Nonstandard Methods in Analysis

An elementary approach to Stochastic Differential Equations

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- In most applications of NSA to analysis, only elementary tools and techniques of nonstandard calculus seems to be necessary.
- The advantages of a theory which includes infinitasimals rely more on the possibility of making new models rather than in the dimostration techniques.

These two points will be illustrated using α-**theory** in the study of **Brownian motion.**

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V. Benci, A Construction of a Nonstandard Universe, in *Advances in Dynamical System and Quantum Physics*, S. Albeverio, R. Figari, E. Orlandi, A. Teta ed.,(Capri, 1993), 11–21, World Scientific, (1995).

V Benci, M Di Nasso, Alpha-theory: an elementary axiomatics for nonstandard analysis. Expo. Math. 21 (2003), no. 4, 355–386.

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ANDERSON, ROBERT M. A nonstandard representation for Brownian motion and Itô integration. Bull. Amer. Math. Soc. 82 (1976), no. 1, 99–101.

KEISLER, H. JEROME An infinitesimal approach to stochastic analysis. Mem. Amer. Math. Soc. 48 (1984), no. 297, x+184 pp.

S. ALBEVERIO, J. E. FENSTAD, R. HOEGH-KROHN, AND T. LINDSTRØM, *Non-standard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986.

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stochastic differential equation

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stochastic differential equation

The nonstandard mathematical model which I will present here is based on the notion of

stochastic grid equation

The basic point

We do not want that every single object or result of the **standard model** have its analogous in the **nonstandard model**.

We want to compare only the final result (namely the Fokker-Plank equation).

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Without this request, usually, the nonstandard models are more complicated that the standard ones since they are forced to follows a development not natural for them.

The abstract scheme

$$\boxed{\textbf{FACTS TO EXPLAIN-DESCRIBE}}$$
$$\Downarrow$$
$$\boxed{\textbf{MATHEMATICAL MODEL}}$$
$$\Downarrow$$
$$\boxed{\textbf{RESULTS WHICH MIGHT COMPARED WITH "REALITY"}}$$

The abstract scheme

$$\boxed{\textbf{BROWNIAN MOTION}}$$
$$\Downarrow$$
$$\boxed{\textbf{MATHEMATICAL MODEL}}$$
$$\Downarrow$$
$$\boxed{\textbf{HEAT EQUATION and FOKKER-PLANK EQUATION}}$$

Our program

Starting from a naive idea of Brownian motion, and using α-theory, we deduce the Fokker-Plank equation in a simple and rigorous way.

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Starting from a naive idea of Brownian motion, and using α-theory, we deduce the Fokker-Plank equation in a simple and rigorous way.

It is possible to keep every things to a simple level since all the theory of stochastic grid equations is treated as a hyperfinite theory and it is not translated in a "*standard model*".

The only standard object is the final one: the Fokker-Plank equation.

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The "existence" of i leads to new mathematical objects such as holomorphic functions etc.
In a similar way, the "existence" of α leads to new mathematical objects such as internal sets (and functions) etc.

α1. Extension Axiom.
Every sequence $\varphi(n)$ can be uniquely extended to $\mathbb{N} \cup \{\boldsymbol{\alpha}\}$. The corresponding value at $\boldsymbol{\alpha}$ will be denoted by $\varphi(\boldsymbol{\alpha})$. If two sequences φ, ψ are different at all points, then $\varphi(\boldsymbol{\alpha}) \neq \psi(\boldsymbol{\alpha})$.

Aximatic introduction of α-theory (B., Di Nasso)

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α2. Composition Axiom.
If φ and ψ are sequences and if f is any function such that compositions $f \circ \varphi$ and $f \circ \psi$ make sense, then

$$\varphi(\boldsymbol{\alpha}) = \psi(\boldsymbol{\alpha}) \Rightarrow (f \circ \varphi)(\boldsymbol{\alpha}) = (f \circ \psi)(\boldsymbol{\alpha})$$

α3. Real Number Axiom.
If $c_m : n \mapsto r$ is the constant sequence with value r, then $c_m(\boldsymbol{\alpha}) = r$; and if $1_{\mathbb{N}} : n \mapsto n$ is the immersion of $\mathbb{N}$ in $\mathbb{R}$, then $1_{\mathbb{R}}(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \notin \mathbb{R}$.

