SOLUZIONI DEI QUESITI DELLA III PROVA IN ITINERE
Matematica per Biotecnologie 4 Maggio 2010

1-a $F(x) = \log(1+e^{(\sin x)^2}) = f(g(h(k(x))))$ $f(g) = \lg(1+g)$ $g(h) = e^h$ $h(k) = k^2$ $k(x) = \sin x$

$F'(\pi) = \frac{dF}{dx}(\pi) = \frac{df}{dg}(g(h(k(\pi))))\cdot\frac{dg}{dh}(h(k(\pi)))\cdot\frac{dh}{dk}(k(\pi))\cdot\frac{dk}{dx}(\pi)$

$\frac{dh}{dk}(k(\pi)) = 2k(\pi) = 2\sin\pi = 0 \quad \Rightarrow F'(\pi) = 0$

1-b $F(x) = x\sin x\, e^{3x^3} = f(x)\cdot g(x)\cdot h(x)$ $f(x) = x$, $g(x) = \sin x$, $h(x) = e^{3x^3}$

$F'(x) = f'(x)(g(x)\cdot h(x)) + f(x)(g(x)h(x))' = f'\cdot g\cdot h + f(g'h + gh') =$

$= f'gh + fg'h + fgh' =$

$= \sin x\, e^{3x^3} + x\cos x\, e^{3x^3} + x\sin x\cdot 9x^2 e^{3x^3}$

1.c $y = f(x) = \sqrt[3]{\lg x} = (\lg x)^{\frac{1}{3}}$ retta tangente al grafico nel punto $(x_0, f(x_0))$

$y = f'(x_0)(x - x_0) + f(x_0)$

$x_0 = e$ $f(x_0) = 1$ $f'(x_0) = \frac{1}{3}\frac{(\lg x_0)^{-\frac{2}{3}}}{x_0}\Big|_{x_0=e} = \frac{1}{3e}$

$y = \frac{1}{3e}(x-e) + 1$ $y = \frac{x}{3e} + \frac{2}{3}$

2.a $\frac{\partial e^{(z^2-y^2)x}}{\partial z}(1,2,3) = \frac{d}{dz}e^{(z^2-4)\cdot 1}\Big|_{z=3} = 2z\,e^{z^2-4}\Big|_{z=3} = 6e^5$

2.b $F(x,y) = \cos xy$ $\frac{\partial F}{\partial x} = \frac{d\cos}{dw}(x\cdot y)\frac{\partial w}{\partial x} = -\sin(x\cdot y)\cdot y$ $\frac{\partial F}{\partial y} = -\sin(xy)\cdot x$

$\doteq \cos w$

$w(x,y) = x\cdot y$

$\frac{\partial^2 F}{\partial x^2} = -\cos(x\cdot y)\cdot y^2$ $\frac{\partial^2 F}{\partial y^2} = -\cos(x\cdot y)\cdot x^2$

$\frac{\partial^2 F}{\partial x\partial y} = \frac{\partial^2 F}{\partial y\partial x} = \frac{\partial}{\partial y}(-\sin(xy)\cdot y) = \frac{\partial}{\partial y}(-\sin x\cdot y)\cdot y - \sin(xy)\frac{\partial y}{\partial y} =$

$= -\cos xy\cdot xy - \sin xy$

3.a $z = f(x,y) = x^3 + y^3$ equazione piano tangente al grafico in $(x_0, y_0, f(x_0,y_0))$

$z = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + f(x_0,y_0)$ $x_0 = 1, y_0 = 1$ $\frac{\partial f}{\partial x}(1,1) = 3x^2\Big|_{x=1} = 3$ $\frac{\partial f}{\partial y}(1,1) = 3$

$z = 3(x-1) + 3(y-1) + 2$ $z = 3x + 3y - 4$ $3x + 3y - z - 4 = 0$

3.b $f(x,y) = x^y + y^x$ Se f è differenziabile in (x_0,y_0) e il suo gradiente $\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right) \neq (0,0)$ allora $\nabla f(x_0,y_0) \perp \{(x,y) : f(x,y) = f(x_0,y_0)\}$ in (x_0, y_0). Quindi l'equazione retta tangente all'insieme in (x_0,y_0) è

