $a \in A$ we have

$$\begin{aligned} d(a,B) &= \min_{b \in B} d(a,b) \\ &\leq \min_{b \in B} \left(d(a,c) + d(c,b)\right) \forall c \in C \\ &= d(a,c) + \min_{b \in B} d(c,b) \forall c \in C \\ &= d(a,c) + d(c,B) \forall c \in C \\ &\leq d(a,c) + d(C,B) \forall c \in C. \end{aligned}$$

The second term in the last expression does not depend on c, so minimizing over c gives

$$d(a,B) \leq d(a,C) + d(C,B).$$

We have shown that for any $a \in A$,

$$d(a,B) \leq d(a,C) + d(C,B).$$

Maximizing over a on the right gives

$$d(a,B) \leq d(A,C) + d(C,B).$$

Maximizing on the left gives the desired

$$d(A,B) \leq d(A,C) + d(C,A).$$

A sketch of the proof of completeness.

Let A_n be a sequence of compact non-empty subsets of X which is Cauchy in the Hausdorff metric. Define the set A to be the set of all $x \in X$ with the property that there exists a sequence of points $x_n \in A_n$ with $x_n \to x$. It is straighforward to prove that A is compact and non-empty and is the limit of the A_n in the Hausdorff metric.

Contractions and the Hausdorff metric.

Suppose that $K : X \to X$ is a contraction. Then K defines a transformation on the space of subsets of X (which we continue to denote by K):

$$K(A) = \{Kx | x \in A\}.$$

Since K continuous, it carries $\mathcal{H}(X)$ into itself.

Contractions induce contractions in the Hausdorff metric.

Let c be the Lipschitz constant of K. Then

$$\begin{aligned} d(K(A),K(B)) &= \max_{a\in A}[\min_{b\in B} d(K(a),K(b))] \\ &\leq \max_{a\in A}[\min_{b\in B} cd(a,b)] \\ &= cd(A,B). \end{aligned}$$

Similarly, $d(K(B),K(A)) \leq c\ d(B,A)$ and hence

$$h(K(A),K(B)) \leq c\ h(A,B). \tag{4}$$

In other words, a contraction on X induces a contraction on $\mathcal{H}(X)$ with the same Lipschitz constant.

The previous remark together with the following observation is the key to Hutchinson's remarkable construction of fractals:

Proposition.

Let $T_1, \ldots, T_n$ be a collection of contractions on $\mathcal{H}(X)$ with Lipschitz constants $c_1, \ldots, c_n$, and let $c = \max c_i$. Define the transformation T on $\mathcal{H}(X)$ by

$$T(A) = T_1(A) \cup T_2(A) \cup \cdots \cup T_n(A).$$

Then T is a contraction with Lipschitz constant c.

Proof.

Proof. By induction, it is enough to prove this for the case $n = 2$. By

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}, \qquad (3)$$

we have

$$\begin{aligned} h(T(A), T(B)) &= h(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B)) \\ &\leq \max\{h(T_1(A), T_1(B)), h(T_2(A), T_2(B))\} \\ &\leq \max\{c_1 h(A, B), c_2 h(A, B)\} \\ &= h(A, B) \max\{c_1, c_2\} = c \cdot h(A, B) \end{aligned}$$

.

Hutchinson's theorem.

Putting the previous facts together we get Hutchinson's theorem:

Theorem

Let $K_1, \ldots, K_n$ be contractions on a complete metric space and let c be the maximum of their Lifschitz contants. Define the Hutchinson operator, K, on $\mathcal{H}(X)$ by

$$K(A) = K_1(A) \cup \cdots \cup K_n(a).$$

Then K is a contraction with Lipschtz constant c.

Affine examples.

We describe several examples in which X is a subset of a vector space and each of the T_i in Hutchinson's theorem are affine transformations of the form

$$T_i : x \mapsto A_i x + b_i$$

where $b_i \in X$ and A_i is a linear transformation.

The classical Cantor set.

Take $X = [0, 1]$, the unit interval. Take

$$T_1 : x \mapsto \frac{x}{3}, \qquad T_2 : x \mapsto \frac{x}{3} + \frac{2}{3}.$$

These are both contractions, so by Hutchinson's theorem there exists a unique closed fixed set C. This is the Cantor set.

Cantor's original construction.

