GROTHENDIECK RINGS OF VARIETIES AND STACKS

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Abstract. The Grothendieck ring of varieties over a field k is a ring that is additively generated by isomorphism classes of varieties over k, modulo a "cut and paste" relation with respect to closed subvarieties, and where the product is cartesian product. Its main application is currently Kontsevich's theory of motivic integration, but it is also related to several other things (for example, point counting over a finite field). There is a variant of this construction that replaces varieties with algebraic stacks.

The talk will be mostly a survey about these Grothendieck rings. I will focus first on the Grothendieck ring of varieties and some interesting questions about it, and eventually I will move on to the variant that involves algebraic stacks (I will not assume that people know about stacks beforehand). Towards the end I will mention a problem that I have been working on lately, about computing the class of the classifying stack BG for an algebraic group G, and that is "morally" related to the Noether problem (rationality of the field of invariants) for G.

Contents

1. Introduction

The talk will be in two parts: first part about varieties, second part about stacks. The second part will not be "for experts only".

2. The Grothendieck ring of varieties

Let us fix a field k (not necessarily algebraically closed). Throughout the talk, "variety" will mean separated (reduced) scheme of finite type over k. Each such thing has an open cover where it becomes isomorphic to the locus cut out by finitely many polynomials in $\mathbb{A}^n_k$, but for example it might not be quasi-projective, globally. Let us denote by Var_k the set of isomorphism classes of varieties over k (this is a big set).

The Grothendieck ring of varieties over k, denoted by $\mathrm{K}_0(\mathrm{Var}_k)$, is defined as follows: it is generated by isomorphism classes $[X]$ of varieties over k, modulo the relations

$$[X] = [Y] + [U]$$

for every closed subvariety $Y \subseteq X$ with open complement U. The sum is a "formal" sum modulo these relations, and the product is defined by product of varieties over k

$$[X] \cdot [Y] = [X \times_k Y]$$

and extended by linearity.

The additive unit element is 0 (the empty sum) and the multiplicative unit is $1 = [\operatorname{Spec} k]$ (the class of a point).

Remark 2.1. If you add "reduced", then you should set $[X] \cdot [Y] = [(X \times_k Y)_{\mathrm{red}}]$ if k is not perfect. In any case, for any X the closed embedding $X_{\mathrm{red}} \subseteq X$ has empty complement, so $[X_{\mathrm{red}}] = [X]$.

Also, if you give the same definition but only allow quasi-projective varieties, you will get the same ring.

2.1. **Examples.** An important element of this ring is the *Lefschetz motive* $\mathbb{L} = [\mathbb{A}^1_k]$.

Example 2.2.

1. Since $\mathbb{A}^n_k \cong \mathbb{A}^1_k \times_k \cdots \times_k \mathbb{A}^1_k$, we see that $[\mathbb{A}^n_k] = \mathbb{L}^n$.
2. Let us compute $[\mathbb{P}^n_k]$. Using $\mathbb{P}^n_k = \mathbb{P}^{n-1}_k \bigcup \mathbb{A}^n_k$, we obtain $[\mathbb{P}^n_k] = [\mathbb{P}^{n-1}_k] + \mathbb{L}^n$, and inductively we get
$$[\mathbb{P}^n_k] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1 = (\mathbb{L}^{n+1} - 1)/(\mathbb{L} - 1).$$
3. More generally, assume that we write X as a disjoint union of locally closed subsets $X = \sqcup_i X_i$ (this is sometimes called a *stratification*). Then $[X] = \sum_i [X_i]$.

   In particular say that X has an *affine paving*, i.e. a decomposition as above with $X_i \cong \mathbb{A}^{n_i}_k$, then $[X]$ is a polynomial in $\mathbb{L}$.

