LOGARITHMIC GEOMETRY AND SOME APPLICATIONS

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Contents

1. Introduction

I want to talk about log geometry. Since, despite being increasingly popular, it's not yet in every introductory algebraic geometry textbook, a big part of the talk will be a survey/advertisement about it. Towards the end I will tell you something about some recent work of my own and D. Carchedi, S. Scherotzke and N. Sibilla, time permitting.

Here is the plan of the talk.

§1. What is logarithmic geometry?
§2. Why care? (i.e. applications to moduli theory)
§3. Kato-Nakayama space and infinite root stack

2. What is logarithmic geometry?

Short answer: it is a variant of algebraic geometry, where on top of the usual algebraic (or geometric...) aspect, you have an additional structure (the "log" part). It was introduced in the late 80s by Fontaine-Illusie, Deligne-Faltings and K. Kato. There are many other names for later developments, way too many to be written down.

It was originally used for arithmetic purposes [Kat89] (log crystalline cohomology and crystals with log poles, in particular in the non-smooth case of semi-stable degenerations), and later it spread to touch various areas, in particular moduli theory (see the survey [ACG$^+$13]), which is the aspect I am most interested in. My talk will be accordingly biased.

The object of interest in log geometry are log schemes.

Definition 2.1.

$$\text{“log scheme”} = \text{a scheme } X + \text{“extra stuff”}.$$

The extra stuff in the above equation typically either

- keeps track of a "boundary" on X (compactification), or
- has infinitesimal information about a family, of which X is a fiber (degeneration).

The two aspects are connected: typically when one compactifies a moduli space, points in the boundary correspond to degenerate objects.

Remark 2.2. Also, about the "boundary" thing, log geometry gives way to talk about "manifolds with boundary" (for example a half closed cylinder) in algebraic geometry [GM]. This will come back at the end of the talk, hopefully.

I will build towards the definition of a log scheme through an example of the "compactification" situation.

Let us say we have a quasi-projective smooth variety Y over an algebraically closed field of characteristic zero, and we want to do something with it. Maybe to do this something it would be better if Y were proper (for example there's some cohomology groups involved, and we'd like them to be finite-dimensional).

We can compactify $Y \subseteq X$, where X is smooth and proper and $X \setminus Y = D$ is a (simple) normal crossings divisor [Kol07], and work with X.

Remark 2.3. Recall that the fact that D is normal crossings means that étale locally $D \subseteq X$ looks like the union of k coordinate hyperplanes $\{x_1 \cdots x_k = 0\}$ in $\mathbb{A}^n$, with $n \geq k$.

So we do stuff on X, but eventually we want to go back to Y. How do we do that?

Example 2.4. Here's an example of how this happens in practice. One can define a sheaf of logarithmic differential forms $\Omega_X(\log D)$ on X, that étale locally, where $D \subseteq X$ looks like $V(x_1 \cdots x_k) \subseteq \mathbb{A}^n$, is generated by

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_k}{x_k}, dx_{k+1}, \ldots, dx_n.$$

In other words we are considering meromorphic differential forms that, together with their exterior derivative, have at most poles of order one along D. Note that these are all regular on Y.

This construction extends to a whole logarithmic de Rham complex $\Omega^\bullet_X(\log D)$, and one has the following "log de Rham" theorem [Gro66, Del71].

Theorem 2.5. *Say we are in the situation above, and over the complex numbers. Then there is an isomorphism* $\mathrm{H}^i(\Omega^\bullet_X(\log D)) \cong \mathrm{H}^i_{sing}(Y, \mathbb{C})$.

This is an instance where by "keeping track of D" we can recover information about the complement $Y = X \setminus D$. Note that this implies that the log de Rham cohomology is independent of the compactification X, something that is not at all clear a priori.

Log geometry gives a nice machine to keep track of the embedding $Y \subseteq X$ in the "intrinsic geometry" of the situation (i.e. without just artificially working with the pair (X, Y), or (X, D), or something similar). A more general case than this normal crossings situation is when $Y \subseteq X$ is a toroidal embedding [KKMSD73], i.e. étale locally it looks like the embedding of the open torus in a toric variety.

