Esercitazioni di Calcolo differenziale ed Integrazione 3-4/12/01 (1)

- $Q(x) = \sum_{i,j=1}^{n} b_{ij} x^i x^j = (B \cdot x \cdot x) \qquad B = (b_{ij})_{1\le i,j\le n}$

$$\nabla Q(x) = (B + {}^tB)x \quad , \quad HQ(x) = B + {}^tB$$

- Se $B = {}^tB = A$ simmetrica

$$\nabla Q(x) = 2Ax \quad , \quad HQ(x) = 2A .$$

- $\rho(x_1, \dots, x_n) = |x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$

$$\frac{\partial \rho}{\partial x_k} = \frac{1}{2} \frac{2\sum_{i=1}^{n}\frac{\partial x_i}{\partial x_k}\cdot x_i}{(x_1^2 + \dots + x_n^2)^{\frac{1}{2}}} = \frac{x_k}{|x|} = \frac{x_k}{\rho} \qquad ; \underline{x} \neq 0$$

$$\nabla \rho = \frac{x}{|x|} = \frac{x}{\rho} =: \hat{x}$$

$$\frac{\partial^2 \rho}{\partial x_h \partial x_k} = \frac{\partial}{\partial x_h}\left(\frac{x_k}{\rho}\right) = \frac{\frac{\partial x_k}{\partial x_h}\cdot \rho - x_k \cdot \frac{x_h}{\rho}}{\rho^2} = \frac{\delta_{hk}}{\rho} - \frac{1}{\rho}\frac{x_k}{\rho}\cdot\frac{x_h}{\rho}$$

$$H\rho = \frac{1}{|x|}\left(I_{n\times n} - \hat{x}\otimes\hat{x}\right) = \frac{1}{\rho}\left(I_{n\times n} - \hat{x}\otimes\hat{x}\right)$$

<u>NOTAZIONE</u> $\quad a = {}^t(a_1 \cdots a_n) \quad b = (b_1 \cdots b_n) \ : \ a\otimes b =: (a_i b_j)_{1\le i,j\le n}$

$$(a\otimes b)v = a(b\cdot v) \quad ; \quad (a\otimes b)(v,w) = (a\cdot w)(b\cdot v) = \sum_{i,j=1}^{n} a_i w_i b_j v_j$$

In particolare se $|a| = 1$, $a = \hat{a}$; $a\otimes a$ da la proiezione ortogonale di v sulla retta di direzione a: In particolare

$|x| H\rho(x)$ è la matrice associata alla proiezione ortogonale sul piano tangente nel punto $\hat{x}$ alla sfera $S^{n-1} = \{ z \in \mathbb{R}^n : |z| = 1\}$.

- $\nabla(\varphi(|x|)) = \varphi'(|x|)\cdot\frac{x}{|x|} \quad ; \quad H(\varphi(|x|)) = \left(\varphi''(|x|) - \frac{\varphi'(|x|)}{|x|}\right)\frac{x}{|x|}\otimes\frac{x}{|x|} + \frac{\varphi'(|x|)}{|x|} I_d$

- $\Delta f(\underline{x}) =: \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \dots + \frac{\partial^2 f}{\partial x_n^2}(x) = \operatorname{tr} Hf(\underline{x})$

- $\Delta(\varphi(|x|)) = \varphi''(|x|) + (n-1)\frac{\varphi'(|x|)}{|x|}$ in particolare si ha

$$\Delta(\log(x^2+y^2)) = 0 \quad ; \quad \Delta\left(\frac{1}{|x|^{n-2}}\right) = 0 \ , n > 2$$

<u>esercizio</u> Sia R una matrice $n\times n$ ortogonale ${}^tR\cdot R = I_d$ e sia $\varphi(\underline{x}) = {}^t\left(\sum_{j=1}^{n} R_{ij}x_j\right)_{1\le i\le n}$ l'applicazione lineare associata. Si ha:

$$\Delta(f\circ\varphi) = (\Delta f)\circ\varphi .$$

Per $n=2$, se $\det R = 1$ φ è una rotazione: $R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.