   For example for $\mathbb{P}_n$, we have
$$\mathbb{P}^n_k = \mathbb{A}^n_k + \mathbb{A}^{n-1}_k + \cdots + \mathbb{A}^1_k + \mathbb{A}^0_k.$$
   Then
$$[X] = \mathbb{L}^{n_1} + \mathbb{L}^{n_2} + \cdots \mathbb{L}^{n_r}.$$
   This also applies for example to grassmannians.
4. Let us compute the class of the algebraic group GL_n, with a non-completely rigourous argument: matrices in GL_n correspond to bases of k^n (via the column of the matrix, for example). To choose the first column, you can take any vector in k^n minus the origin. This has class $(\mathbb{L}^n - 1)$. For the second column, you can choose any vector of k^n minus the span of the first column, and this has class $(\mathbb{L}^n - \mathbb{L})$. Iterating, we obtain
$$[\mathrm{GL}_n] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1}).$$

If you've ever computed the cardinality of $\mathrm{GL}_n(\mathbb{F}_q)$, this and the previous formulas should remind you of something (more about this in a bit).

The way to formalize the non-rigourous reasoning in the last example is to prove the following:

Proposition 2.3. *Let $E \to X$ be a vector bundle of rank r. Then $[E] = \mathbb{L}^n[X]$.*

More generally, assume that we have a fibration $Y \to X$ for the Zariski topology with fiber F (i.e. Zariski locally on X, Y is isomorphic to $X \times F$). Then $[Y] = [F][X]$.

Proof. Take an open set $U \subseteq X$ over which the fibration is trivial and use noetherian induction. □

For example, $\mathbb{A}_k^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a Zariski-locally trivial fibration with fiber $\mathbb{G}_m$, so $\mathbb{L}^{n+1} - 1 = [\mathbb{P}_k^n](\mathbb{L} - 1)$.

Something more general than this holds, where $Y \to X$ is only (Zariski-locally) trivial with fiber F on a stratification of X.

2.2. **Euler characteristics.** The idea of this definition is the following:

Definition 2.4. A *generalized Euler characteristic* is a function $f\colon \mathrm{Var}_k \to R$, that respects the formulas above, i.e. $f(X) = f(X')$ if $X \cong X'$, $f(X) = f(Y) + f(U)$ for $Y \subseteq X$ closed with complement U, and $f(X \times_k Y) = f(X) \cdot f(Y)$.

For every generalized Euler characteristic, because of the "universal" definition of the ring $\mathrm{K}_0(\mathrm{Var}_k)$ there will be a ring homomorphism $\mathrm{K}_0(\mathrm{Var}_k) \to R$ that factors the function $f\colon \mathrm{Var}_k \to R$. The natural map $\mathrm{Var}_k \to \mathrm{K}_0(\mathrm{Var}_k)$ is the "universal" generalized Euler characteristic.

Example 2.5.
- Let k be a subfield of $\mathbb{C}$. For a variety X, set $\chi_c(X) = \sum_i (-1)^i \dim H_c^i(X, \mathbb{C})$. This is a gEc, where $H_c^i(X, \mathbb{C})$ denotes compactly supported cohomology. One can also use ordinary cohomology, but it is not immediate to show that $\chi(X) = \chi_c(X)$.
- Let $k = \mathbb{F}_q$ be a finite field. The function $\mathrm{Var}_k \to \mathbb{Z}$ defined on generators by $X \mapsto \#X(\mathbb{F}_q)$ is a gEc, sometimes called the "point counting measure".

Because of this universal property, if two varieties X and Y have the same class $[X] = [Y]$ in $\mathrm{K}_0(\mathrm{Var}_k)$, then every Euler characteristic will take the same value on X and Y. Thus, it is interesting to understand when exactly is $[X] = [Y]$.

Note that if X and Y can be decomposed as $\sqcup_i X_i$ and $\sqcup_i Y_i$ for locally closed subvarieties, and $X_i \cong Y_i$ for every i, then $[X] = [Y]$.

Conjecture 2.6 (Larsen-Lunts)**.** *Is the converse true?*

Spoiler: this is true in some cases, but the (recent) answer in the general case is "no". More on this later.