In these cases, the "machine" happens as follows: consider the sheaf on X

$$M_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_{U \cap Y} \in \mathcal{O}_Y^\times(U \cap Y)\}$$

i.e. the sheaf of sections of $\mathcal{O}_X$ that are invertible outside of D. Note that this is not a sheaf of rings (addition is not guaranteed to preserve the property), but the inclusion $i\colon M_Y \subseteq \mathcal{O}_X$ realizes M_Y as a sheaf of sub-monoids of $\mathcal{O}_X$, such that moreover $\mathcal{O}_X^\times \subseteq M_Y$.

What does this sheaf keep track of? Well, in the normal crossings case, if we are near a single branch of D (with equation g), then a function that is invertible outside of D is of the form $c \cdot g^k$, where c is a unit and $k \geq 0$. If we are in a point where two branches of D meet (with equations g and h), then every such function is $c \cdot g^k \cdot h^j$, where c is a unit and $k, j \geq 0$. In the first case the relevant piece of info is the monoid $\mathbb{N}$, in the second case it's $\mathbb{N}^2$ (note that this ignores the units). You can imagine what happens when more than two branches of D meet. In situations where D is not normal crossings but we have a more general toroidal embedding, you can also imagine that non-free monoids will show up.

In short, in this case the sheaf M_Y keeps track of invertible functions on Y, that locally extend to regular functions on the boundary (this should remind of characters and monomial functions on toric varieties).

Let's give the definition of a log scheme.

Definition 2.6. A log scheme (X, M, α) is a scheme X with a sheaf of monoids M (for the étale topology) and a morphism $\alpha\colon M \to \mathcal{O}_X$ where $\mathcal{O}_X$ is seen as a monoid with the multiplication of sections, and such that α identifies the units (i.e. $\alpha|_{\alpha^{-1}\mathcal{O}_X^\times}\colon \alpha^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$ is an isomorphism).

Remark 2.7. This sheaf M should be thought of as containing "(possibly new) regular functions" that we are keeping track of.

There is a notion of morphism of log schemes, that is easy to guess (I won't go into details about this). Thus log schemes form a category LogSch.

Example 2.8. In the example above, we have $i\colon M_Y \subseteq \mathcal{O}_X$, and M_Y physically contains all the units $\mathcal{O}_X^\times$. In general, in order to have nice functoriality properties, we need to allow non-injective maps α.

Example 2.9. Every scheme X is a log scheme, by taking $M = \mathcal{O}_X^\times$ (which is the smallest log structure one can have) and α the inclusion. This is called the trivial log structure. This gives a fully faithful embedding Sch $\subseteq$ LogSch, so the theory of log schemes is an enlargement of the usual theory of schemes.

This example suggests that the non-trivial part of the log structure is all in the quotient $\overline{M} = M/\mathcal{O}_X^\times$, usually called the characteristic monoid.

Remark 2.10. In fact (in presence of a mild hypothesis - that the log structure is quasi-integral) one can reformulate the definition [BV12] by replacing α with a symmetric monoidal functor $L\colon \overline{M} \to \mathrm{Div}_X$, where Div_X is the fibered category of line bundles with a global section on the small étale site of X. The link is in the fact that $\mathrm{Div}_X = [\mathcal{O}_X/\mathcal{O}_X^\times]$, and we get L by taking a "quotient" of α by $\mathcal{O}_X^\times$.

Looking at the stalks of the sheaf $\overline{M}$ gives a rough way to visualize the log scheme X. The next example was basically already worked out above.

Example 2.11. Consider $X = \mathbb{A}^2$ with the log structure coming from the divisor D that is the union of the coordinate axes. Then the stalk of $\overline{M}$ at $p \in X$ is

- trivial if p is outside of D (this always happens),
- isomorphic to $\mathbb{N}$ if p is on D but not the origin, and the generator is the image of x or y,
- isomorphic to $\mathbb{N}^2$ if p is the origin, generated by the images of x and y.