To relate it to Cantor's original construction, let us go back to the proof of the contraction fixed point theorem applied to T acting on $\mathcal{H}(X)$. It says that if we start with any non-empty compact subset A_0 and keep applying T to it, i.e. set $A_n = T^n A_0$ then $A_n \to C$ in the Hausdorff metric, h. Suppose we take the interval I itself as our A_0. Then

$$A_1 = T(I) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

in other words, applying the Hutchinson operator T to the interval $[0, 1]$ has the effect of deleting the "middle third" open interval $(\frac{1}{3}, \frac{2}{3})$.

Applying T once more gives

$$A_2 = T^2[0,1] = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1].$$

In other words, A_2 is obtained from A_1 by deleting the middle thirds of each of the two intervals of A_1 and so on. This was Cantor's original construction. Since $A_{n+1} \subset A_n$ for this choice of initial set, the Hausdorff limit coincides with the intersection.

Using triadic expansions.

But of course Hutchinson's theorem (and the proof of the contractions fixed point theorem) says that we can start with *any* non-empty closed set as our initial "seed" and then keep applying T. For example, suppose we start with the one point set $B_0 = \{0\}$. Then $B_1 = TB_0$ is the two point set

$$B_1 = \{0, \frac{2}{3}\},$$

B_2 consists of the four point set

$$B_2 = \{0, \frac{2}{9}, \frac{2}{3}, \frac{8}{9}\}$$

and so on. We then must take the Hausdorff limit of this increasing collection of sets.

To describe the limiting set C from this point of view, it is useful to use triadic expansions of points in $[0,1]$. Thus

$$\begin{aligned} 0 &= .0000000\cdots \\ 2/3 &= .2000000\cdots \\ 2/9 &= .0200000\cdots \\ 8/9 &= .2200000\cdots \end{aligned}$$

and so on. Thus the set B_n will consist of points whose triadic exapnsion has only zeros or twos in the first n positions followed by a string of all zeros.

Thus a point will lie in C (be the limit of such points) if and only if it has a triadic expansion consisting entirely of zeros or twos. This includes the possibility of an infinite string of all twos at the tail of the expansion. for example, the point 1 which belongs to the Cantor set has a triadic expansion $1 = .222222\cdots$. Simialrly the point $\frac{2}{3}$ has the triadic expansion $\frac{2}{3} = .0222222\cdots$ and so is in the limit of the sets B_n. But a point such as $.101\cdots$ is not in the limit of the B_n and hence not in C. This description of C is also due to Cantor.

Notice that for any point a with triadic expansion $a = .a_1 a_2 a_2 \cdots$

$$T_1 a = .0a_1 a_2 a_3 \cdots , \quad \text{while} \quad T_2 a = .2a_1 a_2 a_3 \cdots .$$

Thus if all the entries in the expansion of a are either zero or two, this will also be true for $T_1 a$ and $T_2 a$. This shows that the C (given by this second Cantor description) satisfies $TC \subset C$. On the other hand,

$$T_1(.a_2 a_3 \cdots) = .0a_2 a_3 \cdots , \qquad T_2(.a_2 a_3 \cdots) = .2a_2 a_3 \cdots$$

which shows that $.a_1 a_2 a_3 \cdots$ is in the image of T_1 if $a_1 = 0$ or in the image of T_2 if $a_1 = 2$. This shows that $TC = C$. Since C (according to Cantor's second description) is closed, the uniqueness part of the fixed point theorem guarantees that the second description coincides with the first.

Self-similarity.

The statement that $TC = C$ implies that C is "self-similar", a notion emphasized and popularized by Mandelbrot.

The Sierpinski Gasket

Consider the three affine transformations of the plane:

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}, \quad T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$T_3 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The fixed point of the Hutchinson operator for this choice of T_1, T_2, T_3 is called the Sierpinski gasket, S.

If we take our initial set A_0 to be the right triangle with vertices at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then each of the $T_i A_0$ is a similar right triangle whose linear dimensions are one-half as large, and which shares one common vertex with the original triangle.

In other words,

$$A_1 = TA_0$$

is obtained from our original triangle be deleting the interior of the (reversed) right triangle whose vertices are the midpoints of our original triangle. Just as in the case of the Cantor set, successive applications of T to this choice of original set amounts to sussive deletions of the "middle" and the Hausdorff limit is the intersection of all them: $S = \bigcap A_i$.