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α4. Internal Set Axiom.
If ψ *is a sequence of sets, then also* $\psi(\boldsymbol{\alpha})$ *is a set and*

$$\psi(\boldsymbol{\alpha}) = \{\varphi(\boldsymbol{\alpha}) : \varphi(n) \in \psi(n) \text{ for all } n\}.$$

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α5. Pair Axiom.
If $\vartheta(n) = \{\varphi(n), \psi(n)\}$ for all n, then $\vartheta(\boldsymbol{\alpha}) = \{\varphi(\boldsymbol{\alpha}), \psi(\boldsymbol{\alpha})\}$.

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Then, in α-theory there exists a **unique** *Mathematical Universe* with three kind of sets:

- **standard sets**: they can be constructed without postulating the existence of "α".
- **internal sets**: they are constructed according to the rule defined by Axiom 4.
- sets which are not standard nor internal.

The application of α-theory to the study of Brownian motion is contained in the following works:

RAGO, EMILIANO Una deduzione dell'equazione di Fokker-Planck con metodi Nonstandard, Thesis, University of Pisa, (2001).

BENCI V., GALATOLO S., GHIMENTI M. An elementary approach to Stochastic Differential Equations using the infinitesimals, *to appear.*

Basic notions

The *hyperfinite grid* $\mathbb{H}_\alpha$ is defined as the ideal value of the set

$$\mathbb{H}_n = \left\{ \frac{k}{n} : k \in \mathbb{Z},\ -\frac{n^2}{2} \leq k < \frac{n^2}{2} \right\};$$

namely,

$$\mathbb{H} := \mathbb{H}_\alpha = \left\{ k\Delta : k \in \mathbb{Z}^*,\ -\frac{\alpha^2}{2} \leq k < \frac{\alpha^2}{2} \right\}$$

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$$\Delta := \frac{1}{\alpha}$$

Clearly $\mathbb{H}$ is an hyperfinite set with $|\mathbb{H}| = \alpha^2$. Given $a, b \in \mathbb{H}$, we set

$$\begin{aligned} [a, b]_{\mathbb{H}} &= \{x \in \mathbb{H} : a \leq k \leq b\} \\ [a, b)_{\mathbb{H}} &= \{x \in \mathbb{H} : a \leq k < b\} \end{aligned}$$

Definition

An internal function

$$\xi : \mathbb{H} \rightarrow \mathbb{R}^*$$

is called *grid function*.

Definition

Given a grid function $\xi : \mathbb{H} \rightarrow \mathbb{R}^*$, we define its grid derivative $\frac{\Delta\xi}{\Delta t}$ as

$$\frac{\Delta\xi}{\Delta t}(t) = \frac{\xi(t+\Delta) - \xi(t)}{\Delta};$$

Definition

The grid integral of ξ is defined as

$$\mathbb{I}\left[\xi\right] = \Delta \sum_{t \in \mathbb{H}} \xi\left(t\right);$$

if $\Gamma \subset \mathbb{H}$ is a hyperfinite set we define its grid integral as

$$\mathbb{I}_{\Gamma}\left[\xi\right] = \Delta \sum_{t \in \Gamma} \xi\left(t\right)$$

Definition

of A grid function ξ is called integrable in $[a, b]$ $(a, b \in \mathbb{R})$ if $\mathbb{I}_{[a,b]}[\xi]$ is finite; in this case, we set

$$\int_a^b \xi(s)\, ds_\Delta := sh\left(\mathbb{I}_{[a,b)}[\xi]\right) = sh\left(\Delta \sum_{t \in \mathbb{H} \cap [a,b)} \xi(t)\right)$$

To every real function

$$f : [a, b] \to \mathbb{R}$$

it is possible to associate its natural extension

$$f^* : [a, b]^* \to \mathbb{R}^*$$

and a grid function

$$\tilde{f} : [a, b]_{\mathbb{H}} \to \mathbb{R}^* \tag{1}$$

obtained as restriction of f^* to $[a, b]_{\mathbb{H}}$.

