Automatic continuity of nonstandard measures

David A. Ross
Department of Mathematics
University of Hawai'i at Manoa
Honolulu, HI 96822 USA

June 2, 2008


Free!

1 What does Nonstandard Analysis give you "for free"?

- Quantifier simplification
- Proof strength (Henson, Kaufman, Keisler)
- Weak limits
- Ideal objects (*eg* Measures; Neometric spaces of Keisler/Fajardo)
- Automatic uniformization (*eg*, Gordon Keller's proof that Amenable varieties of groups are uniformly amenable)
- Automatic continuity of measures ◀

Assumption: Nonstandard model is as saturated as it needs to be, but at least $\aleph_1$–saturated

Remark: There are interesting FA measures that do not extend to a σ–additive measure, eg:

- Nonprincipal ultrafilters on ω
- Amenable finitely generated groups

2 Loeb Measures

- Let $(\Omega, \mathcal{A}, \mu)$ be an internal finitely additive finite *-measure.
  - Ω is an internal set
  - $\mathcal{A}$ is an internal *–algebra on Ω
  - $\mu : \mathcal{A} \longrightarrow^* [0, \infty)$ is an internal function satisfying (i) $\mu(\emptyset) = 0$, (ii) $\mu(\Omega)$ is finite, and and (iii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ are disjoint.
- **Note:** $\mathcal{A}$ is (externally) an algebra on Ω, and $st \circ \mu = {}^\circ\mu$ is an "actual" finitely-additive measure on $(\Omega, \mathcal{A})$.
- If $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of elements of $\mathcal{A}$ indexed by the standard natural numbers, and the intersection $\bigcap_n A_n$ is empty, then by $\aleph_1$–saturation there is a finite N such that $\bigcap_{n \leq N} A_n = \emptyset$. ($\therefore {}^\circ\mu$ is σ–additive on $\mathcal{A}$.)
- The Carathéodory extension criterion is therefore satisfied trivially, and $(\Omega, \mathcal{A}, {}^\circ\mu)$ extends to a countably-additive measure space $(\Omega, \mathcal{A}_L, \mu_L)$, (a *Loeb space*) where $\mathcal{A}_L$ is the smallest (external) sigma-algebra containing $\mathcal{A}$.
- **A useful fact:** If $E \in \mathcal{A}_L$, and $\epsilon > 0$ is standard, then $\exists A_i, A_o \in \mathcal{A}$ such that $A_i \subseteq E \subseteq A_o$ and $\mu(A_o) - \mu(A_i) < \epsilon$,

3 Nonnull subsets of a finite, finitely-additive measure space

Theorem (F.A. Borel-Cantelli). *Let $(X, \mathcal{A}, \mu)$ be a finite, finitely-additive measure, and for $n \in \mathbb{N}$ let $A_n \in \mathcal{A}$. Suppose that for some $\epsilon > 0$, $\mu(A_n) > \epsilon$ for all n. Then there is an increasing sequence of natural numbers $\{n_m : m \in \mathbb{N}\}$ such that for every $N \in \mathbb{N}$,* $\mu\Big(\bigcap\limits_{m=1}^{N} A_{n_m}\Big) > 0.$

Equivalently: If a countable collection of sets is uniformly nonnull, then there is an infinite subcollection that any finite subcollection of *it* has nonnull intersection.

Case 1 μ is actually $\sigma-$additive.

\begin{Graduate exercise}

Put $B = \bigcup\{\bigcap_{i\in I} A_i : I \subseteq \mathbb{N}, I$ finite, $\mu(\bigcap_{i\in I} A_i) = 0\}$

This union is *over* at most countably many nullsets, $\therefore \mu(B) = 0$.

Put $A'_n = A_n \setminus B$ for each n

Note: If $I \subseteq \mathbb{N}$ is finite, $\mu(\bigcap_{i\in I} A_i) = 0$ if and only if $\bigcap_{i\in I} A'_i = \emptyset$.

$\therefore$ suffices to find an increasing sequence n_m such that $\bigcap_{m=1}^{N} A'_{n_m} \neq \emptyset$ for every N

As in easy half of Borel-Cantelli Lemma, $\mu(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A'_n) > \epsilon$

let $x \in \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A'_n$; there is an increasing sequence n_m such that $x \in A'_{n_m}$, done.

