DIVISOR THEORY ON TROPICAL AND LOG SMOOTH CURVES

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ABSTRACT. Tropical geometry is a relatively new branch of algebraic geometry, that aims to prove facts about algebraic varieties by studying their "tropicalizations", which are piecewise linear objects, amenable to combinatorial study. A prominent topic in recent research in the area, that is leading to new insights about "classical" open questions, is a theory of divisors on tropical curves. In this talk I will survey some of the related ideas, and explain how they are connected to line bundles on log smooth curves (joint work with Foster, Ranganathan and Ulirsch).

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1. INTRODUCTION

I want to talk a bit about recent techniques for studying degenerations of line bundles and linear series to nodal curves, coming from tropical and logarithmic geometry. There is a nice survey by Matt Baker and Dave Jensen ("Degeneration of linear series..."), where you can find a lot more. I will stay on a fairly elementary level, and try to avoid the technicalities.

2. DEGENERATIONS OF LINE BUNDLES ON NODAL CURVES

Let me start from line bundles on nodal curves.

If $C \to T$ is a family of smooth curves, then the Jacobians of the fibers also form a family of abelian varieties $\underline{\mathrm{Jac}}(C) \to T$, with nice properties (the morphism is proper). This gives a universal Jacobian over the whole moduli stack, which is a "relative" abelian variety.

It is interesting to consider the situation where the fibers of the family are singular (nodal). Sometimes, degenerating a smooth high genus curve to a nodal one with rational components helps. For example, the Brill-Noether theorem was proved by a degeneration argument (involving also limit linear series). This is also related to the problem of compactifying the universal Jacobian $\mathcal{J}ac^d_{g,n} \to \mathcal{M}_{g,n}$ (or its rigidification).

Assume that $C \to \operatorname{Spec} R$ is a flat family of curves, with smooth generic fiber, and nodal central fiber C_0/k. Here R is a DVR, with quotient field F and residue field k. Let us also assume that the total space C is regular, and that the irreducible components of the central fiber are smooth (these are sometimes called regular "strictly" or "strongly" semistable degenerations, I think). Can we still construct a relative Jacobian? Is it still proper? These questions are difficult. Two basic questions: given a line bundle $\mathcal{L}_F$ on the generic fiber C_F, does it always extend across the central fiber? Does it extend uniquely? I will focus on the second aspect in this talk.

The very first problem is that line bundles very often do not extend uniquely. Assume that $C_0 = C_1 \cup C_2 \cup \cdots \cup C_r$ are the irreducible components. Given a line bundle $\mathcal{L}_F$ on C_F, there is always at least one extension (take a Weil divisor to represent the line bundle on $C_F \subseteq C$, and take the closure in C). If $\mathcal{L} \in \mathrm{Pic}(C)$ is an extension, then for every i also $\mathcal{L} \otimes \mathcal{O}_C(C_i)$ is an extension of $\mathcal{L}_F$, and the two line bundles are not isomorphic over the total family (nor are the restrictions to the central fiber). Thus, the "naive" Jacobian will not be separated (the same object can have different limits). Of course you can twist by integral linear combinations of these divisors C_i, and in the situation that I've put myself in, two line bundles extending the same $\mathcal{L}_F$ differ from one of these twists (think of them as Weil divisors again).

In joint work with Foster, Ranganathan and Ulirsch, we relate these twists by components of the central fiber to chip firing on graphs. Part of our observation was already implicit in earlier work of Amini and Baker, what we do is put the observation in a more systematic framework, using logarithmic geometry (which I'll try to get to, towards the end of the talk).

A first quantity to look at for degenerations of $\mathcal{L}_F$ to the central fiber, is the degree d_i of the restriction of $\mathcal{L}$ to the component C_i. This vector $(d_1, \ldots, d_r) \in \mathbb{Z}^r$ is called the *multidegree* of $\mathcal{L}$.

Twisting by $\mathcal{O}_C(C_i)$ changes the multidegree, according to how the components intersect. More precisely, if we denote by $\alpha_{ij} = C_i \cdot C_j$ the number of points where the two components intersect, twisting by $\mathcal{O}_C(C_i)$ increases d_j by α_{ij} and decreases d_i by $\sum_j \alpha_{ij}$ (note that the "total degree" stays the same). Of course the multidegree does not completely determine the degeneration, unless the components are all rational (because otherwise a line bundle on a smooth curve is not determined by its degree). The "all rational" case is called "maximally degenerate", and the geometry of the nodal curve is completely encoded by the combinatorics of the "dual graph", which is coming up next.

This is related to the divisor theory on the dual graph of the central fiber C_0, in a way that I will now explain.

