Log geometry (with a slight view towards tropical geometry) and root stacks

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Log geometry

- (born in) Arithmetic geometry (log crystalline cohomology), work of Fontaine-Illusie, Kato.
- Hodge theory
- Tropical/non-Archimedean geometry
- Moduli theory
- Mirror symmetry and log GW invariants
- etc

(many other names. **Some** of them: Deligne, Faltings, Kato (a different one), Nakayama, Ogus, Olsson, Abramovich, Chen, Gross, Siebert, . . .)

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Toric case: $P = \sigma^\vee \cap M$, where $\sigma \subseteq N_\mathbb{R}$ is a rational polyhedral cone and $M \simeq N^\vee$ is a lattice.

The scheme $X_P = \operatorname{Spec} k[P]$ has a natural structure of log scheme, the “additional stuff” in this case is just “given by” P.

Another example: X a smooth variety, $D \subseteq X$ an effective Cartier divisor ($\approx$ codim. 1 subvariety with a nice equation)

(example: compactify some $U \subseteq X$ by adding a simple normal crossing divisor $D = X \setminus U$ at the boundary. Do stuff on X, and then go back to U, so need to "keep track" of D)

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One way to do it:

$$M_{(X,D)} = \{f \in \mathcal{O}_X \mid f|_{X \setminus D} \text{ is invertible}\} \subseteq \mathcal{O}_X$$

This is a sheaf of submonoids, and contains all the units $\mathcal{O}_X^*$ (and recovers D in good cases).

In this case the "additional stuff" is the sheaf $M_{(X,D)}$ together with the map to $\mathcal{O}_X$.

Definition

A log scheme (X, M_X) is

- a scheme X
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- a sheaf of monoids M_X with a map $\alpha_X \colon M_X \to (\mathcal{O}_X, \cdot)$ (pre-log)
- such that $\alpha_X^{-1}(\mathcal{O}_X^*) \xrightarrow{\simeq} \mathcal{O}_X^*$ is an isomorphism (i.e. the units are the same).

The sheaf M_X contains “distinguished” or “new” “regular functions” that you want to keep track of.

Example

If X is a scheme, then $(X, \mathcal{O}_X^*)$ is a log scheme (trivial log structure).

The sheaf $\overline{M}_X = M_X/\mathcal{O}_X^*$ (characteristic sheaf) contains the "non-trivial" part of the log structure.

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There is a notion of morphism of log schemes $(X, M_X) \to (Y, M_Y)$: a morphism $f \colon X \to Y$ of schemes with

$$\begin{array}{ccc} f^{-1}M_Y & \longrightarrow & M_X \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \end{array}$$

The functor $X \mapsto (X, \mathcal{O}_X^*)$ embeds schemes in log schemes.

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If P is a monoid, $X_P = \operatorname{Spec} k[P]$ has a log structure:

$$P \to k[P] = \Gamma(\mathcal{O}_{X_P})$$

induces a pre-log structure (by sheafifying)

$$P_{X_P} \to \mathcal{O}_{X_P}.$$

You can “logify” in a universal way to get

$$M_{X_P} \to \mathcal{O}_{X_P}.$$

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- require that charts exist locally on X
- impose "niceness" conditions on the log structure using the local models. For example
  - P integral
  - P finitely generated (fine = finitely generated and integral)
  - P saturated (fs = fine and saturated)

The category of fs log schemes is particularly popular.

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Example: if $D \subseteq X$ is NC, then (étale) locally it looks like $\{x_1 \cdot \cdots \cdot x_r = 0\} \subseteq \mathbb{A}^n$, and $\mathbb{N}^r \to \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ that sends e_i to x_i is a chart for the log structure.

So how do you visualize a (fs) log scheme?

One way: look at $\overline{M}_X$.

- there is a largest open subscheme $U \subseteq X$ such that $\overline{M}_X|_U = 0$ (i.e. the log structure is trivial on U). (U might be empty)

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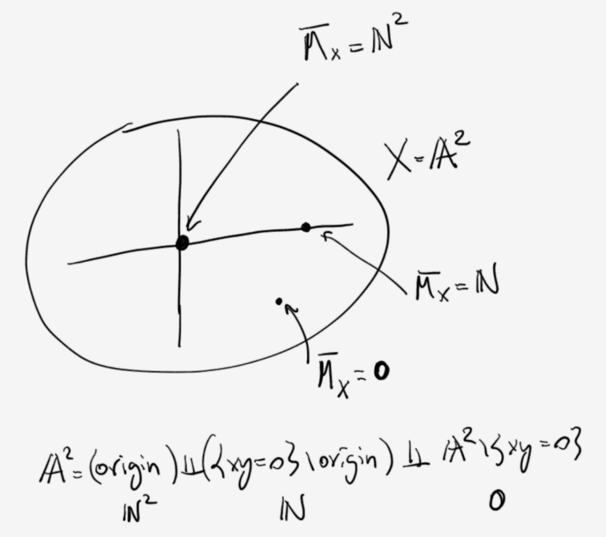
- more generally $\overline{M}_X$ is locally constant on a stratification of X.

So $X = \bigsqcup_i X_i$ and we have a single monoid P_i on each constructible piece X_i.