Remark 2.7. This universal property is nice and natural, but note that this ring is not likely to have nice properties or be easy to study in general. For example, I think that one can show that it is not finitely generated over $\mathbb{Z}$. It is also not reduced.

Generally speaking getting a firm handle on things on this ring is not easy, morally because it contains a wealth of information.

Remark 2.8. There is a conjectural abelian category of motives over k, that has an analogous universal property with respect to "cohomology theories" instead of "Euler characteristics". The idea here is that $\mathrm{K}_0(\mathrm{Var}_k)$ should be the K_0 of this abelian category (as the notation somewhat suggests).

Here is another example of an Euler characteristic. Let k be a subfield of $\mathbb{C}$.

Proposition 2.9. *There is a ring homomorphism* $P\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathbb{Z}[t]$, *such that for* X *smooth and projective, the polynomial* $P([X])$ *is the Poincaré polynomial* $\sum_i t^i \cdot \dim H^i(X, \mathbb{C})$.

Note that $P(X)(-1) = \chi(X)$. For X arbitrary, the polynomial $P(X)$ is called the *virtual Poincaré polynomial*. We have that $P(\mathbb{L}) = t^2$, since $\mathbb{L} = [\mathbb{P}^1_k] - 1$, and $P([\mathbb{P}^1_k]) = 1 + t^2$, $P(1) = 1$.

Now let C be a smooth projective curve of genus $g > 0$. Then $P([C]) = 1 + gt + t^2$. Note that this implies that $[C]$ is not a polynomial in $\mathbb{L}$.

In fact, for an irreducible curve over an algebraically closed field, $[C] = \mathbb{L} + \beta$ with $\beta \in \mathbb{Z}$ if and only if C is rational.

2.3. **Interesting facts.** Related to point counting, recall that if X is a scheme of finite type over $k = \mathbb{F}_q$, its Zeta function is the formal power series

$$Z(X) = \exp\left(\sum_{n \geq 1} t^n \cdot \frac{\#X(\mathbb{F}_{q^n})}{n}\right).$$

This is a rational function, satisfies a functional equation, etc. (Weil conjectures, ...)

There is a motivic version of this, due to Kapranov. Let k be again an arbitrary field.

For a quasi-projective variety X, consider the symmetric powers $\mathrm{Sym}^n X$. Recall that these are defined as X^n modulo the permutation action of S_n. These are again quasi-projective varieties.

Define

$$Z_{\mathrm{mot}}(X) = \sum_{n \geq 0} t^n \cdot [\mathrm{Sym}^n X] \in 1 + t\mathrm{K}_0(\mathrm{Var}_k)[[t]].$$

Proposition 2.10. *This defines a group homomorphism* $\mathrm{K}_0(\mathrm{Var}_k) \to (1 + t\mathrm{K}_0(\mathrm{Var}_k)[[t]], \cdot)$

If k is a finite field, then by using the point-counting homomorphism, $Z_{\mathrm{mot}}(X)$ specifies to the Hasse-Weil zeta function. It is maybe natural to expect that $Z_{\mathrm{mot}}(X)$ is also rational (in some sense... it is not completely clear what it should mean!).

This is the case for smooth projective curves (Kapranov), but not in general: Larsen-Lunts showed that for a smooth projective surface, $Z_{\mathrm{mot}}(X)$ is rational if and only if X has negative Kodaira dimension. So rationality is not "motivic", in most cases.

Another interesting connection is with *stable birationality.*

Definition 2.11. Two varieties X and Y are stably birational if for some $n, m \geq 0$ we have that $X \times_k \mathbb{P}^n_k$ and $Y \times_k \mathbb{P}^m_k$ are birationally isomorphic. A variety X is stably rational if it is stably birational to $\operatorname{Spec} k$ (or to any projective space).

Let SBir_k denote the monoid of stable birational equivalence classes of varieties over k, where the operation is product. Let $\mathbb{Z}[\mathrm{SBir}_k]$ be the monoid algebra.