3. Divisors on (metric) graphs and metrized complexes

I will start with the simplest situation of an "abstract" graph, and then later refine it by considering metric graphs, and metrized complexes.

Recall that the dual graph of the central fiber is defined as follows: it has a vertex v_i for each irreducible component C_i of C_0, and an edge e_{ij} connecting v_i and v_j, for every node in $C_i \cap C_j$ (note that there might be more than one, and you could allow $i = j$ - let us exclude this for now). These graphs are already important in relation to the moduli space of stable curves $\overline{M}_g$, because they describe the combinatorics of how the components of the boundary intersect.

On such a graph, there is a divisor theory that has remarkable similarities with the usual situation on a smooth algebraic curve. This is due to Baker and Norine, although some of the stuff about graphs is more classical, the new thing that they proved was the Riemann-Roch theorem (along with some stuff about the Abel-Jacobi map).

3.1. Divisor theory on a graph.

Assume that in general $G = (V, E)$ is a graph.

Definition 3.1. A divisor on G is a formal combination $\sum_{v \in V} D(v) \cdot v$ on the vertices of G, where $e_v \in \mathbb{Z}$. Usually one thinks of such a thing as a configurations of "chips" or "dollars" on the vertices of the graph. A negative number signifies a debt. Let us denote by $\mathrm{Div}(G)$ the abelian group of divisors on G (for the obvious operation).

There are two basic moves that one can do to such a configuration ("chip firing" moves): a borrowing move at v, where each adjacent vertex gives a dollar to v along each edge, or the opposite lending move, where each vertex adjacent to v gets a dollar from v along each edge. This should remind you of what was happening to the multi-degree (edges are nodes, so points of intersection of the two components).

Draw an example

Formally, two divisors D and D' on G are equivalent if one can be obtained from the other via these chip firing moves. One can also formulate this by defining "principal divisors" as divisors of "zeros and poles" of functions $f: V \to \mathbb{Z}$ on the vertex set of G, and mod out by them, the result is the same. Let us denote by $\mathrm{Pic}(G)$ the quotient group $\mathrm{Div}(G)/\sim$.

There is a degree morphism $\deg: \mathrm{Div}(G) \to \mathbb{Z}$, defined by sending $D = \sum_{v \in V} D(v) \cdot v$ to $\sum_{v \in V} D(v)$, and equivalent divisors have the same degree, so deg descends to $\deg: \mathrm{Pic}(G) \to \mathbb{Z}$. In terms of chip firing, the degree is the total number of chips. The preimage of 0, denoted $\mathrm{Pic}^0(G)$ is the *Jacobian* of the graph G. Fun fact: the cardinality of $\mathrm{Jac}(G)$ (which is a finite abelian group) is the number of spanning trees in the graph (trees that touch every vertex). This number is also called the *complexity* of the graph, and the Jacobian also goes by other names.

3.1.1. *Riemann-Roch.*

Let us talk about Riemann-Roch for graphs. Recall that for a smooth projective curve C over some field k, Riemann-Roch says that

$$l(D) - l(K - D) = \deg D - g + 1$$

where g is the genus of the curve, and

$$l(D) = \dim_k H^0(C, \mathcal{O}_C(D)) - 1 = \dim \mathbb{P}(H^0(C, \mathcal{O}_C(D))).$$

To formulate the analogue for graphs, we need to make sense of $l(D)$ and of the genus g.

The genus is the easiest part: the graph G has a first betti number $b_1(G)$, that is by definition the genus. This is the first betti number of a topological realization of the graph, and can also be defined combinatorially, as #edges − #vertices + 1.

Let us define the *combinatorial rank* $r(D)$ for $D \in \mathrm{Div}(G)$.

Definition 3.2. A divisor is effective if $D(v) \geq 0$ for all $v \in V$.

We set $r(D) = -1$ if D is not equivalent to an effective divisor. Otherwise, $r(D)$ is the maximum of the natural numbers r such that $D - E$ is linearly equivalent to an effective divisor, for every effective divisor E of degree r. One can check that for curves, this same definition gives the number $l(D)$.

In terms of chip firing, a divisor D is linearly equivalent to an effective divisor if, starting from the configuration corresponding to D and using the chip firing moves, we can get to a configuration where no vertex is in debt. The condition above says that $r(D) = r$ if, by subtracting r dollars in any possible way from D, we can still get to a position where no vertex is in debt.

Let K_G be the canonical divisor of G, defined as $K_G = \sum_{v \in V}(\#\text{edges adjacent to } v - 2) \cdot v$.

