Convoluzione e teoremi di approssimazione

Scelta di ϕ_ε

Sia $\varphi \in C_c^\infty(\mathbb{R}^d)$ una funzione tale che:

$$\varphi \geq 0 \text{ su } \mathbb{R}^d \ ; \qquad \text{il supporto di } \varphi \text{ è contenuto in } B_1 \ ; \qquad \int_{B_1} \varphi(x)\,dx = 1 \ ; \qquad \varphi(x) = \varphi(-x) \ .$$

Per ogni $\varepsilon > 0$ definiamo la funzione

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(x/\varepsilon).$$

Allora,

$$\phi_\varepsilon \geq 0 \text{ su } \mathbb{R}^d \ ; \qquad \text{il supporto di } \phi_\varepsilon \text{ è contenuto in } B_\varepsilon \ ; \qquad \int_{B_\varepsilon} \phi_\varepsilon(x)\,dx = 1 \ ; \qquad \phi_\varepsilon(x) = \phi_\varepsilon(-x) \ .$$

Convoluzione in $L^p(\mathbb{R}^d)$

Lemma 1. *Sia $u \in L^p(\mathbb{R}^d)$ con $p \in [1, +\infty]$. Allora $u * \phi_\varepsilon \in L^p(\mathbb{R}^d)$ e*

$$\|u * \phi_\varepsilon\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)} \ .$$

Dimostrazione. Nel caso $p = +\infty$, abbiamo:

$$\big|u * \phi_\varepsilon(y)\big| = \left|\int_{\mathbb{R}^d} u(y-x)\phi_\varepsilon(x)\,dx\right| \leq \|u\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \phi_\varepsilon(x)\,dx = \|u\|_{L^\infty(\mathbb{R}^d)}.$$

Quando $p = 1$, abbiamo:

$$\begin{aligned}
\int_{\mathbb{R}^d} \big|u * \phi_\varepsilon(y)\big|\,dy &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} u(y-x)\phi_\varepsilon(x)\,dx\right| dy \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(y-x)|\phi_\varepsilon(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(y-x)|\,dy\,\phi_\varepsilon(x)\,dx \\
&= \int_{\mathbb{R}^d} |u(y)|\,dy \int_{\mathbb{R}^d} \phi_\varepsilon(x)\,dx = \int_{\mathbb{R}^d} |u(y)|\,dy.
\end{aligned}$$

Nel caso $p \in (1, +\infty)$, abbiamo che per la disuguaglianza di Jensen

$$\begin{aligned}
\int_{\mathbb{R}^d} \big|u * \phi_\varepsilon(y)\big|^p\,dy &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} u(y-x)\phi_\varepsilon(x)\,dx\right|^p dy \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(y-x)|^p\phi_\varepsilon(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(y-x)|^p\,dy\,\phi_\varepsilon(x)\,dx \\
&= \int_{\mathbb{R}^d} |u(y)|^p\,dy \int_{\mathbb{R}^d} \phi_\varepsilon(x)\,dx = \int_{\mathbb{R}^d} |u(y)|^p\,dy.
\end{aligned}$$

□

Lemma 2. *Sia $u \in L^p(\mathbb{R}^d)$ e $\phi \in C_c^\infty(\mathbb{R}^d)$. Allora $u * \phi \in C^\infty(\mathbb{R}^d)$ e*

$$\partial_j(u * \phi) = u * \partial_j\phi \quad \textit{per ogni} \quad j = 1, \dots, d.$$