Proposition 2.12. *There is a surjective ring homomorphism* $\mathrm{K}_0(\mathrm{Var}_k) \to \mathbb{Z}[\mathrm{SBir}_k]$ *sending the class of a smooth projective variety to its stable birational equivalence class, whose kernel is the ideal generated by* $\mathbb{L}$*. Hence* $\mathbb{Z}[\mathrm{SBir}_k] \cong \mathrm{K}_0(\mathrm{Var}_k)/(\mathbb{L})$*.*

Why do you have to kill $\mathbb{L}$??? Because the image of $[\mathbb{P}^1_k]$ and of 1 has to be the same, and $[\mathbb{P}^1_k] = \mathbb{L} + 1$.

On the opposite side of the spectrum, sometimes it is important to invert $\mathbb{L}$. For example in motivic integration, you want to define a measure on the arc space of an algebraic variety, with values in $\mathrm{K}_0(\mathrm{Var}_k)$, but normalized in a way that the measure of affine space is 1, so you want to divide by $\mathbb{L}$. This will also be important for the Grothendieck ring of stacks.

A question is natural at this point. Is $\mathbb{L}$ a zero-divisor in $\mathrm{K}_0(\mathrm{Var}_k)$? Poonen proved in 2002 that $\mathrm{K}_0(\mathrm{Var}_k)$ is not a domain. His example used some abelian varieties, such that $A \times A \cong B \times B$ but A is not isomorphic to B (more or less). Still, it is possible that $\mathbb{L}$ is a non-zerodivisor. This was widely believed to be true.

Very recently (2014) Borisov proved that $\mathbb{L}$ is in fact a zero divisor, and his proof also implies that the "cut and paste" question of Larsen-Lunts has a negative answer.

He shows that for two smooth derived equivalent non-birational Calabi-Yau 3folds X and Y (pfaffian-grassmannian double mirror correspondence) we have

$$([X] - [Y])(\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = 0.$$

3. The Grothendieck ring of stacks

3.1. **Algebraic stacks.** Now I have to briefly tell you what algebraic stacks are, without really telling you.

Algebraic stacks are a category of "spaces" that contains algebraic varieties and schemes. They are sometimes kind of shunned because to be very precise about them you need a not-so-small chunk of category theory (you need to talk about categories fibered in groupoids that satisfy descent).

Here is the sentence that you will hear in any talk that uses algebraic stacks for an audience that is not likely to know about them: you should think about algebraic stacks as some sort of algebraic varieties, where points are allowed to have intrinsic "stabilizer groups" attached to them. Caveat: this is only a first approximation of the truth, but it will be sufficient for today.

So, every algebraic variety or scheme is an algebraic stack, where all these stabilizer groups are trivial.

Algebraic stacks were invented because sometimes you want a space that parametrizes some kinds of objects and with certain properties, and that space does not exist as a variety or a scheme. Sometime it exists as a stack.

Remark 3.1. The most famous example is the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points. You might have heard about the construction of the coarse moduli space $\overline{M}_{g,n}$, a projective variety whose (geometric) points correspond to isomorphism classes of stable curves over k. This used to make people happy, and still does it for someone.

But for example, the coarse moduli space is not smooth (even though the deformation theory of nodal marked curves is smooth, in a sense), and does not have a universal family. Singular points come from "extra automorphisms" of the corresponding curves. The algebraic stack $\overline{\mathcal{M}}_{g,n}$ is smooth, and does have a universal family. In particular, it explains why $\overline{M}_{g,n}$ has a well-defined intersection product with rational coefficients.

3.1.1. *Quotient stacks.* A big source of examples and intuition about algebraic stacks is given by group actions.