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Clearly, if f is continuous, the α-integral of $\tilde{f}$ coincides with the Riemann integral of f. Notice that every (bonded) function has its α-integral.

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In order to develop the the theory, the grid derivative of $x(t)$ needs to be infinite, but not too big, namely

$$\frac{\Delta x}{\Delta t}(t) \cong \sqrt{\alpha}$$

The main tool: Ito's formula

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Theorem (The Ito's Formula for grid-functions)

Let $\varphi \in C_0^3(\mathbb{R}^2)$ and $x(t)$ be a grid function such that

$$\left| \frac{\Delta x}{\Delta t}(t) \right| \leq \eta \alpha^{2/3}, \quad \eta \sim 0. \tag{2}$$

Then

$$\frac{\Delta}{\Delta t}\varphi(t,x) \sim \varphi_t(t,x) + \varphi_x(t,x)\frac{\Delta x}{\Delta t} + \frac{\Delta}{2}\varphi_{xx}(t,x) \cdot \left(\frac{\Delta x}{\Delta t}\right)^2.$$

Here φ_t, φ_x and φ_{xx} denote the usual partial derivative of φ.

Idea of the proof

$$\begin{aligned}
\frac{\Delta}{\Delta t}\varphi(t,x(t)) &= \frac{\varphi(t+\Delta,x(t+\Delta))-\varphi(t,x(t+\Delta))}{\Delta} \\
&\quad + \frac{\varphi(t,x(t+\Delta))-\varphi(t,x(t))}{\Delta} \\
&\sim \varphi_t(t,x(t)) + \frac{\varphi(t,x(t+\Delta))-\varphi(t,x(t))}{\Delta}
\end{aligned}$$

Idea of the proof

But

$$\varphi(t, x(t+\Delta)) = \varphi\left(t, x(t) + \Delta\frac{\Delta x}{\Delta t}(t)\right),$$

and $\left|\Delta\frac{\Delta x}{\Delta t}(t)\right| \leq \eta\alpha^{2/3}\Delta = \eta\Delta^{1/3}$ is infinitesimal.

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and $\left|\Delta\frac{\Delta x}{\Delta t}(t)\right| \leq \eta\alpha^{2/3}\Delta = \eta\Delta^{1/3}$ is infinitesimal.
Then

$$\begin{aligned} & \varphi\left(t, x(t) + \Delta\frac{\Delta x}{\Delta t}(t)\right) \\ = \;& \varphi(t, x(t)) + \varphi_x(t, x(t))\Delta\frac{\Delta x}{\Delta t}(t) + \frac{1}{2}\varphi_{xx}(t, x(t))\left(\Delta\frac{\Delta x}{\Delta t}(t)\right)^2 \\ & + \frac{1}{3!}\varphi_{xxx}(t, x(t))\left(\Delta\frac{\Delta x}{\Delta t}(t)\right)^3 + \varepsilon\left(\Delta\frac{\Delta x}{\Delta t}(t)\right)^3 \end{aligned}$$

Idea of the proof

Hence

$$
\begin{aligned}
& \frac{\varphi(t,x(t+\Delta)) - \varphi(t,x(t))}{\Delta} \\
= \quad & \varphi_x(t,x(t))\frac{\Delta x}{\Delta t}(t) + \frac{\Delta}{2}\varphi_{xx}(t,x(t)) \cdot \left(\frac{\Delta x}{\Delta t}(t)\right)^2 \\
& + \frac{\Delta^2}{6}\varphi_{xxx} \cdot \left(\frac{\Delta x}{\Delta t}(t)\right)^3 + \varepsilon\Delta^2 \left(\frac{\Delta x}{\Delta t}(t)\right)^3
\end{aligned}
$$

Grid Differential Equations

A Grid Differential Equations has the following form

$$\begin{cases} \frac{\Delta x}{\Delta t}(t) = f(t, x(t)), & t \in \mathbb{H} \\ x(t_0) = x_0 \end{cases}$$

where f is **any** internal function.