Theorem 3.3 (Baker-Norine, Riemann-Roch for graphs). *We have the equality*

$$r(D) - r(K_G - D) = \deg D - g(G) + 1.$$

One consequence, is that if a divisor D has $\deg D \geq g$, then by using the chip firing moves one can get to a configuration where no vertex is in debt.

Remark 3.4. I will explain now how this is related to tropical geometry. The proof of Baker-Norine is entirely combinatorial, and it would be interesting to have an "algebraic" proof, that mimics the classical one in the tropical setting. Even sheaf cohomology is missing for now in the tropical world!

Also, this is quite orthogonal with respect to the "usual" Riemann-Roch theorem. There are no implications between the two. The Riemann-Roch theorem for metrized complexes, on the other hand, generalizes both.

The multidegree gives a sort of (very coarse) "tropicalization" of divisors, $\mathrm{Trop}\colon \mathrm{Pic}(C) \to \mathrm{Pic}(G)$. Tropicalization is a process that replaces a variety (or algebro-geometric object) with a piecewise linear "shadow", that still retains some information, and is amenable to combinatorial handling.

The ranks $l(D)$ and $r(\mathrm{Trop}(D))$ are related: we always have $l(D) \leq r(\mathrm{Trop}(D))$, and equality holds when the central fiber is maximally degenerate. This is important, because it allows to have information about the rank of a divisor on a smooth curve via combinatorial methods. This type of argument also helps in studying linear series.

3.2. **Metric graphs and metrized complexes.**

3.2.1. *Metric graphs.* Now let us be a bit more sophisticated. This combinatorial picture can be mixed with the algebraic geometry, to allow for more general degenerations than these strongly semistable ones, and for finite extensions of the base DVR.

Assume that $C \to \operatorname{Spec} R$ is a degeneration of curves, and only assume that the central fiber is a nodal curve. In this setting, the dual graph of the central fiber loses information, about how exactly the curve is degenerating. For example, if t is a uniformizer of R, around a node of C_0 the family will have some equation of the form $xy - at^n$ where a is a unit, and we might want to record the integer n. This is encoded in a specified length for the edge corresponding to the node in the dual graph.

In this way, we are led to consider *metric graphs* Γ, where on top of the structure of a graph, every edge has a length in $\mathbb{R}_{>0}$, and we consider as "equivalent" graphs with an edge length function that admit a common refinement.

Draw an example

For some people, a "tropical curve" is a metric graph. If the fraction field of R is a complete non-archimedean field, the tropical curve associated with a degeneration as above can be identified with a subset of the Berkovich space of the smooth projective curve C_F/F (possibly base-changed to $\overline{F}$), called a "skeleton". If you've never seen Berkovich spaces, these are some kind of analytic spaces over a non-archimedean field. One can think of $(C_{\overline{F}})_{\text{an}}$ as a union of all skeleta of all possible semistable models (not strictly semistable) of the curve $C_{\overline{F}}/\overline{F}$.

For a metric graph Γ, when defining divisors one also allows points in the interior of the edges, i.e. a divisor is a formal sum $\sum_{v\in\Gamma} D(v)\cdot v$, where $D(v)=0$ for all but finitely many points. Chip firing is more complicated, and best expressed in terms of principal divisors: given a piecewise linear function $f\colon \Gamma\to\mathbb{R}$ with integral slope, one defines $\operatorname{ord}_v(f)$ as the sum of the incoming slopes along the edges containing v.

Draw an example

The divisor of f is $\sum_{v\in\Gamma}\operatorname{ord}_v(f)\cdot v$, and we mod out by these principal divisors. This time there is a tropicalization map $\operatorname{Trop}\colon \operatorname{Pic}(C_{\overline{F}})\to\operatorname{Pic}(\Gamma)$ that extends $\operatorname{Trop}\colon\operatorname{Pic}(C)\to\operatorname{Pic}(G)$, where G is a "model" for Γ.

The Jacobian in this case turns out to be a real torus. There are also Abel-Jacobi maps, the Torelli theorem, and a number of other things. There is also a Riemann-Roch theorem, formulated exactly as before.

3.2.2. *Metrized complexes.* Let me also briefly mention *metrized complexes*. The point here is that even recording the lengths loses information if the components of the central fiber are not rational, and this even more refined point of view retains all the information about the components of the central fiber.

Assume k is algebraically closed.

Definition 3.5. A metrized complex $\mathfrak{C}$ of algebraic curves over k is a metric graph Γ as above with a specified model G (i.e. you know who the vertices are), together with a smooth, irreducible, projective curve C_v over k for each vertex $v\in V$, and a set of points $x_v^e\in C_v(k)$, in bijection with the half-edges e incident to v.