You might have noticed that $\mathbb{A}^2$ is a toric variety, and the union of the axes is the toric boundary. In fact every normal toric variety $X(\Delta)$ has a canonical log structure, associated to the embedding $T \subseteq X(\Delta)$ of the open torus. The monoids that appear as stalks are closely related to lattice points in the dual cones of the cones appearing in the fan Δ. One important example of this is an affine toric variety $X_P = \operatorname{Spec} k[P]$ where P is a fine saturated sharp monoid (in standard notation, $P = \sigma^\vee \cap M$).

Usually one works with log schemes (X, M, α) for which the log structure "comes" locally from the canonical log structure of X_P for monoids as above. In other words étale locally on X there is one such monoid P with a morphism $X \to \operatorname{Spec} k[P]$, and the log structure of X is pulled back from $\operatorname{Spec} k[P]$ (log structures can be pulled back). Note that the image might be entirely contained in the boundary divisor, so the stalks of $\overline{M}$ need not be trivial in an open subset in general!

These log schemes are called "fine and saturated" (sometimes "fs" for short).

Before going to applications, let me mention how the second interpretation of a log structure, on retaining information about a family, comes up. Assume that we have a flat family $X \to \mathbb{A}^1$ where the fibers X_t for $t \neq 0$ are smooth, and the central fiber X_0 is a simple normal crossings variety (for example take $X = V(xy - t)$ and $X \to \mathbb{A}^1$ the projection to the t-coordinate).

Then X and $\mathbb{A}^1$ admit log structures for which the map $X \to \mathbb{A}^1$ becomes a "well-behaved" morphism of log schemes (in the sense of log smoothness, coming next): they are the divisorial log structures given by $X_0 \subseteq X$ and $\{0\} \subseteq \mathbb{A}^1$. If we restrict the log structure of X to X_0, we get a log structure on X_0 that is not of "divisorial type", since it is not generically trivial! This is a log structure that "remembers" some information about the family $X \to \mathbb{A}^1$.

3. Applications to moduli theory and enumerative geometry

Moving on, one can develop an EGA-style theory of these log schemes, and reinterpret some classical notions in this new category. The most important example of this for applications to moduli theory is log smoothness [Kat89].

In this section we will denote a log scheme just by X, and $\underline{X}$ will denote the underlying scheme.

There is a notion of log smoothness for a morphism of log schemes $f: X \to Y$ that generalizes usual smoothness, in the sense that if X and Y are trivial log schemes (i.e. they have the trivial log structure coming from $\underline{X}$ and $\underline{Y}$), then f is log smooth if and only if $\underline{f}: \underline{X} \to \underline{Y}$ is smooth. Sometimes though, $\underline{f}$ is not a smooth morphism of schemes $\underline{X} \to \underline{Y}$, but there is a way to equip the schemes with log structures and promote $\underline{f}$ to a morphism of log schemes $f: X \to Y$ that is log smooth.

For example, if P is any fine saturated sharp monoid, the affine toric variety $\operatorname{Spec} k[P]$ with its canonical log structure is log smooth over $\operatorname{Spec} k$ (equipped with the trivial log structure), despite almost always not being smooth in the usual sense.

This log smoothness property is connected with infinitesimal liftings of maps and deformation theory in the log setting, as smoothness is in the classical setting. Also relevant here is the sheaf of logarithmic differentials $\Omega_X^{\log}$, that one can define for a general log scheme. If $\underline{X}$ is smooth and the log structure of X is given by a normal crossings divisor $D \subseteq \underline{X}$, then $\Omega_X^{\log}$ is exactly the sheaf of logarithmic differential forms $\Omega_{\underline{X}}(\log D)$ that we encountered before.

Log smooth maps behave like classically smooth maps in some respects. K. Kato writes that a log structure is "a "magic" by which a degenerate scheme begins to behave as being non-degenerate" [Kat].

Now let's get to moduli theory. The most famous example of a moduli problem is probably the moduli space (stack) of smooth curves of genus g with n marked points $\mathcal{M}_{g,n}$. This is not proper, and has a Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ that parametrizes (stable) nodal curves. The complement $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is a normal crossings divisor.

Now assume you wanted to compactify $\mathcal{M}_{g,n}$, but you didn't have the idea to use nodal curves. Log geometry would help you in the following way. The philosophy is "log smoothness allows only "good" degenerations".