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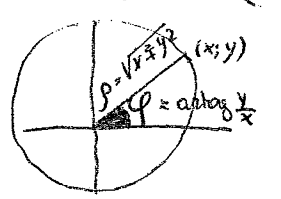
- coordinate polari

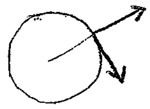


- $T(\rho,\varphi) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix}$ $\quad T:]0;+\infty] \times [0;2\pi[\longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$

$$T^{-1}(x,y) = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan\frac{y}{x} \end{pmatrix} \quad x>0,\ y\geq 0 \qquad \begin{pmatrix} \sqrt{x^2+y^2} \\ \frac{\pi}{2} \end{pmatrix} \quad x=0,\ y>0$$

$$\begin{pmatrix} \sqrt{x^2+y^2} \\ \pi+\arctan\frac{y}{x} \end{pmatrix} \quad x<0 \qquad \begin{pmatrix} \sqrt{x^2+y^2} \\ \frac{3}{2}\pi \end{pmatrix} \quad x=0,\ y<0 \qquad \begin{pmatrix} \sqrt{x^2+y^2} \\ 2\pi+\arctan\frac{y}{x} \end{pmatrix} \quad x>0,\ y<0$$

- NOTAZIONE



$$\tilde{g}(\rho,\varphi) = (g\circ T)(\rho,\theta) = g(\rho\cos\varphi, \rho\sin\varphi)$$

- $\dfrac{\partial\tilde{f}}{\partial\rho} = \widetilde{\dfrac{\partial f}{\partial x}}\cos\varphi + \widetilde{\dfrac{\partial f}{\partial y}}\sin\varphi$, $\quad \dfrac{\partial\tilde{f}}{\partial\varphi} = -\widetilde{\dfrac{\partial f}{\partial x}}\rho\sin\varphi + \widetilde{\dfrac{\partial f}{\partial y}}\rho\cos\varphi$

in effetti:

$$\left(\frac{\partial\tilde{f}}{\partial\rho}, \frac{\partial\tilde{f}}{\partial\varphi}\right) = d(f\circ T) = df\cdot dT = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial\tilde{f}}{\partial\rho}\cos\varphi - \frac{\partial\tilde{f}}{\partial\theta}\frac{\sin\varphi}{\rho} \ ; \qquad \frac{\partial f}{\partial y} = \frac{\partial\tilde{f}}{\partial\rho}\sin\varphi + \frac{\partial\tilde{f}}{\partial\varphi}\frac{\cos\varphi}{\rho}$$

$$= \frac{\partial\tilde{f}}{\partial\rho}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial\tilde{f}}{\partial\varphi}\frac{y}{x^2+y^2} \ ! \qquad \frac{\partial f}{\partial y} = \frac{\partial\tilde{f}}{\partial\rho}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial\tilde{f}}{\partial\varphi}\frac{x}{x^2+y^2}$$

$$\nabla_{\rho\theta}\tilde{f} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\rho\sin\varphi & \rho\cos\varphi \end{pmatrix}\nabla_{xy} f$$

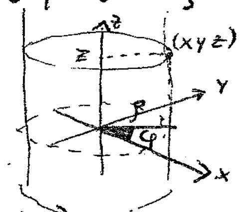
<u>esercizio</u> si provi che $\widetilde{\dfrac{\partial^2 f}{\partial x^2}} + \widetilde{\dfrac{\partial^2 f}{\partial y^2}} = \dfrac{\partial^2\tilde{f}}{\partial\rho^2} + \dfrac{\partial^2\tilde{f}}{\partial\varphi^2}\cdot\dfrac{1}{\rho^2} + \dfrac{\partial\tilde{f}}{\partial\rho}\dfrac{1}{\rho} =$

$$= \frac{1}{\rho^2}\left(\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{f}}{\partial\rho}\right) + \frac{\partial^2\tilde{f}}{\partial\varphi^2}\right).$$

- coordinate cilindriche

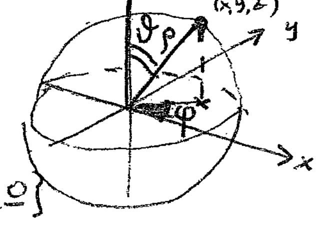
$$T(r,\varphi,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad T:]0;+\infty] \times [0,2\pi[\times \mathbb{R} \to \mathbb{R}^3 \setminus \{x^2+y^2=0\}$$

$r = \sqrt{x^2+y^2}$



- coordinate sferiche

$$T(\rho,\varphi,\theta) = \begin{pmatrix} \rho\cos\varphi\sin\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\theta \end{pmatrix} \qquad T:]0,+\infty] \times [0;2\pi[\times [0;\pi] \to \mathbb{R}^3 \setminus \{(0,0,0)\}$$



- coordinate radiali

$$T(\rho,\nu) = \rho\cdot\nu \quad : T:]0,+\infty] \times S^{n-1} \to \mathbb{R}^n \setminus \{\underline{0}\}$$

$$\rho = |x| , \quad \nu = \frac{x}{|x|} = \hat{x} = \nabla\rho$$

$$\tilde{f}(\rho;\nu) = f(\rho\cdot\nu)$$

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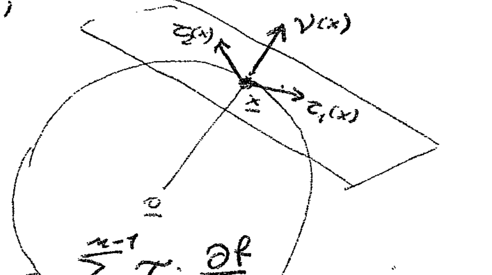
- Derivate normali e tangenziali