The genus $g(\mathfrak{C})$ is the sum of the genus of the graph Γ and of the

The idea is that the C_v are the components of the special fiber, and the specified points are the nodes. The "half-edges" bit is to allow for self-intersecting components.

Draw an example

The divisor theory on such a thing will be a mix of the divisor theory of the graph, and of the curves corresponding to the vertices.

Definition 3.6. A divisor $\mathfrak{D}$ on $\mathfrak{C}$ is an element (D_Γ, D_v) of $\operatorname{Div}(\Gamma)\oplus\bigoplus_v\operatorname{Div}(C_v)$, such that $\deg D_v = D_\Gamma(v)$. The degree of such a thing is $\deg(D_\Gamma)$.

We can define principal divisors in this case as well: a rational function $\mathfrak{f}$ on $\mathfrak{C}$ is a "rational function" f_Γ on Γ, as defined before, together with a rational function f_v on C_v for every v. The divisor of such a thing is

$$\mathrm{div}(f_\Gamma) \oplus \sum_v (\mathrm{div}(f_v) + \mathrm{div}_v(f_\Gamma))$$

where $\mathrm{div}_v(f_\Gamma) = \sum_{e \in v} \mathrm{slope}_e(f_\Gamma) \cdot x_v^e$ with $\mathrm{slope}_e(f_\Gamma)$ being the outgoing slope of f_Γ from v in direction e.

As usual, two divisors are linearly equivalent if the difference is principal. The degree is invariant for linear equivalence.

A divisor is effective if D_Γ and the D_v are effective. We define the rank $r(\mathfrak{D})$ as for divisors on graphs.

There is also a canonical divisor $\mathfrak{K}$, given by $K_\Gamma \oplus \sum_v (K_v + A_v)$, where $K_\Gamma = \sum_v (\mathrm{val}_G(v) + 2g_v - 2) \cdot v$, and for every vertex v, the divisor K_v is a canonical divisor on C_v, and A_v is the sum of the distinguished points on C_v corresponding to the nodes.

Theorem 3.7 (Amini-Baker, Riemann-Roch for metrized complexes)**.** *We have the equality*

$$r(\mathfrak{D}) - r(\mathfrak{K} - \mathfrak{D}) = \deg \mathfrak{D} - g(\mathfrak{C}) + 1$$

This specializes to both the classical Riemann-Roch on curves, and the one on graphs. Also, for any semistable model $\mathcal{C}$ of C over the valuation ring of F, one has an associated metrized complex $\mathfrak{C}$, and there is a specialization morphism $\tau_* \colon \mathrm{Div}(C) \to \mathrm{Div}(\mathfrak{C})$ (extending the tropicalization map mentioned before). In this setting as well we have a specialization inequality $l(D) \leq r(\tau_*(D))$, that strengthens the one mentioned before.

Amini and Baker go further, and define a notion of limit linear series for these objects, even for curves not of compact type (that correspond to the case where the "tropical Jacobian" is trivial, so the combinatorics is not needed).

4. Log geometry and log line bundles

Very briefly, let me tell you what we do in our paper with the other guys.

We point out that this business of chip firing and degenerations of line bundles is related to the *logarithmic picard group*, and we translate the Amini-Baker Riemann-Roch theorem to a Riemann-Roch theorem for log line bundles on log smooth curves.

There is this theory of *logarithmic schemes*. These are schemes that have some extra structure, that is keeping track of either a boundary divisor, or of some data relative to a degeneration, of which your scheme is a fiber. Non-smooth morphisms sometimes become *log smooth* (i.e. smooth in the sense of log geometry) when equipped with the correct log structures. In particular families of nodal curves have canonical log structures (on the base and on the total space), that make the family log smooth.

The log Picard group is an analogue of the Picard group for log schemes. For nodal curves, it turns out to be exactly the quotient of the "usual" Picard group of the nodal curve with respect to twisting by the components (chip firing). We define a notion of rank $r(\mathcal{L})$ for a log line bundle $\mathcal{L}$ (similar to the ones described above), and prove a Riemann-Roch theorem, by reducing to metrized complexes. The canonical divisor in this case corresponds to the relative sheaf of log differentials of the curve, that is the same as the dualizing sheaf.

Theorem 4.1 (Foster-Ranganathan-T.-Ulirsch)**.** *We have the equality*

$$r(\mathcal{L}) - r(\omega_{log} \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1$$

Our point is that log schemes are somewhat more natural objects than these metrized complexes, and their theory is much more developed. We plan to use this point view to study limit linear series on the logarithmic side. It would also be interesting to have an "algebraic" proof of this Riemann-Roch theorem, for example by interpreting the LHS as an Euler characteristic (as in the classical case), but this is missing so far.