Inoltre, abbiamo le disuguaglianze

$$\|u * \phi\|_{L^\infty} \leq \|u\|_{L^p}\|\phi\|_{L^q} \qquad e \qquad \|\nabla(u * \phi)\|_{L^\infty} \leq \|u\|_{L^p}\|\nabla\phi\|_{L^q},$$

dove

$$q = \frac{p}{p-1} \ .$$

Dimostrazione. Siccome
$$(u*v)(y)=\int_{\mathbb{R}^d}u(y-x)v(x)\,dx=\int_{\mathbb{R}^d}u(z)v(y-z)\,dz=(v*u)(y)\,,$$
abbiamo che
$$\begin{aligned}(u*\phi)(y+h)&-(u*\phi)(y)-h\cdot(u*\nabla\phi)(y)\\&=\int_{\mathbb{R}^d}\Big(\phi(y+h-x)-\phi(y-x)-h\cdot\nabla\phi(y-x)\Big)u(x)\,dx.\end{aligned}$$
D'altra parte, esistono $t,s\in(0,1)$ tali che
$$\begin{aligned}\phi(y+h-x)-\phi(y-x)-h\cdot\nabla\phi(y-x)&=h\cdot\nabla\phi(y+th-x)-h\cdot\nabla\phi(y-x)\\&=th\cdot\nabla^2\phi(y+sth-x)h\,.\end{aligned}$$
In particolare, se B è una palla che contiene il supporto di ϕ, allora per $|h|<<1$
$$\big|\phi(y+h-x)-\phi(y-x)-h\cdot\nabla\phi(y-x)\big|\le|h|^2\|\nabla^2\phi\|_{L^\infty}\mathbb{1}_B(y-x).$$
Di conseguenza,
$$\big|\phi(y+h-x)-\phi(y-x)-h\cdot\nabla\phi(y-x)\big|\le|h|^2\|\nabla^2\phi\|_{L^\infty}\int_{\mathbb{R}^d}\mathbb{1}_B(y-x)|u(x)|\,dx\,.$$
Questo dimostra che $u*\phi$ è differenziabile in ogni $y\in\mathbb{R}^d$ e che
$$\nabla(u*\phi)=u*(\nabla\phi).$$
Le ultime due disuguaglianze seguono dalla definizione di $u*\phi$ e la disuguaglianza di Hölder. □

Proposizione 3. *Sia $p\in[1,+\infty)$. Allora, per ogni funzione $u\in L^p(\mathbb{R}^d)$ si ha*
$$\lim_{\varepsilon\to0}\|u*\phi_\varepsilon-u\|_{L^p(\mathbb{R}^d)}=0\,.$$

Dimostrazione 1. Sia $u_n\in C_c(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ una successione di funzioni continue che converge a u in $L^p(\mathbb{R}^d)$. Allora, per ogni $\varepsilon>0$ si ha che
$$\begin{aligned}\|u*\phi_\varepsilon-u\|_{L^p(\mathbb{R}^d)}&\le\|u_n*\phi_\varepsilon-u_n\|_{L^p(\mathbb{R}^d)}\\&\qquad+\|(u-u_n)*\phi_\varepsilon-(u-u_n)\|_{L^p(\mathbb{R}^d)}\\&\le\|u_n*\phi_\varepsilon-u_n\|_{L^p(\mathbb{R}^d)}\\&\qquad+\|(u-u_n)*\phi_\varepsilon\|_{L^p(\mathbb{R}^d)}+\|u-u_n\|_{L^p(\mathbb{R}^d)}\\&\le\|u_n*\phi_\varepsilon-u_n\|_{L^p(\mathbb{R}^d)}+2\|u-u_n\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$
Ora, siccome per $\varepsilon\to0$, la famiglia $u_n*\varepsilon$ tende uniformemente ed in $L^p(\mathbb{R}^d)$ a u_n, abbiamo la tesi. □