Let G be an algebraic group over k (it can be a finite group, for example) acting on a variety X. It is useful to be able to construct quotients (i.e. orbit spaces), and you might have heard that this is a big industry in algebraic geometry (Geometric Invariant Theory). The short story: it is not always (and this is not rare) possible to have a nice quotient X/G as a variety or even a scheme. The GIT solution is to be content to just define a quotient for a smaller (dense open usually) subset of X, where the group action is well-behaved (semistable points). There is a quotient map $X \to X/G$, but this does not alway have nice properties. Even if the action of G on X is without stabilizers (so that you expect the fibers of the projection to be isomorphic to G, as varieties), it is not always true that $X \to X/G$ is a G-principal bundle.

In the stacks world, every group action has a quotient stack $[X/G]$, and there is a projection $X \to [X/G]$ that is a G-principal bundle. The geometry of $[X/G]$ "is" the G-equivariant geometry of X, so objects on the quotient are objects on X + a compatible group action on them. For example, vector bundles on $[X/G]$ are G-equivariant vector bundles on X.

Points of $[X/G]$ correspond to orbits of G on X, and the group associated to such a point is exactly the "stabilizer group of the orbit". So the "geometry" of the quotient stack is a mix of the geometry of an orbit space and some data about the stabilizers of the group action.

In the extreme case where $X = \operatorname{Spec} k$ (and G is of course acting trivially), you get the "classifying stack" $BG = [\operatorname{Spec} k/G]$ of G. This construction is the analogue of classifying spaces in topology, where you take a simply connected space EG with a free action of G, and take the quotient $BG = EG/G$ (for example $S^1 = \mathbb{R}/\mathbb{Z}$ is the classifying space of $\mathbb{Z}$). Apart from simple examples, in topology BG is typically infinite dimensional. In AG, the complication is in fact that it is a stack.

3.2. **The Grothendieck ring of stacks.** The same construction that I introduced for varieties produces a Grothendieck ring of stacks.

Define $\mathrm{K}_0(\mathrm{Stack}_k)$ to be the ring generated by isomorphism classes $[X]$ of algebraic stacks of finite type over k with affine diagonal, subject to the relation $[X] = [Y] + [U]$ for every closed substack $Y \subseteq X$, and $[E] = [\mathbb{A}^n_k \times_k X]$ for every vector bundle $E \to X$ of rank n, and with product defined by $[X][Y] = [X \times_k Y]$ (so that $[E] = \mathbb{L}^n[X]$ in the formula above).

The condition about vector bundles follows from the other relations for varieties, but not for stacks, and we certainly want it.

There is a natural map $\mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Stack}_k)$. One can prove that from the relations above, it follows that if $Y \to X$ is a GL_n-principal bundle, then $[Y] = [X][\mathrm{GL}_n]$ (vector bundles are very strictly related to principal bundles for GL_n).

Since $\operatorname{Spec} k \to B\mathrm{GL}_n$ is a GL_n-torsor, we have $[\mathrm{GL}_n][B\mathrm{GL}_n] = 1$, so

$$[\mathrm{GL}_n] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1}) = \mathbb{L}^k(\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1) \cdots (\mathbb{L} - 1)$$

is invertible in $\mathrm{K}_0(\mathrm{Stack}_k)$. This means that $\mathbb{L}$ and all polynomials $\mathbb{L}^n - 1$ are also invertible. It is a remarkable fact that this is all you need to invert if you start from $\mathrm{K}_0(\mathrm{Var}_k)$.

Proposition 3.2 (Ekedahl)**.** *The ring $\mathrm{K}_0(\mathrm{Stack}_k)$ is isomorphic to the localization of $\mathrm{K}_0(\mathrm{Var}_k)$ on the multiplicative subset generated by $\mathbb{L}$ and $\mathbb{L}^n - 1$ for $n \in \mathbb{N}$.*

3.3. **Computing $[BG]$.** Computing the class $[BG]$ for a group G in the Grothendieck ring of stacks is an interesting question.

An algebraic group G is *special* if every G-torsor is Zariski-locally trivial. If $X \to Y$ is a G-torsor for G special, then $[X] = [G][Y]$. This holds for X and Y varieties, but also for stacks.