$\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = 0$ $x_0 = y_0 = 1$ $\frac{\partial f}{\partial x}(1,1) = \left(\frac{y}{x}x^y + (\lg y)y^x\right)_{(1,1)} = 1 = \frac{\partial f}{\partial y}(1,1)$

$x + y - 2 = 0$

[Altrimenti: nelle ipotesi l'insieme ha tangente in $(1,1)$, essendo simmetrico per $y = x$ in $(1,1)$ la tangente deve essere ortogonale a $y = x$: $y = -x + 2$]

4a $\log(1+\sin(x^4))$ $\quad [t=\sin x^4]$

$\log(1+t) = t - \frac{t^2}{2} + O(t^3)$

$t=\sin x^4 = \sin y = y - \frac{y^3}{6} + O(y^5) = [y=x^4]$

$= x^4 + O(x^{12})$

$\lg(1+t) = x^4 + O(x^{12}) - \frac{(x^4+O(x^{12}))^2}{2} + O(x^{12})$

$= x^4 - \frac{1}{2}(x^8+O(x^{16})) + O(x^{12}) = x^4 - \frac{x^8}{2} + O(x^{12})$

$O(x^{12}) \le o(x^{10})$ per unicità il polinomio di Taylor di centro x_0 e grado 10 è

$x^4 - \frac{x^8}{2}$

4b

$$F(x,y) = F(x_0,y_0) + \frac{\partial F}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)\cdot(y-y_0) +$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(x_0,y_0)\cdot(y-y_0)^2 + \frac{\partial^2 F}{\partial x\partial y}(x_0,y_0)\cdot(x-x_0)(y-y_0) +$$
$$+ o((x-x_0)^2+(y-y_0)^2)$$

$F(x,y) = \cos xy \qquad x_0=\pi \qquad y_0 = 1$

$F(\pi,1) = -1 \qquad \frac{\partial F}{\partial x}(\pi,1) = \frac{\partial F}{\partial y}(\pi,1) = -\sin\pi = 0$

$\frac{\partial^2 F}{\partial x^2}(\pi,1) = -\cos\pi = 1 \qquad \frac{\partial^2 F}{\partial y^2} = -\pi^2\cos\pi = \pi^2 \qquad \frac{\partial^2 F}{\partial x\partial y}(\pi,1) = -\sin\pi - \pi\cos\pi = \pi$

$$-1 + \frac{1}{2}(x-\pi)^2 + \frac{1}{2}\pi^2(y-1)^2 + \pi(x-\pi)(y-1) =$$

$$= \dots = \frac{x^2}{2} + y^2\frac{\pi^2}{2} + \pi xy - 2\pi x - 2\pi^2 y + 2\pi^2 - 1$$

5.a $\int_{-1}^{1}\sin\sqrt{2x+7}\,dx \left[\begin{array}{l} y=\sqrt{2x+7} \quad dy = \frac{dx}{\sqrt{2x+7}},\ dx = y\,dy \\ \sqrt{5}\le y\le\sqrt{9}=3\end{array}\right] = \int_{\sqrt5}^{3} \underset{f}{y}\,\underset{g'}{\sin y}\,dy = \qquad g=-\cos y$

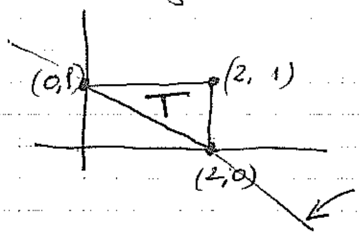
$= fg\Big]_{\sqrt5}^{3} - \int_{\sqrt5}^{3} f'g\,dy = -y\cos y\Big]_{\sqrt5}^{3} + \int_{\sqrt5}^{3}\cos y\,dy = -y\cos y\Big]_{\sqrt5}^{3} + \sin y\Big]_{\sqrt5}^{3} =$

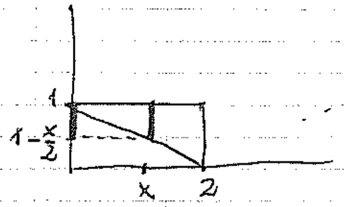
$= -3\cos 3 + \sqrt5\cos\sqrt5 + \sin 3 - \sin\sqrt5$.