Dimostrazione 2 (nel caso $p\in(1,+\infty)$). Per ogni funzione $\psi\in C_c^\infty(\mathbb{R}^d)$ abbiamo:
$$\begin{aligned}\int_{\mathbb{R}^d}(u*\phi_\varepsilon)(x)\psi(x)\,dx&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}u(x-y)\phi_\varepsilon(y)\,dy\right)\psi(x)\,dx\\&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}u(y)\phi_\varepsilon(x-y)\,dy\right)\psi(x)\,dx\\&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}u(y)\phi_\varepsilon(y-x)\,dy\right)\psi(x)\,dx\\&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\psi(x)\phi_\varepsilon(y-x)\,dx\right)u(y)\,dy=\int_{\mathbb{R}^d}(\psi*\phi_\varepsilon)(y)u(y)\,dy.\end{aligned}$$
Siccome $\psi*\phi_\varepsilon$ converge a ψ uniformemente ed in $L^q(\mathbb{R}^d)$, otteniamo che
$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^d}(u*\phi_\varepsilon)(x)\psi(x)\,dx=\int_{\mathbb{R}^d}u(x)\psi(x)\,dx\,,$$
e quindi u_ε converge a u debolmente in $L^p(\mathbb{R}^d)$. In particolare, per la semicontinuità della norma L^p
$$\|u\|_{L^p(\mathbb{R}^d)}\le\liminf_{\varepsilon\to0}\|u*\phi_\varepsilon\|_{L^p(\mathbb{R}^d)}.$$

D'altra parte, abbiamo mostrato che, per ogni $\varepsilon > 0$,

$$\|u * \phi_\varepsilon\|_{L^p(\mathbb{R}^d)} \le \|u\|_{L^p(\mathbb{R}^d)}.$$

Quindi

$$\|u\|_{L^p(\mathbb{R}^d)} = \lim_{\varepsilon\to 0} \|u * \phi_\varepsilon\|_{L^p(\mathbb{R}^d)},$$

ma questo implica che la convergenza è forte in L^p. □

Densità delle funzioni $C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$

Lemma 4. *Siano $p \in [1, +\infty]$, $u \in W^{1,p}(\mathbb{R}^d)$ e siano $\partial_j u \in L^p(\mathbb{R}^d)$, $j = 1, \dots, d$, le derivate deboli di u. Allora, per ogni $\psi \in C_c^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \big((\partial_j\, u) * \phi_\varepsilon\big)(x)\psi(x)\,dx = \int_{\mathbb{R}^d} \big(u * (\partial_j \phi_\varepsilon)\big)(x)\psi(x)\,dx = -\int_{\mathbb{R}^d} \big(u * \phi_\varepsilon\big)(x)\partial_j\psi(x)\,dx$$

In particolare,

$$u * \phi_\varepsilon \in W^{1,p}(\mathbb{R}^d)$$

e valgono le uguaglianze

$$(\partial_j\, u) * \psi_\varepsilon = u * \partial_j \psi_\varepsilon = \partial_j(u * \phi_\varepsilon).$$

Inoltre,

$$\|\partial_j(u * \phi_\varepsilon)\|_{L^p(\mathbb{R}^d)} \le \|\partial_j u\|_{L^p(\mathbb{R}^d)} \qquad \text{per ogni} \qquad j = 1, \dots, d.$$

Dimostrazione.

$$\begin{aligned}
\int_{\mathbb{R}^d} \big((\partial_j\, u) * \phi_\varepsilon\big)(x)\psi(x)\,dx &= \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \partial_j u(x-y)\phi_\varepsilon(y)\,dy\bigg)\psi(x)\,dx \\
&= \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \partial_j u(y)\phi_\varepsilon(x-y)\,dy\bigg)\psi(x)\,dx \\
&= \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\psi(x)\,dx\bigg)\partial_j u(y)\,dy \\
&= \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(y-x)\psi(x)\,dx\bigg)\partial_j u(y)\,dy.
\end{aligned}$$

Per la definizione di derivata debole

$$\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(y-x)\psi(x)\,dx\bigg)\partial_j u(y)\,dy = -\int_{\mathbb{R}^d} \partial_{y_j}\bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(y-x)\psi(x)\,dx\bigg)u(y)\,dy.$$