Theorem

The Cauchy problem for a Grid Differential equation has always a unique solution

Stochastic Grid Equations

A Stochastic Grid Equation is simply a family of grid differential equations having the following form

$$\begin{cases} \frac{\Delta x}{\Delta t}(t) = f(t,x) + h(t,x)\xi(t), \\ x(0) = x_0, \\ \xi \in \mathcal{R}. \end{cases}$$

where $\mathcal{R}$ is a **hyperfinite** set.

We want to study the statistical behavior of the set of solutions of the above Cauchy problems

$$\mathcal{S} = \left\{ x_\xi(t) : \xi \in \mathcal{R} \right\};$$

More precisely we want to describe the behavior of the density function

$$\rho : [0,1]_{\mathbb{H}} \times \mathbb{H}^* \to \mathbb{Q}^*$$

defined as follows

$$\rho(t,x) = \frac{\left| \left\{ x_\xi \in \mathcal{S} : x \le x_\xi(t) < x + \Delta \right\} \right|}{\Delta \left| \mathcal{R} \right|}.$$

Definition

A *stochastic class of white noises* (or simply a withe noise) is the internal set of grid functions defined by

$$\mathcal{R} = \mathcal{R}_{\alpha}$$

where

$$\mathcal{R}_n = \left\{-\sqrt{n}, +\sqrt{n}\right\}^{[0,1]_{\mathbb{H}_n}}$$

Thus, given a grid function ξ, we have that

$$\xi \in \mathcal{R} \Leftrightarrow \forall x,\ \xi(x) = \pm\sqrt{\alpha}$$

The main result

We will prove that $\forall\varphi \in \mathcal{D}\left([0,1) \times \mathbb{R}\right)$,

$$\int\int \left(\varphi_t + f\varphi_x + \varphi_{xx}h^2\right)\rho \; dx_\Delta \; dt_\Delta + \varphi(0,x_0) = 0$$

under the assumption that f and h are internal functions, bounded on bounded sets.

Standard interpretation

This result has a meaningful interpretation which makes sense also using the standard language.

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The *standard objects* which allow a bridge between Grid Functions and Standard Universe are the **distributions**

In fact, it is possible to associate a distribution T_ξ to a grid function ξ via the following formula:

$$\langle T_\xi, \varphi \rangle = \int_A \xi\varphi \, ds_\Delta = sh\left(\Delta \cdot \sum_{t \in A_{\mathbb{H}}} \xi(t)\varphi(t)\right), \varphi \in \mathcal{D}.$$

provided that $\xi\varphi$ is integrable.

Thus we have obtained the following result:

Theorem

Assume that $\mathcal{R}$ is a white noise and that $f(t,x)$ and $h(t,x)$ are continuous functions. Then the distribution T_ρ relative to the density function ρ is a measure and satisfies the Fokker-Plank equation

$$\frac{dT_\rho}{dt} + \frac{d}{dx}\left(f(t,x)\,T_\rho\right) - \frac{1}{2}\frac{d^2}{dx^2}\left(h(t,x)^2\,T_\rho\right) = 0. \tag{3}$$

$$T_\rho(0,x) = \delta \tag{4}$$

in the sense of distribution.

If $f(t,x)$ and $h(t,x)$ are smooth functions, by standard results in PDE, we know that, for $t > 0$, the distribution T_ρ coincides with a smooth function $u(t,x)$.