There is a notion of log smooth curve, and of a family of log smooth curves over a log scheme. One can define a moduli stack $\mathcal{LM}_{g,n}$ over the category of fine saturated log schemes parametrizing such families. Here g is the genus of the underlying curve, and n is another invariant, sometime called "type", that of course is going to correspond to marked points.

The following things (morally) happen:

- the moduli stack $\mathcal{LM}_{g,n}$ is already proper,
- every log smooth curve is nodal (i.e. the underlying curve is),
- every family of nodal curves admits canonical log structures (on the base and on the total space) for which it becomes a family of log smooth curves.

All of this together more or less says that $\mathcal{LM}_{g,n}$ "automatically recovers" the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$.

In the above I was actually begin a bit careless about some points, hence the "morally" (what does it mean that $\mathcal{LM}_{g,n}$ is proper, exactly?). Here is a precise way to state what happens: the boundary $\partial\overline{\mathcal{M}}_{g,n}$ gives a log structure to $\overline{\mathcal{M}}_{g,n}$. The resulting log stack $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ represents the functor $\mathcal{LM}_{g,n}$ on log schemes, in the sense that there is an equivalence

$$\mathcal{LM}_{g,n} \cong \operatorname{Hom}(-, (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}))$$

where Hom denotes morphisms of log stacks.

There is a way to extract $\overline{\mathcal{M}}_{g,n}$ as the substack of minimal objects of $\mathcal{LM}_{g,n}$, but I don't want to go into details about this. Basically, the log structures that one can put on a family of nodal curves in order to make it log smooth are infinitely many, and the canonical one is the minimal. Because of this, $\mathcal{LM}_{g,n}$ is in some sense "way bigger" than $\overline{\mathcal{M}}_{g,n}$, and taking the substack of minimal objects fixes this.

There are other examples in which this philosophy of singling out good degenerations through log smoothness was applied fruitfully, mostly by Olsson (log K3 surfaces, log abelian varieties, toric Hilbert scheme...).

Once you've seen this, you might think of applications to Gromov-Witten theory, especially in the "degeneration" setting: say you have a smooth projective variety X_t that is degenerating to a mildly singular one X_0 (over $\mathbb{A}^1$, for example). The simplest example is a degeneration where X_0 is a union $Y_1 \cup_D Y_2$ of smooth varieties Y_1 and Y_2 along a common smooth divisor D in both.

In this setting one hopes to be able to compute GW invariants of the smooth fiber X_t by computing the ones of the special fiber X_0 (which then had better be simpler, or more structured, than X_t). Note that strictly speaking one needs to worry about defining the GW theory of X_0, since it is not clear how to do so.

To define invariants of X_0, one strategy is to study relative invariants of the pairs (Y_1, D) and (Y_2, D), and then put together their contributions. Relative invariants are defined over moduli spaces of objects that intersect D "nicely". To get a proper moduli space, one has to allow Y_1 and Y_2 to degenerate along D, via a bubbling process that basically comes from a blowup. This theory of "expanded degenerations" was developed by J. Li [Li01].

Log geometry gives an alternative way: the map $X \to \mathbb{A}^1$ corresponding to the family can be made log smooth by choosing appropriate log structures, and if the parallel with classical smoothness holds, it should be possible to define logarithmic GW invariants for the fibers of any log smooth morphism $X \to Y$ (with some additional hypotheses). The log GW invariants should agree with the ones defined using expanded degenerations.

This programme has been carried out to success only very recently [AC14, GS13], and a full logarithmic version of the degeneration formula is currently being worked out, to my understanding.

4. Kato-Nakayama space and infinite root stacks

Now that hopefully you're convinced that log geometry is nice and useful, let me tell you a bit about my most recent work.

In the past there have been some attempts to capture the "log" aspect of the geometry of a log scheme through incarnations in more familiar terrain. With Carchedi, Scherotzke and Sibilla we proved a comparison result between two of those [CSST].

The Kato-Nakayama space [KN99] is a topological space $X_{\log}$ associated to a fine saturated log analytic space X (that for example might be the analytification of a log scheme). It has a map $\tau \colon X_{\log} \to X$, and the fiber of τ over $x \in X$ is homeomorphic to $(S^1)^k$, where k is the rank of the abelian group $\overline{M}^{\mathrm{gp}}_{X,x}$. Note that $k = 0$ where the log structure is trivial, so over this open subset of X we have that τ is an isomorphism.