5.b $A = \text{area}(\{(x,y): \log x \le y \le 0\})$ $[\lg x\le y\le 0 \Rightarrow \lg x \le 0 \Leftrightarrow 0<x\le 1]$ = area sopragrafico di $y=\lg x$ e asse orizzontale sul segmento $]0,1]$ =

$= \lim_{\varepsilon\to0} -\int_{\varepsilon}^{1}\lg x\,dx = \lim_{\varepsilon\to0} -[x\lg x - x]_{\varepsilon}^{1} = \lim_{\varepsilon\to0} -[-1-\varepsilon\lg\varepsilon+\varepsilon] = 1$

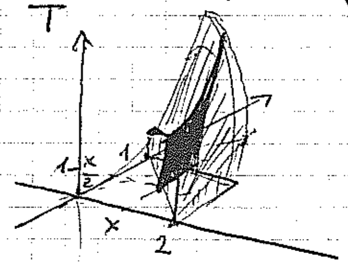
6 $\iint_T (xy)\, dx\,dy$ T triangolo di vertici $(0,1)$, $(2,1)$, $(2,0)$





$$T = \left\{(x,y) \;:\; 0 \le x \le 2 \;\text{ e }\; 1-\tfrac{x}{2} \le y \le 1\right\}$$

$$\iint_T xy\, dx\,dy = \int_0^2 dx \left(\int_{1-\frac{x}{2}}^{1} y\, dy \right) = \int_0^2 x \left[\frac{y^2}{2}\right]_{1-\frac{x}{2}}^{1} dx =$$



$$= \int_0^2 x\left(\frac{1}{2} - \frac{(1-\frac{x}{2})^2}{2}\right) dx =$$

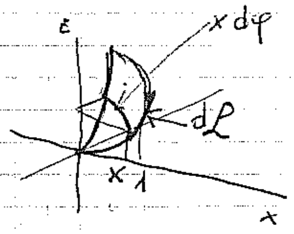
$$= \int_0^2 x\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{x^2}{8} + \frac{x}{2}\right) dx = \int_0^2 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx =$$

$$= \left.\frac{x^3}{6}\right]_0^2 - \left.\frac{x^4}{32}\right]_0^2 = \frac{8}{6} - \frac{16}{32} = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$

7 Area superficie di rotazione attorno all'asse verticale dell'insieme $\left\{(x,z) : \frac{x^2}{2} = z \le \frac{1}{2}\right\}$ =

= Area superficie di rotazione attorno all'asse verticale dell'insieme $\left\{(x,z) : \frac{x^2}{2} = z \le \frac{1}{2} \text{ e } x \ge 0\right\}$

= Area superficie di rotazione attorno all'asse verticale del grafico $\left(x, \frac{x^2}{2}\right) = \gamma(x)$, $0 \le x \le 1$



$$= 2\pi \int_0^1 x\, d\mathcal{L}_\gamma = 2\pi \int_0^1 x\sqrt{1+x^2}\, dx =$$

$$d\mathcal{L}_\gamma = |\gamma'| = \sqrt{\gamma_1'^2 + \gamma_2'^2}\, dx$$
$$= \sqrt{1+x^2}\, dx$$

$$= \pi \int_0^1 \sqrt{1+x^2}\, d(x^2) = \pi\, \frac{2}{3} \left.(1+x^2)^{\frac{3}{2}}\right]_0^1 = \pi \frac{2}{3}\left(2^{3/2} - 1\right) =$$

$$= \frac{2}{3}\pi\left(2\sqrt{2} - 1\right)$$