If $f(t,x)$ and $h(t,x)$ are smooth functions, by standard results in PDE, we know that, for $t > 0$, the distribution T_ρ coincides with a smooth function $u(t,x)$. Then, for any $t > 0$, ρ defines a smooth function u by the formula

$$\forall \varphi \in \mathcal{D}\left((0,1) \times \mathbb{R}\right), \int \int \rho\varphi \, dx_\Delta dt_\Delta = \int \int u\varphi \, dx \, dt$$

and u satisfies the Fokker-Plank equation in $(0,1) \times \mathbb{R}$ in the usual sense:

$$\frac{du}{dt} + \frac{d}{dx}\left(f(t,x)u\right) - \frac{1}{2}\frac{d^2}{dx^2}\left(h(t,x)^2 u\right) = 0.$$

The conclusion of our Theorem hold not only if the "stochastic class" $\mathcal{R}$ defined as above, but for any class $\mathcal{R}$ which satisfies suitable properties. For example we can take

$$\mathcal{R} = \mathcal{R}_\alpha; \quad \mathcal{R}_n := \left\{q_1\sqrt{n}, ..., q_k\sqrt{n}\right\}^{[0,1]_{\mathbb{H}_n}}; \quad k \in \mathbb{N}$$

with $q_i \in \mathbb{R}^*$,

$$\sum_{i=1}^{k} q_i = 0; \ \sum_{i=1}^{k} q_i^2 = 1.$$

Probabilistic interpretation

In classical mathematics and also in some Nonstandard approach to this topic, the most delicate part relies in the notion of probability measure in an infinite dimensional metric space, namely the space of all the orbits.

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In classical mathematics and also in some Nonstandard approach to this topic, the most delicate part relies in the notion of probability measure in an infinite dimensional metric space, namely the space of all the orbits.

In our approach we have not used the notion of probability but rather that of *descriptive statistics*

Probabilistic interpretation

In classical mathematics and also in some Nonstandard approach to this topic, the most delicate part relies in the notion of probability measure in an infinite dimensional metric space, namely the space of all the orbits.

How can we introduce a probabilistic interpretation of the Fokker-Plank equation?

Probabilistic interpretation

In a world where infinitesimals are allowed, it makes sense to define the probability function

$$P : [\mathcal{P}(\Omega)]^* \to [0,1]^* \cap \mathbb{Q}^*$$

in the following way

$$P(E) = \frac{|E|}{|\Omega|}$$

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In this approach, there is no need to define the Lieb measure.

Probabilistic interpretation

In fact, we have that

$$P_{[a,b)} := P\left(x_{\xi}(t) \in [a,b)\right) = \mathbb{I}_{[a,b)}\left(\rho\left(\cdot,t\right)\right)$$

namely the probability is a hyperrational number; if you do not like it you may take the standard part:

$$sh\left(P_{[a,b)}\right) := \int_a^b \rho\left(x,t\right)\ dx_{\Delta} = \int_a^b u\left(x,t\right)\ dx$$

Idea of the proof

Chosen an arbitrary $\varphi \in \mathcal{C}([0,1] \times \mathbb{R})$ bounded in the second variable, we have that

$$\varphi(1, x_\xi(1)) - \varphi(0, x_0) = \Delta \sum_{t \in [0,1-\Delta]_{\mathbb{H}}} \frac{\Delta \varphi}{\Delta t}(t, x_\xi(t)),$$

Now we assume that $\varphi \in \mathcal{D}([0,1) \times \mathbb{R}))$; the by the Ito grid formula

$$-\varphi(0, x_0) \sim \Delta \sum_{t \in [0,1)_{\mathbb{H}}} \left[\varphi_t + \varphi_x \cdot \frac{\Delta x}{\Delta t} + \frac{\Delta}{2} \varphi_{xx} \cdot \left(\frac{\Delta x}{\Delta t} \right)^2 \right]$$