\end{Graduate exercise}

Theorem (F.A. Borel-Cantelli). *Let $(X, \mathcal{A}, \mu)$ be a finite, finitely-additive measure, and for $n \in \mathbb{N}$ let $A_n \in \mathcal{A}$. Suppose that for some $\epsilon > 0$, $\mu(A_n) > \epsilon$ for all n. Then there is an increasing sequence of natural numbers $\{n_m : m \in \mathbb{N}\}$ such that for every $N \in \mathbb{N}$,* $\mu\big(\bigcap_{m=1}^{N} A_{n_m}\big) > 0$.

Case 2 μ is **not** assumed to be σ−additive

\begin{Free Lunch}

Pass from $(X, \mathcal{A}, \mu)$ to the σ−additive Loeb measure μ_L on $({}^*X, {}^*\mathcal{A}_L)$.

For each $n \in \mathbb{N}$, $\mu_L({}^*A_n) = \mu(A_n) > \epsilon$

By Case 1, there is an increasing subsequence n_m in $\mathbb{N}$ such that for any $N \in \mathbb{N}$, $\mu_L\big(\bigcap_{m=1}^{N} {}^*A_{n_m}\big) > 0$.

When N is standard,

$$\mu\Big(\bigcap_{m=1}^{N} A_{n_m}\Big) = \mu_L\Big(\bigcap_{m=1}^{N} {}^*A_{n_m}\Big) > 0,$$

done.

\end{Free Lunch}

Theorem. *(Banach)Let X be a set, $B(X)$ be all bounded real functions on X, and $\{f_n : n \in \mathbb{N}\}$ be a uniformly bounded sequence. The following are equivalent:*

(i) $\{f_n\}_n$ coverges weakly to 0;

(ii) for any sequence $\{x_k : k \in \mathbb{N}\}$ in X, $\lim_{n\to\infty} \liminf_{k\to\infty} f_n(x_k) = 0$

Weak convergence to zero here means that for any positive linear functional T on $B(X)$, $Tf_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark: If X is finite, then it is trivial to verify that (ii) is equivalent to $f_n \longrightarrow 0$ pointwise on X.

Easy direction: $(\neg ii \Rightarrow \neg i)$

By $(\neg ii)$ there is a sequence x_k in X, a positive real number r, and an increasing sequence n_m of natural numbers such that $\liminf_{k\to\infty} |f_{n_m}(x_k)| > r$ for all m.

For each $m \in \mathbb{N}$ there is a $N \in \mathbb{N}$ such that for all $k > N, |f_{n_m}(x_k)| > r$.

$\therefore$ For all standard $m \in \mathbb{N}$ and any *infinite* $k \in ({}^*\mathbb{N} \setminus \mathbb{N}), |{}^*f_{n_m}(x_k)| > r$. Fix such a k.

Define $T : B(X) \longrightarrow \mathbb{R}$ by $T(g) = {}^{\circ *}g(x_k)$.

T is a positive linear functional.

For standard $m \in \mathbb{N}$, $0 < r < |{}^*f_{n_m}(x_k)| \approx |T(f_{n_m})|$, so $Tf_n \not\longrightarrow 0$ as $n \longrightarrow \infty$, done.

Theorem. *(Banach)Let X be a set, $B(X)$ be all bounded real functions on X, and $\{f_n : n \in \mathbb{N}\}$ be a uniformly bounded sequence. The following are equivalent:*

(i) $\{f_n\}_n$ coverges weakly to 0;

(ii) for any sequence $\{x_k : k \in \mathbb{N}\}$ in X, $\lim_{n\longrightarrow\infty} \liminf_{k\longrightarrow\infty} f_n(x_k) = 0$

Proof of $(\neg i \Rightarrow \neg ii)$

By $(\neg i)$ there is a positive linear functional T such that $Tf_n \not\to 0$ as $n \longrightarrow \infty$.

Note: If (through some miracle) T is given by integration against a measure μ then the rest is trivial:

By the Bounded Convergence Theorem, for some $x \in X$ $f_n(x) \nrightarrow 0$.

Put $x_k = x$ for all k, then x_k witnesses failure of (ii).