$\nu : \mathbb{R}^n \setminus \{0,0,0\} \to S^{n-1} \quad \nu(x) = \frac{x}{|x|} = \hat{x}$; $\tau_1(x) \cdot \tau_{n-1}(x)$ base ortonorm. $\nu \cdot x = 1$



$$\nabla_\nu f(x) = \nu \otimes \nu \cdot \nabla f = \frac{\partial f}{\partial \underline{\nu}} \cdot \underline{\nu} = \Big(\sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \nu^i\Big)\nu^i$$

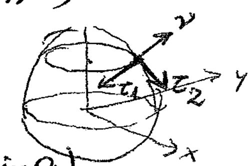
$$\nabla_\tau f(x) = \nabla f(x) - \frac{\partial f}{\partial \nu} \cdot \nu = \nabla f - \nu \otimes \nu \cdot \nabla f =$$

$$= (I_d - \nu \otimes \nu)\nabla f = \sum_{i=1}^{n-1} (\underline{\tau_i} \otimes \underline{\tau_i}) \nabla f = \sum_{i=1}^{n-1} \underline{\tau_i} \frac{\partial f}{\partial \underline{\tau_i}}$$

$\nabla_\tau f \perp \nabla_\nu f$

- In coordinate polari: $\frac{\partial f}{\partial \underline{\nu}} = \frac{\partial f}{\partial \rho}$, $\nu(x,y) = \frac{{}^t(x,y)}{\sqrt{x^2+y^2}} = {}^t(\cos\varphi, \sin\varphi)$

$\tau_1(x) = {}^t(\sin\varphi\ -\cos\varphi) = \frac{{}^t(y,-x)}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial \tau_1} = -\frac{1}{\rho}\frac{\partial f}{\partial \varphi}$ NOTA $\det(\tau_1, \nu) = 1$



- In coordinate sferiche: $\frac{\partial f}{\partial \nu} = \frac{\partial f}{\partial \rho}$;

$\tau_1(x) = {}^t(\sin\varphi, -\cos\varphi, 0) \quad \tau_2(x) = {}^t(\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)$

$\frac{\partial f}{\partial \tau_1} = -\frac{1}{\rho\sin\theta}\frac{\partial f}{\partial \varphi}$; $\frac{\partial f}{\partial \tau_2} = \frac{1}{\rho}\frac{\partial f}{\partial \theta}$. NOTA $\det(\tau_1\ \tau_2\ \nu) = 1$

- Sia M una matrice simmetrica, $(m_{ij})_{1\le i,j\le n}$, $n \times n$. Siano:

$d_k = \det\begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix} \quad 1 \le k \le n$; allora:

$Mx \cdot x > 0 \quad \forall x \ne 0$ sse, tutti gli autovalori sono positivi sse, $\forall k \quad d_k > 0$

$Mx \cdot x < 0 \quad \forall x \ne 0$ sse, tutti gli autovalori sono negativi sse, $\forall k \ (-1)^k d_k > 0$

- Sia $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = \prod_{i=1}^n (\lambda - \lambda_i)$ un polinomio di radici $\lambda_1, \dots, \lambda_n$,

le radici sono tutte (con parte reale) non negativa sse, coeff. a segni alterni $((-1)^k a_{n-k} \geq 0)$

le radici " " " non positiva sse, coeff. non negativi $(a_k \geq 0)$

$$p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) = \lambda^n + \sum_{k=1}^n (-1)^k \lambda^{n-k} \sum_{i_1 < \dots < i_k} \prod_{j=1}^k \lambda_{i_j}$$

e.g. $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$

- Sia M matrice simmetrica $(m_{ij})_{1\le i,j\le n}$, $n\times n$: $p(\lambda) = \det(\lambda I - M) =$

$= \lambda^n - \lambda^{n-1}\sum_{i=1}^n \lambda_i + \dots + (-1)^n \prod_{i=1}^n \lambda_i = \lambda^n - \mathrm{Tr}M\,\lambda^{n-1} + \dots + (-1)^n \det M$.