Quindi

$$\begin{aligned}
\int_{\mathbb{R}^d} \big((\partial_j\, u) * \phi_\varepsilon\big)(x)\psi(x)\,dx &= -\int_{\mathbb{R}^d} \partial_{y_j}\bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(y-x)\psi(x)\,dx\bigg)u(y)\,dy \\
&= -\int_{\mathbb{R}^d} \partial_{y_j}\bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(x)\psi(y-x)\,dx\bigg)u(y)\,dy \\
&= -\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(x)\partial_j\psi(y-x)\,dx\bigg)u(y)\,dy \\
&= -\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(y-x)\partial_j\psi(x)\,dx\bigg)u(y)\,dy \\
&= -\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\partial_j\psi(x)\,dx\bigg)u(y)\,dy \\
&= -\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \phi_\varepsilon(x-y)u(y)\,dy\bigg)\partial_j\psi(x)\,dx = -\int_{\mathbb{R}^d} (u * \phi_\varepsilon)(x)\partial_j\psi(x)\,dx\,.
\end{aligned}$$

Analogamente

$$\begin{aligned}\int_{\mathbb{R}^d}\big((\partial_j u)*\phi_\varepsilon\big)(x)\psi(x)\,dx &= -\int_{\mathbb{R}^d}\partial_{y_j}\bigg(\int_{\mathbb{R}^d}\phi_\varepsilon(y-x)\psi(x)\,dx\bigg)u(y)\,dy\\ &= -\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}\partial_j\phi_\varepsilon(y-x)\psi(x)\,dx\bigg)u(y)\,dy\\ &= \int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}\partial_j\phi_\varepsilon(x-y)\psi(x)\,dx\bigg)u(y)\,dy\\ &= \int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}\partial_j\phi_\varepsilon(x-y)u(y)\,dy\bigg)\psi(x)\,dx = \int_{\mathbb{R}^d}\big((\partial_j\phi_\varepsilon)*u\big)(x)\psi(x)\,dx.\end{aligned}$$ □

Teorema 5. *Per $p\in[1,+\infty)$, le funzioni $C^\infty(\mathbb{R}^d)\cap W^{1,p}(\mathbb{R}^d)$ sono dense in $W^{1,p}(\mathbb{R}^d)$.*

Densità delle funzioni $C_c^\infty(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$

Lemma 6. *Siano $p\in[1,+\infty]$, $u\in W^{1,p}(\mathbb{R}^d)$ e $\psi\in C_c^\infty(\mathbb{R}^d)$. Allora*

$$u\psi\in W^{1,p}(\mathbb{R}^d)\qquad e\qquad \nabla(u\psi)=\psi\nabla u+u\nabla\psi.$$

Dimostrazione. Abbiamo che

$$u\psi\in L^p(\mathbb{R}^d)\qquad\text{e}\qquad \psi\nabla u+u\nabla\psi\in L^p(\mathbb{R}^d).$$

Quindi, basta dimostrare che $\psi\nabla u+u\nabla\psi$ sia il gradiente debole di $u\psi$. Preso un campo $\Phi\in C_c^1(\mathbb{R}^d)$, calcoliamo

$$\begin{aligned}\int_{\mathbb{R}^d}u\psi\,\mathrm{div}\,\Phi\,dx &= \int_{\mathbb{R}^d}u\,\mathrm{div}\,(\psi\Phi)\,dx-\int_{\mathbb{R}^d}u\nabla\psi\cdot\Phi\,dx\\ &= -\int_{\mathbb{R}^d}\nabla u\cdot(\psi\Phi)\,dx-\int_{\mathbb{R}^d}u\nabla\psi\cdot\Phi\,dx = -\int_{\mathbb{R}^d}(\psi\nabla u+u\nabla\psi)\cdot\Phi\,dx\end{aligned}$$ □

Lemma 7. *Sia $\psi\in C_c^\infty(\mathbb{R}^d)$ una funzione tale che*
- *il supporto di ψ è contenuto in B_1;*
- *$0\le\psi\le 1$;*
- *$\psi\equiv 1$ in $B_{1/2}$.*