We can also start with the one element set

$$B_0 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Using a binary expansion for the x and y coordinates, application of T to B_0 gives the three element set

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} .1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ .1 \end{pmatrix} \right\}.$$

The set $B_2 = TB_1$ will contain nine points, whose binary expansion is obtained from the above three by shifting the x and y expansions one unit to the right and either inserting a 0 before both expansions (the effect of T_1), insert a 1 before the expansion of x and a zero before the y or vice versa.

Proceeding in this fashion, we see that B_n consists of 3^n points which have all 0 in the binary expansion of the x and y coordinates, past the n-th position, and which are further constrained by the condition that at no earler point do we have both $x_i = 1$ and $y_1 = 1$. Passing to the limit shows that S consists of all points for which we can find (possible inifinite) binary expansions of the x and y coordinates so that $x_i = 1 = y_i$ never occurs.

For example $x = \frac{1}{2}, y = \frac{1}{2}$ belongs to S because we can write $x = .10000\cdots, y = .011111\ldots$. Again, from this (second) description of S in terms of binary expansions it is clear that $TS = S$.

A one line code for creating the Sierpinski gasket.

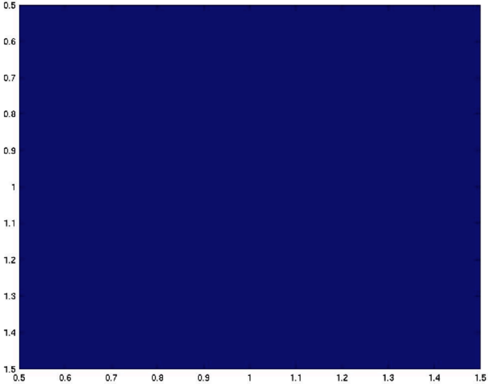
The following slide gives a matlab m file for doing the first seven approximations to the Sierpinksi gasket as a movie. Notice that that iterative scheme is encoded in the single line

J=[J J;J zeros($2^i, 2^i$)];.

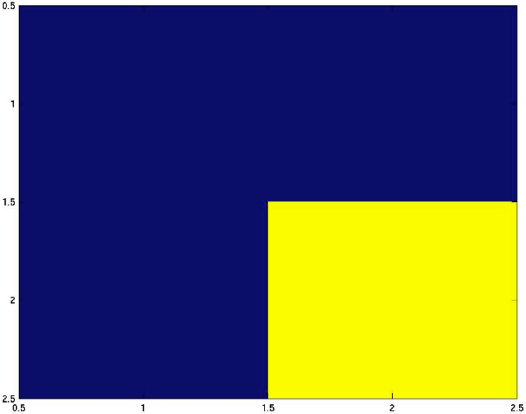
The other instructions are for the graphics, etc. This shows the power of Hutchinsons theorem and also raises the philosophical question as to the notion of simplicity.

```
J=[10];
image(J);colormap(colorcube(17))
pause(3)
for i=0:6
J=[J J;J zeros(2^i,2^i)];
image(J);
colormap(colorcube(17));
pause(3)
end
```

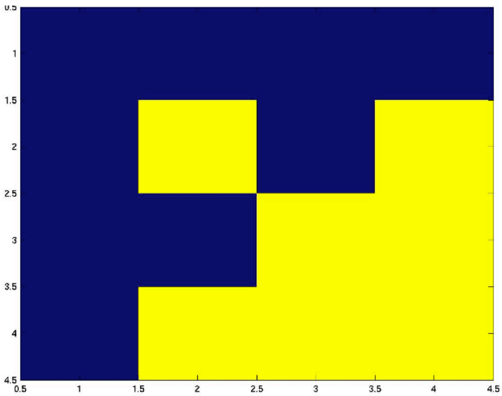
Here are the first seven images:



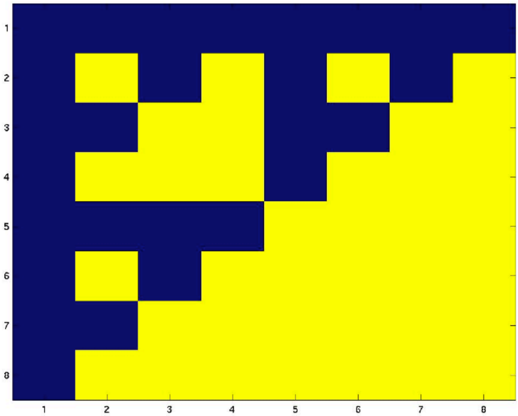
Stage 1.