- Se $\det M \ne 0$: $Mx \cdot x > 0$ sse i coeff. di $p(\lambda)$ sono a segni alterni
  $Mx \cdot x < 0$ sse i coeff. di $p(\lambda)$ sono non negativi
- Se $\det M < 0$, n dispari non può essere $Mx \cdot x > 0 \ \forall x \ne 0$. $\overset{+}{-}\ \overset{+}{-}\ \cdots\ \overset{+}{-}$

3-4/12/01 (4)

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \end{cases} \qquad \exists\, \frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}$$

(Cecconi-Piccinini-Stampacchia: esercizi: pag 171 - 18.7.A)

1) $|f(x,y)| \leq x^2+y^2 \Rightarrow \exists\, df(0,0) \equiv 0 \Rightarrow \exists\, \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$

2) $\forall (x,y) \neq (0,0) \quad \frac{\partial f}{\partial x}(x,y) = -\frac{\partial f}{\partial y}(x,y)$

3) $$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y \to 0} \frac{1}{y}\frac{\partial f}{\partial x}(0,y)$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x \to 0} \frac{1}{x}\frac{\partial f}{\partial x}(x,0)$$

4) $$\frac{\partial f}{\partial x}(0,y) \underset{y\neq 0}{=} \lim_{x\to 0} \frac{f(x,y) - f(0,y)}{x} = \lim_{x\to 0} \frac{y(x^2-y^2)}{x^2+y^2} = -y$$

$$\frac{\partial f}{\partial y}(x,0) \underset{x\neq 0}{=} \lim_{y\to 0} \frac{f(x,y) - f(x,0)}{y} = \lim_{y\to 0} \frac{x(x^2-y^2)}{x^2+y^2} = x$$

5) $$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{-y}{y} = -1 \neq \frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0}\frac{x}{x} = 1 \quad \blacksquare$$

Trovare i massimi e minimi relativi di $f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 20$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 3x^2 - 6y - 39 = 0 \\ \frac{\partial f}{\partial y}(x,y) = 2y - 6x + 18 = 0 \end{cases} \text{sse} \begin{cases} x^2 - 2y - 13 = 0 \\ y - 3x + 9 = 0 \end{cases} \text{sse} \begin{cases} x^2 - 6x + 5 = 0 \\ y = 3x - 9 \end{cases} \text{sse} \begin{cases} x = 5 \\ y = 6 \end{cases} \text{o} \begin{cases} x = 1 \\ y = -6 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -6 \quad Hf = \begin{pmatrix} 6x & -6 \\ -6 & 2 \end{pmatrix}$$

$(5,6)$: $\begin{pmatrix} 30 & -6 \\ -6 & 2 \end{pmatrix}$ $\det\begin{pmatrix} 30 & -6 \\ -6 & 2 \end{pmatrix} > 0$ $\operatorname{tr}\begin{pmatrix} 30 & -6 \\ -6 & 2 \end{pmatrix} > 0 \Rightarrow (5,6)$ minimo relativo

$(1,-6)$; $\begin{pmatrix} 6 & -6 \\ -6 & 2 \end{pmatrix}$ $\det\begin{pmatrix} 6 & -6 \\ -6 & 2 \end{pmatrix} < 0$ $\operatorname{tr}\begin{pmatrix} 6 & -6 \\ -6 & 2 \end{pmatrix} > 0 \Rightarrow (1,-6)$ sella.

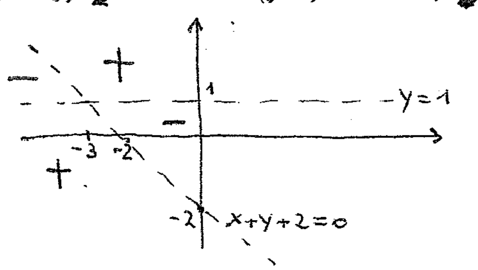
$f(5,6) = \ldots\ldots$

Trovare i massimi e minimi relativi di $f(x,y) = \arctan\{(y-1)(x+y+2)\}$

$$\frac{\partial f}{\partial x} = \frac{(y-1)}{1+(y-1)^2(x+y+2)^2} \; ; \; \frac{\partial f}{\partial y} = \frac{x+y+2+y-1}{1+(y-1)^2(x+y+2)^2} = \frac{x+2y+1}{1+(y-1)^2(x+y+2)^2} \; ;$$

$\frac{\partial f}{\partial x} = 0$ sse $y = 1$; $\frac{\partial f}{\partial y} = 0$ sse $x + 2y + 1 = 0$; $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$ sse $(x,y) = (-3,1)$

$f(x,y) \geq 0$ sse $(y-1)(x+y+2) \geq 0$ sse $\begin{cases} y > 1 \\ x+y+2 > 0 \end{cases}$ o $\begin{cases} y < 1 \\ x+y+2 < 0 \end{cases}$



Poichè $f(-3,1) = 0$ ma in ogni suo intorno si hanno punti per cui $f(x,y) > 0 = f(-3,1)$ e punti per cui $f(x,y) < 0 = f(-3,1)$ $(-3,1)$ è un punto di sella.
Quindi f non ha nè punti di minimo relativo nè punti di massimo relativo.