Consideriamo le riscalate

$$\psi_R(x):=\psi(x/R).$$

Allora, per ogni $u\in W^{1,p}(\mathbb{R}^d)$ con $p\in[1,+\infty)$, abbiamo che

$$\lim_{R\to\infty}\|u\psi_R-u\|_{W^{1,p}(\mathbb{R}^d)}=0.$$

Dimostrazione. Osserviamo che per definizione

$$\psi_R\uparrow 1\quad\text{su}\quad\mathbb{R}^d\qquad\text{e}\qquad\|\nabla\psi_R\|_{L^\infty(\mathbb{R}^d)}\to 0\,.$$

In particolare, per il teorema di Lebesgue, abbiamo che

$$\lim_{R\to+\infty}\|u\psi_R-u\|_{L^p(\mathbb{R}^d)}=0\,.$$

Per quanto riguarda la convergenza dei gradienti, osserviamo che

$$\begin{aligned}\|\nabla(\psi_R u)-\nabla u\|_{L^p(\mathbb{R}^d)} &= \|\psi_R\nabla u+u\nabla\psi_R-\nabla u\|_{L^p(\mathbb{R}^d)}\\ &\le \|\psi_R\nabla u-\nabla u\|_{L^p(\mathbb{R}^d)}+\|u\nabla\psi_R\|_{L^p(\mathbb{R}^d)}\,,\end{aligned}$$

e quindi $\nabla(\psi_R u)$ converge a ∇u in $L^p(\mathbb{R}^d)$. □

Teorema 8. *Le funzioni $C_c^\infty(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$ sono dense in $H^1(\mathbb{R}^d)$.*

Dimostrazione. Segue da Esercizio 7 e Teorema 5. □

TEOREMI DI APPROSSIMAZIONE IN $W^{1,p}(\Omega)$

Sia $\Omega \subset \mathbb{R}^d$ un aperto. Per ogni $\delta > 0$ definiamo

$$\Omega_\delta := \Big\{ x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \delta \Big\}.$$

Lemma 9. *Siano $p \in [1, +\infty)$, $u \in W^{1,p}(\Omega)$, $j \in \{1, \dots, d\}$. Dati*

$$\delta > 0 \qquad e \qquad \varepsilon < \frac{\delta}{2}\,,$$

abbiamo che per ogni $\psi \in C_c^\infty(\Omega_\delta)$,

$$\int_{\mathbb{R}^d} \big((\partial_j\, u) * \phi_\varepsilon\big)(x)\psi(x)\,dx = \int_{\mathbb{R}^d} \big(u * (\partial_j \phi_\varepsilon)\big)(x)\psi(x)\,dx = -\int_{\mathbb{R}^d} \big(u * \phi_\varepsilon\big)(x)\partial_j\psi(x)\,dx$$

In particolare,

$$u * \phi_\varepsilon \in W^{1,p}(\Omega_\delta) \qquad e \qquad \partial_j(u * \phi_\varepsilon) = (\partial_j\, u) * \psi_\varepsilon = u * \partial_j \psi_\varepsilon\,.$$

Inoltre, valgono le disuguaglianze

$$\|u * \phi_\varepsilon\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} \qquad e \qquad \|\partial_j(u * \phi_\varepsilon)\|_{L^p(\Omega_\delta)} \le \|\partial_j u\|_{L^p(\Omega)}.$$

Teorema 10. *Siano Ω un aperto in $\mathbb{R}^d$ e $p \in [1, +\infty)$. Allora, per ogni funzione $u \in W^{1,p}(\Omega)$ esiste una successione u_n di funzioni $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ tale che*

$$\lim_{n\to\infty} \|u_n - u\|_{L^p(\Omega)} = 0 \qquad e \qquad \lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\Omega_\delta)} = 0 \qquad \textit{per ogni} \qquad \delta > 0.$$

Di conseguenza, abbiamo anche

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(D)} = 0 \qquad \textit{per ogni} \qquad D \Subset \Omega.$$