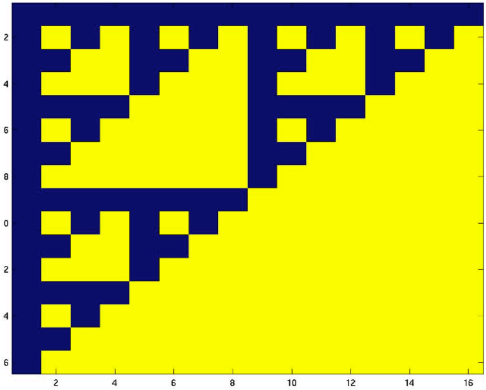
Stage 2.



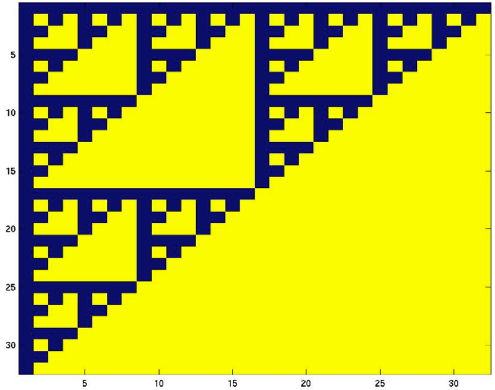
Stage 3.



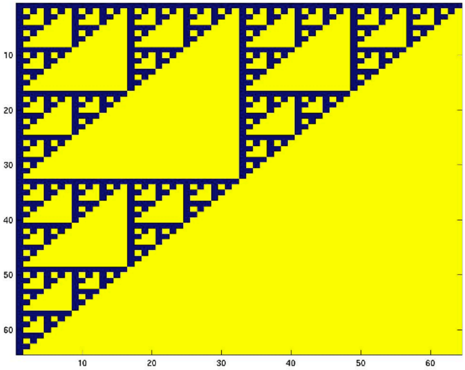
Stage 4.



Stage 5.



Stage 6.



Stage 7.

Let X be a metric space. Let $\mathcal{B}$ be a countable collection of balls B_i of radii $r_i < \epsilon$ which cover X, meaning that

$$X \subset \bigcup_i B_i.$$

(We assume that at least one such countable cover exists.)

For each non-negative real number t let

$$m_t(\mathcal{B}) := \sum_i r_i^t$$

where this sum might be infinite. Define

$$M_{t,\epsilon}(X) = g.l.b.\{m_t(\mathcal{B})\}$$

where we are taking the greatest lower bound over all possible countable covers of X by balls of radius at most ϵ. So $M_{t,\epsilon}(X)$ can be infinite, or be any non-negative real number, including 0.

By its definition, $M_{t,\epsilon}$ is non-decreasing as $\epsilon \to 0$, so there is a limit (again possibly infinite or zero or any non-negative real number). Call this limit $M_t(X)$.

Theorem

If $0 < s < t$ and $M_s(X) < \infty$ then $M_t(X) = 0$.

Proof.

For any cover $\mathcal{B}$ by balls of radius at most ϵ we have

$$\sum_i r_i^t = \sum_i r_i^{t-s} r_i^s \leq \epsilon^{t-s} \sum_i r_i^s.$$

So $M_{t,\epsilon} \leq \epsilon^{t-s} M_{s,\epsilon}$. Passing to the limit gives the theorem. □

The contrapositive assertion is that if $0 < s < t$ and $M_t(X) > 0$, then $M_s(X) = \infty$.

Taken together they imply that there is a number d (possibly zero or ∞) such that $M_t(X) = 0$ for all $t > d$ and $M_s(X) = \infty$ for $s < d$. This number d is called the **Hausdorff dimension** of X.

Easy arguments will show that the Hausdorff dimension of $\mathbb{R}^n$ is n.

Contracting ratio lists.

A finite collection of real numbers

$$(r_1, \ldots, r_n)$$

is called a **contracting ratio list** if

$$0 < r_i < 1 \quad \forall\ i = 1, \ldots, n.$$

Theorem

Let $(r_1, \ldots, r_n)$ be a contracting ratio list. There exists a unique non-negative real number s such that

$$\sum_{i=1}^{n} r_i^s = 1. \tag{5}$$

The number s is 0 if and only if $n = 1$.