In this incarnation the nontrivial stalks $\overline{M}_{X,x}$ are replaced by real tori. The space $X_{\log}$ was used by Kato and Nakayama to obtain and study a version of "Betti cohomology" for log schemes.

Example 4.1. Assume $X = \mathbb{A}^1$, with the divisorial log structure. Then $\overline{M}_{X,x}$ is trivial unless x is the origin, in which case it is $\mathbb{N}$.

One can see that in this case $X_{\log} \cong \mathbb{R}_{\geq 0} \times S^1$ is a half-closed cylinder. The map $X_{\log} \to \mathbb{A}^1 = \mathbb{C}$ is given in coordinates by $(r, a) \mapsto r \cdot a$. In this case (and more generally when the log structure is given by a normal crossings divisor $D \subseteq X$) $X_{\log}$ is the real oriented blowup of X along D [Gil].

Note that in this example $X_{\log}$ is a manifold with boundary, and the boundary exactly replaces the divisor D. In this sense log schemes give a way to talk about manifolds with boundary in algebraic geometry. See also [GM].

The other construction that I want to talk about is the infinite root stack $\sqrt[\infty]{X}$. This was recently introduced by myself and Angelo Vistoli [TV14], as a "stacky" incarnation of log structures over a scheme.

For any natural number n, a log scheme has a tame Artin stack $\sqrt[n]{X} \to X$ that parametrizes "n-th roots" of the log structure [BV12].

Assume that the log structure is given by a single smooth Cartier divisor $D \subseteq X$. Then the stack $\sqrt[n]{X}$ parametrizes n-th roots of the divisor D, i.e. pairs (L, s) consisting of a line bundle L and a global section s, such that $(L, s)^{\otimes n} \cong (\mathcal{O}(D), 1_D)$. Locally on an affine open $\operatorname{Spec} A \subseteq X$ where D has a single equation $f \in A$, we have $\sqrt[n]{X} \cong [(\operatorname{Spec} A[t]/(t^n - f))/\mu_n]$.

If $n \mid m$ there is a projection $\sqrt[m]{X} \to \sqrt[n]{X}$. The infinite root stack is the inverse limit of this system, $\sqrt[\infty]{X} = \varprojlim_n \sqrt[n]{X}$. It is a pro-algebraic stack, not of finite type. This infinite root construction should be thought of as "magnifying" the log structure, and is useful for example in relation with parabolic sheaves. Another feature is that one can recover the log scheme X from the infinite root stack.

The reduced fiber of $\sqrt[\infty]{X} \to X$ over a point x is the classifying stack $\mathrm{B}\widehat{\mathbb{Z}}^k$, where k is the rank of the abelian group $\overline{M}^{\mathrm{gp}}_{X,x}$ (the same showing up in the Kato-Nakayama space). There is a clear similarity in the fact that the fiber of the Kato-Nakayama space is $(S^1)^k \cong \mathrm{B}\mathbb{Z}^k$.

This led us to proving the following.

Theorem 4.2 ([CSST]). *Let X be a fine saturated log scheme locally of finite type over $\mathbb{C}$. Then there is a canonical morphism of topological stacks $X_{\log} \to \sqrt[\infty]{X}_{\mathrm{top}}$, that induces an equivalence after profinite completion of both sides.*

Apart from the interest of relating these two objects that live in different worlds, this is nice because it implies that if we want to define a "profinite homotopy type" for a log scheme locally of finite type over $\mathbb{C}$, we can do it in any of these two ways, i.e. by taking the profinite completion of $X_{\log}$ or the one of $\sqrt[\infty]{X}_{\mathrm{top}}$, and we would get the same result.

Over an arbitrary base, one can take the profinite étale homotopy type of $\sqrt[\infty]{X}$, and there's an upcoming comparison result of Carchedi (generalizing a classical theorem of Artin-Mazur), that shows that this also gives the same profinite space when we are over $\mathbb{C}$, but is also defined in general (contrarily to the profinite completion of $X_{\log}$, for example).

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