COUNTING INFINITE POINT-SETS

MARCO FORTI

[Joint research with Mauro Di Nasso.]

We axiomatize a notion of size for collections (*numerosity*) that satisfies the five *common notions* of Euclid's Elements, including the Aristotelian principle that "the whole is greater than the part", under the natural Cantorian definitions of sum, product, and ordering (see [1, 2, 3]). These numerosities turn out to have a much better arithmetic than cardinalities, included as they are in the positive part of a discrete (partially) ordered ring. Therefore equinumerosity cannot be identified with equipotency.

Focusing on finite dimensional point-sets, *i.e.* sets of tuples of points from a given "line" $\mathcal{L}$, we only postulate that natural "geometric" bijections are numerosity-preserving, *inter alia* all biunique "support preserving" transformations of tuples. We thus obtain that the numerosities are a positive subsemiring $\mathfrak{N}$ of the quotient ring $\mathfrak{R} = \mathbb{Z}[[\mathcal{T}]]/\mathfrak{p}$ of the powerseries ring over a set $\mathcal{T}$ of indeterminates of size $|\mathcal{L}|$, modulo a suitable prime ideal $\mathfrak{p}$. We name "Euclidean" such ideals, as well as the corresponding ultrafilters over $\kappa = |\mathcal{L}|$. The ring $\mathfrak{R}$ is then isomorphic to a ring of hyperintegers, namely the ultrapower $\mathbb{Z}^{\kappa}_{\mathcal{U}_{\mathfrak{p}}}$, where $\mathcal{U}_{\mathfrak{p}}$ is the ultrafilter corresponding to $\mathfrak{p}$.

The existence of "Euclidean" ultrafilters on an arbitrary cardinal κ is problematic, and we deal here with three particular cases, leaving the general question to further investigation:

(1) A countable line $\mathcal{L}$. Then the numerosities can be taken to be a cut in the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$, and for $\mathcal{U}$ ultrafilter on $\mathbb{N}$ one has
$\mathcal{U}$ Ramsey $\Rightarrow$ $\mathcal{U}$ Euclidean $\Rightarrow$ $\mathcal{U}$ P-point.

(2) The ordinary line $\mathbb{R}$. Then the countable numerosities can be taken as above, provided that one assume $\mathfrak{c} < \aleph_\omega$, but we conjecture that no Euclidean ultrafilter on an uncountable cardinal can provide linearly ordered uncountable numerosities.

(3) Countable point-sets over an arbitrary line $\mathcal{L}$. Then the numerosities can be taken as above, but the existence of the Euclidean ultrafilter $\mathcal{U}$ over $\kappa = |\mathcal{L}|$ such that $\mathbb{Z}^{\kappa}_{\mathcal{U}} \approx \mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ requires supplementary, although not too strong, set theoretic assumptions, *e.g.* the Singular Cardinal Hypothesis plus $\Box_\kappa$ for each singular cardinal κ of countable cofinality.

References

[1] V. Benci, M. Di Nasso - Numerosities of labelled sets: a new way of counting, *Adv. Math.* **173** (2003), 50–67.

[2] V. Benci, M. Di Nasso, M. Forti - An Aristotelian notion of size, *Ann. Pure Appl. Logic* **143** (2006), 43–53.

[3] V. Benci, M. Di Nasso, M. Forti - An Euclidean measure of size for mathematical universes, *Logique et Analyse* **50** (2007).

(Marco Forti) Dipart. di Matematica Applicata "U. Dini", Università di Pisa, Italy.
E-mail address: forti@dma.unipi.it

(Mauro Di Nasso) Dipart. di Matematica "L. Tonelli", Università di Pisa, Italy.
E-mail address: dinasso@dm.unipi.it