Proof.

If $n = 1$ then $s = 0$ works and is clearly the only solution. If $n > 1$, define the function f on $[0, \infty)$ by

$$f(t) := \sum_{i=1}^{n} r_i^t.$$

We have $f(0) = n$ and $\lim_{t\to\infty} f(t) = 0 < 1$. Since f is continuous, there is some postive solution to (5). To show that this solution is unique, it is enough to show that f is monotone decreasing. This follows from the fact that its derivative is

$$\sum_{i=1}^{n} r_i^t \log r_i < 0.$$

$\square$

Similarity dimension.

There exists a unique non-negative real number s such that

$$\sum_{i=1}^{n} r_i^s = 1. \qquad (5)$$

Definition

The number s in (5) is called the **similarity dimension** of the ratio list $(r_1, \dots, r_n)$.

A map $f : X \to Y$ between two metric spaces is called a **similarity** with similarity ratio r if

$$d_Y(f(x_1), f(x_2)) = rd_X(x_1, x_2) \;\; \forall \; x_1, \;\; x_2 \in X.$$

(Recall that a map is called **Lipschitz** with Lipschitz constant r if we only had an inequality, $\leq$, instead of equality in the above.)

Realizations of a contracting ratio list.

Let X be a complete metric space, and let $(r_1, \ldots, r_n)$ be a contracting ratio list. A collection

$$(f_1, \ldots, f_n), \quad f_i : \ X \to X$$

is called an **iterated function system** which **realizes** the contracting ratio list if

$$f_i : X \to X, \quad i = 1, \ldots, n$$

is a similarity with ratio r_i. We also say that $(f_1, \ldots, f_n)$ is a **realization** of the ratio list $(r_1, \ldots, r_n)$.

Using Hutchinson.

It is a consequence of *Hutchinson's theorem*, that

Theorem

If $(f_1, \ldots, f_n)$ *is a realization of the contracting ratio list* $(r_1, \ldots, r_n)$ *on a complete metric space,* X*, then there exists a unique non-empty compact subset* $K \subset X$ *such that*

$$K = f_1(K) \cup \cdots \cup f_n(K).$$

In fact, Hutchinson's theorem and the contraction fixed point theorem asserts the corresponding result where the f_i are merely assumed to be Lipschitz maps with Lipschitz constants $(r_1, \ldots, r_n)$.

The set K is sometimes called the **fractal** associated with the realization $(f_1, \ldots, f_n)$ of the contracting ratio list $(r_1, \ldots, r_n)$.

Some facts:

$$\dim(K) \leq s \tag{6}$$

where dim denotes Hausdorff dimension, and s is the similarity dimension of $(r_1, \ldots, r_n)$. In general, we can only assert an inequality here, for the the set K does not fix $(r_1, \ldots, r_n)$ or its realization.

For example, we can repeat some of the r_i and the corresponding f_i. This will give us a longer list, and hence a larger s, but will not change K. But we can demand a rather strong form of non-redundancy known as **Moran's condition**: There exists an open set O such that

$$O \supset f_i(O) \ \forall \ i \ \text{ and } \ f_i(O) \cap f_j(O) = \emptyset \ \ \forall \ i \neq j. \tag{7}$$

Then

Theorem

If $(f_1, \ldots, f_n)$ is a realization of $(r_1, \ldots, r_n)$ on $\mathbb{R}^d$ and if Moran's condition holds then

$$\dim K = s.$$

The Hausdorff dimension of the Cantor set and of the Sierpinski gasket.

In both of these examples the Moran condition is satisfied. So

- For the Cantor set we are looking for an s such that $\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1$ which says that $\frac{2}{3^s} = 1$. Taking logarithms gives $\log 2 - s \log 3 = 0$ so the Hausdorff dimension of the Cantor set is $\log 2 / \log 3$.
- For the Sierpinski gasket the equation becomes $3 \cdot \frac{1}{2^s} = 1$ so the Hausdorff dimension of the Sierpinski gasket is $\log 3 / \log 2$.

Felix Hausdorff

Born: 8 Nov 1868 in Breslau, Germany (now Wroclaw, Poland)
Died: 26 Jan 1942 in Bonn, Germany by suicide, to avoid being sent to an extermination camp.