So if G is special, since $\operatorname{Spec} k \to BG$ is a G-torsor, we have $1 = [BG][G]$, i.e. $[BG] = [G]^{-1}$.

The first guess is that it should in general be true that $[BG] = [G]^{-1}$, if G is connected. Ekedahl showed though that for every non-special G there is a G-principal bundle $P \to T$ such that $[P] \neq [G] \cdot [T]$, so multiplicativity for G-torsors fails in general, and on second thought if it fails for some torsor, it is reasonable to expect that it does for the "universal one".

Conjecture 3.3. *There exists a connected linear algebraic group G such that $[BG] \neq [G]^{-1}$.*

The class of $[BG]$ for G connected has been computed in some examples: by Daniel Bergh for PGL_2 and PGL_3 and by Dhillon and Young for SO_{2n}. Me and Angelo Vistoli computed the class of $B\mathrm{SO}_n$ for any n. In all those cases, we have $[BG] = [G]^{-1}$.

If G is finite, the expected class for $[BG]$ is 1. Ekedahl showed that this is the case in many examples, but there are also examples where this fails. Interestingly, in the cases where this fails the obstruction for it being true is given by the same obstruction that prevents the Noether problem for G to have a positive answer.

3.4. **The Noether problem.** The Noether problem was posed by Emmy Noether sometime in the first half of the 20th century. Here is the original formulation.

Let G be a finite group, and k be a field. Consider a purely transcendental extension $k(x_1, \ldots, x_n)$ on which G acts by permuting the variables. We can consider the invariant subfield $k(x_1, \ldots, x_n)^G$. The question is whether this is a purely transcendental extension of k or not. Noether introduced this problem in relation to the inverse Galois problem over $\mathbb{Q}$.

In particular one can look at the "translation" action of G on $k(x_g \mid g \in G)$, and consider the invariant field for this. Noether's problem for G asks if this field is purely transcendental over k or not.

Swan proved that for $G = \mathbb{Z}/47\mathbb{Z}$ over $\mathbb{Q}$, the answer to this question is negative. Later Saltman and Bogomolov found many more examples, over an arbitrary field.

For a non-finite algebraic group G (say connected, and maybe semisimple), the question is the following: given a linear representation V of G with trivial generic stabilizer, is the quotient V/G stably rational? Or retract rational?

There are a lot of results in this direction, but for example there is no known example of a connected linear group G for which V/G is not rational.

The fact that $[BG] = [G]^{-1}$ or not is morally related (by analogy) with the Noether problem for G.

3.5. **Results.** Recently with Roberto Pirisi we have been looking at the class of BG for spin groups. The spin group Spin_n is the universal (2:1) cover of SO_n. It is special for $n \leq 6$, so in that range for sure $[B\mathrm{Spin}_n] = [\mathrm{Spin}_n]^{-1}$. We have proved that this is also the case for $n = 7, 8$, and that also $[BG_2] = [G_2]^{-1}$.

Theorem 3.4 (Pirisi,-). *For $k = \mathbb{C}$ in the Grothendieck ring of stacks we have*

$$[BG] = [G]^{-1}$$

for $G = G_2, \mathrm{Spin}_7, \mathrm{Spin}_8$.

For the general n, we reduce the computation to $[BD_n]$ for a finite subgroup $D_n \subseteq \mathrm{Spin}_n$. We believe that Spin_n for high n has a good change of being an example of a connected group for which $[BG] \neq [G]^{-1}$.

The Noether problem is still open for spin groups. Conjecturally, for $n \geq 15$ quotients of generically free Spin_n-representations are not retract rational (Merkurjev's conjecture). Our hope is to be able to prove that for some n we have $[B\mathrm{Spin}_n] \neq [\mathrm{Spin}_n]^{-1}$ and that this will turn out to be equivalent to a negative answer to Noether's problem.