Theorem. *(Banach)Let X be a set, $B(X)$ be all bounded real functions on X, and $\{f_n : n \in \mathbb{N}\}$ be a uniformly bounded sequence. The following are equivalent:*

(i) $\{f_n\}_n$ coverges weakly to 0;

(ii) for any sequence $\{x_k : k \in \mathbb{N}\}$ in X, $\lim_{n\to\infty} \liminf_{k\to\infty} f_n(x_k) = 0$

Proof of $(\neg i \Rightarrow \neg ii)$

By $(\neg i)$ there is a positive linear functional T such that $Tf_n \not\to 0$ as $n \to \infty$.

$\mu : E \mapsto T(\chi_E)$ is a finite, finitely-additive measure on $(X, \mathcal{P}(X))$

Pass from $(X, \mathcal{A}, \mu)$ to the $\sigma-$additive Loeb measure μ_L on $({}^*X, {}^*\mathcal{A}_L)$

Exercise: For any $f \in B(X)$, $T(f) = \int {}^{\circ *}f_n d\mu_L$.

$\int {}^{\circ *}f_n d\mu_L = T(f_n) \not\to 0$ as $n \to \infty$

By Bounded convergence, there is some $x_\infty \in {}^*X$, $r > 0$, and increasing sequence n_m of natural numbers such that $|{}^{\circ *}f_{n_m}(x_\infty)| > r$ for all $m \in \mathbb{N}$.

For any $N \in \mathbb{N}, x_\infty$ witnesses $(\exists x_N \in {}^*X) \bigwedge_{m=1}^{N} [|{}^*f_{n_m}|(x_N) > r]$.

By transfer $(\exists x_N \in X) \bigwedge_{m=1}^{N} [|f_{n_m}|(x_N) > r]$.

For any $m, N \in \mathbb{N}$ with $N > m, |f_{n_m}(x_N)| > r$, $\quad \therefore \lim_{m\to\infty} \liminf_{k\to\infty} |f_{n_m}(x_k)| > r$.

This contradicts (ii), done.

It is also possible to give an alternate proof of the implication (ii $\Rightarrow$ i) of Theorem 3 by an appeal to Theorem 3. Suppose (i) fails, and obtain T and μ as in the proof above. Then there is an $r > 0$ and an increasing sequence n_m of natural numbers such that $|T(f_{n_m})| > r$. Let $\delta \in \mathbb{R}$ satisfy $0 < \delta < \frac{r}{2T(1)}$; equivalently, $0 < T(\delta) < r/2$. Note that for any $g \in B(X)$ with $-\delta \leq g \leq \delta$, positivity of T ensures that $-T(\delta) = T(-\delta) \leq T(g) \leq T(\delta)$, so $|T(g)| \leq T(\delta) < r/2$. Let $M > 0$ be a bound for all the functions f_n.

For $m \in \mathbb{N}$ put $A_{n_m} = \{x \in X : |f_{n_m}(x)| > \delta\}$. Then $r < |T(f_{n_m})| = |T(f_{n_m}\chi_{A_{n_m}}) + T(f_{n_m}\chi_{A^{\complement}_{n_m}})| \leq |T(f_{n_m}\chi_{A_{n_m}})| + |T(\delta)| \leq MT(\chi_{A_{n_m}}) + r/2$, so $\mu(A_{n_m}) = T(\chi_{A_{n_m}}) > \frac{r}{2M} > 0$ for all m.

By Theorem 3 there is a subsequence (which for simplicity will just be denoted n_m again) such that for every $N \in \mathbb{N}$, $\mu\left(\bigcap_{m=1}^{N} A_{n_m}\right) > 0$. Let $x_N \in \mu\left(\bigcap_{m=1}^{N} A_{n_m}\right)$. For any $m, N \in \mathbb{N}$ with $N > m, x_N \in A_{n_m}$, therefore $|f_{n_m}(x_N)| > \delta$, so $\lim_{m\to\infty} \liminf_{k\to\infty} |f_{n_m}(x_k)| \geq \delta$. This contradicts (ii) and proves the implication.

4 Towards a metatheorem

Is there a metatheorem of the form, “If T is a statement satisfying ★, and T is true for all countably-additive finite measures, then T is true for finitely-additive finite measures?

Yes , if ★ is “expressible in the “probability logic” $L_{\omega_1 P}$ of Hoover and Keisler.

Is there something more practically interesting?