In the case where $D = \{xy = 0\} \subseteq \mathbb{A}^2 = X$,



Geometry ≈ each piece X_i

Combinatorics ≈ the monoids P_i

The way in which the X_i are attached together is a mixture of the two aspects (the are "specialization maps" between the monoids).

A spectrum of log schemes:

Moduli theory

Starting point: there is a notion of log smooth morphism that generalizes usual smoothness.

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- log infinitesimal liftings
- log differentials $\Omega^1_{\log}$
- log deformation theory
- log de Rham cohomology
- etc

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If the log scheme is (X, D) where D is NC, the sheaf $\Omega^1_{\log}$ is exactly the sheaf of differential forms with at most a pole of order 1 along D, locally generated by Ω^1 and $d\log(x_i) = \frac{dx_i}{x_i}$.

One application: moduli of log smooth curves.

You can construct a moduli space (or stack) $\mathcal{M}_{g,n}^{\log}$ of (basic, stable) log smooth curves of genus g and "type" n.
Every smooth curve is log smooth, and this gives $\mathcal{M}_{g,n} \subseteq \mathcal{M}_{g,n}^{\log}$.

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- The moduli space $\mathcal{M}_{g,n}^{\log}$ is proper (degenerations are already there)
- $\mathcal{M}_{g,n} \subseteq \mathcal{M}_{g,n}^{\log}$ is an open immersion
- in the boundary we find exactly stable nodal curves.

So log smoothness “selects” good degenerations of smooth objects.

This idea was applied in many other cases: log K3 surfaces, abelian varieties, toric hilbert schemes....

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Another instance: Gross-Siebert program for mirror symmetry. Idea: degenerate a smooth variety to a union of toric varieties and then use a combinatorial construction.

Log smoothness is used to ensure that the degeneration is nice enough.

Tropical geometry

The relation with tropical geometry is via a tropicalization map (Martin's work).

In the case of a subvariety of a torus $T = \operatorname{Spec} k[M]$ (here k is trivially valued), define

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by requiring

$$\langle \mathrm{trop}(x), m \rangle = -\log |\chi^m|_x.$$

Here $x \in T^{\mathrm{an}}$ corresponds to the seminorm $|\cdot|_x\colon k[M] \to \mathbb{R}$ and N and M are dual via $\langle \cdot, \cdot \rangle$ (Einsiedler-Kapranov-Lind).

Then the tropicalization of $Y \subseteq T$ is the closure of $\mathrm{trop}(Y^{\mathrm{an}})$.

- There is a version of this that replaces T by a T-toric variety X_Δ

$$\operatorname{trop}\colon X_\Delta^{\mathrm{an}} \to N_{\mathbb{R}}(\Delta)$$

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- and a version for fs log schemes of finite type X

$$\mathrm{trop}\colon X^{\beth} \to \overline{\Sigma}_X$$

where $\overline{\Sigma}_X$ is a combinatorial object that depends on X (Ulirsch).

This gives a nice bridge between the two theories.

Root stacks

(infinite) root stacks incorporate the “log geometry” of X into their “bare” geometry.

In

$$\alpha_X \colon M_X \to \mathcal{O}_X$$

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If we mod out by them

$$\overline{\alpha}_X \colon \overline{M}_X \to [\mathcal{O}_X/\mathcal{O}_X^*] = \mathrm{Div}_X$$

where Div_X is the category of line bundles with a global section (L, s).

The functor $\overline{\alpha}_X\colon \overline{M}_X \to \mathrm{Div}_X$ (a "Deligne-Faltings" structure) sends sums into tensor products (is a "symmetric monoidal functor").

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Fix $n \in \mathbb{N}$. Define $\sqrt[n]{X}$ as the (algebraic) stack that parametrizes liftings

$$\begin{array}{ccc} \overline{M}_X & \longrightarrow & \mathrm{Div}_X \\ \downarrow & \nearrow & \\ \frac{1}{n}\overline{M}_X & & \end{array}$$

that you could call "n-th roots" of the log structure.

If $D \subseteq X$ is a smooth divisor, then using a chart we can look at

$$1 \longmapsto (\mathcal{O}_X(D), s_D)$$

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathrm{Div}(X) \\ \downarrow & \nearrow & \\ \frac{1}{n}\mathbb{N} & & \end{array}$$

and $\frac{1}{n}$ will go into (L, s) such that $(L, s)^{\otimes n} \simeq (\mathcal{O}_X(D), s_D)$.

This is the same as Cadman's "stack of n-th roots" of the divisor D.

To consider all roots, take a limit for growing n.

If $n \mid k$, then there is $\sqrt[k]{X} \to \sqrt[n]{X}$, and these form an inverse system. Take

$$\sqrt[\infty]{X} = \varprojlim_{n} \sqrt[n]{X}$$

- not algebraic, but has a flat (fpqc) atlas
- incorporates the log geometry of X in its "bare" geometry
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Theorem (-,Vistoli)

There is a procedure that gives back (X, M_X) from $\sqrt[\infty]{X}$. In particular if $\sqrt[\infty]{X} \simeq \sqrt[\infty]{Y}$, then $(X, M_X) \simeq (Y, M_Y)$.

Thank you for listening!

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