Idea of the proof

Since x_ξ solves our equation, we obtain

$$
\begin{aligned}
-\varphi(0,x_0) &\sim \Delta \sum_{t\in[0,1)_{\mathbb{H}}} \left[\varphi_t + \varphi_x \cdot (f + h\xi) + \frac{\Delta}{2}\varphi_{xx} \cdot (f + h\xi)^2\right] \\
&= \Delta \sum_{t\in[0,1)_{\mathbb{H}}} (\varphi_t + f\varphi_x) + (\varphi_x h + \Delta\varphi_{xx} f)\,\xi \\
&\quad + \Delta \sum_{t\in[0,1)_{\mathbb{H}}} \frac{\Delta}{2}\varphi_{xx} f + \frac{\Delta}{2}\varphi_{xx} h^2\xi^2
\end{aligned}
$$

Idea of the proof

Now we want to compute the *mean* or *expectation value*

$$\mathbb{E}_{\xi\in\mathcal{R}}$$

of each term of the above formula.

The expectation value is defined in the following way:

$$\mathbb{E}_{\xi\in\mathcal{R}}(F_\xi) := \frac{1}{|\mathcal{R}|}\sum F_\xi$$

Idea of the proof

- $$\mathbb{E}_{\xi\in\mathcal{R}}\left[\varphi_t + f\varphi_x\right] \sim \Delta \sum_{x\in\mathbb{H}} \left[\varphi_t + f\varphi_x\right]\rho$$

Idea of the proof

- $$\mathbb{E}_{\xi \in \mathcal{R}} \left[\varphi_t + f \varphi_x \right] \sim \Delta \sum_{x \in \mathbb{H}} \left[\varphi_t + f \varphi_x \right] \rho$$

- $$\mathbb{E}_{\xi} \left[\left(\varphi_x h + \Delta \varphi_{xx} f \right) \xi \right] \sim 0.$$

Idea of the proof

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- $$\mathbb{E}_{\xi}\left[\frac{\Delta}{2}\varphi_{xx} f\right] \sim 0$$

Idea of the proof

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- $$\mathbb{E}_{\xi}\left[\left(\varphi_x h + \Delta\varphi_{xx} f\right)\xi\right] \sim 0.$$

- $$\mathbb{E}_{\xi}\left[\frac{\Delta}{2}\varphi_{xx} f\right] \sim 0$$

- $$\begin{aligned}\mathbb{E}_{\xi}\left[\frac{\Delta}{2}\varphi_{xx} h^2\xi^2\right] &\sim \mathbb{E}_{\xi}\left[\frac{\alpha\Delta}{2}\varphi_{xx} h^2\right] \\ &= \frac{1}{2}\mathbb{E}_{\xi}\left[\varphi_{xx} h^2\right] = \frac{\Delta}{2}\sum_{x\in\mathbb{H}} \varphi_{xx} h^2\rho\end{aligned}$$

Idea of the proof

$$
\begin{aligned}
-\varphi(0,x_0) &= \mathbb{E}_{t,\xi}\left[-\varphi(0,x_0)\right] \\
&\sim \Delta \sum_{t\in[0,1)_{\mathbb{H}}} \left(\mathbb{E}_{\xi}\left[\varphi_t + f\varphi_x\right] + \mathbb{E}_{\xi}\left[\left(\varphi_x h + \Delta\varphi_{xx} f\right)\xi\right]\right) + \\
&\quad + \Delta \sum_{t\in[0,1)_{\mathbb{H}}} \left(\mathbb{E}_{\xi}\left[\frac{\Delta}{2}\varphi_{xx} f\right] + \mathbb{E}_{\xi}\left[\frac{\Delta}{2}\varphi_{xx} h^2 \xi^2\right]\right) \\
&\sim \Delta^2 \sum_{t\in[0,1)_{\mathbb{H}}} \left(\sum_{x\in\mathbb{H}} \left(\varphi_t + f\varphi_x\right)\rho + \varphi_{xx} h^2 \rho\right) \\
&\sim \int\int \left(\varphi_t + f\varphi_x + \varphi_{xx} h^2\right)\rho \; dxdt
\end{aligned}
$$

Then,

$$-\varphi(0,x_0) \sim \int\int \left(\varphi_t + f\varphi_x + \varphi_{xx} h^2\right)\rho \; dxdt$$

The end

Thank you for your attention!