# Exterior ball condition
# and up-to-the-boundary continuity for harmonic functions

## Exterior ball condition

**Definizione 1** (Exterior ball condition)**.** *Let $\Omega$ be an open set in $\mathbb{R}^d$. We say that $\Omega$ satisfies the exterior ball condition if for every boundary point $y \in \partial\Omega$ there is a ball $B_r(x) \subset \mathbb{R}^d$ such that*

$$\overline{B}_r(x) \cap \overline{\Omega} = \{y\}.$$

**Esempio 2.** *Every convex set $\Omega \subset \mathbb{R}^d$ satisfies the exterior ball condition.*

**Esempio 3.** *If $\Omega \subset \mathbb{R}^d$ is an open set of class $C^2$, then $\Omega$ satisfies the exterior ball condition.*

**Esempio 4.** *The domains*

- $\Omega = B_1 \setminus \{(0,0)\}$,
- $\Omega = B_1 \setminus \{(x,0) \in \mathbb{R}^2 \ : \ x \in [0,+\infty)\}$,
- $\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ xy > 0\}$,
- $\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ y \geq |x|\}$,

*do not satisfy the exterior ball condition. In fact, for all of them the origin $(0,0)$ is a boundary point at which one cannot place a ball lying in the complement of $\Omega$.*

---

## Up-to-the-boundary continuity of harmonic functions

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ and let

$$g : \overline{\Omega} \to \mathbb{R}$$

be a continuous function in $g \in H^1(\Omega) \cap C(\overline{\Omega})$. We know that there is a unique weak solution $h \in H^1(\Omega)$ of the problem

$$\Delta h = 0 \quad \text{in} \quad \Omega\ , \qquad h = g \quad \text{on} \quad \partial\Omega,$$

in the sense that

$$h - g \in H^1_0(\Omega),$$

and

$$\int_\Omega \nabla h \cdot \nabla \varphi \, dx = 0 \qquad \text{for every} \qquad \varphi \in C^\infty_c(\mathbb{R}^d).$$

We already know that $h$ is $C^\infty$ in $\Omega$ and that

$$\Delta h(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

In this section we will show that if $g$ is continuous and if $\Omega$ satisfies the external ball condition, then $h$ is also continuous up to the boundary $\partial\Omega$.

**Teorema 5** (Continuity up to the boundary)**.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ that satisfies the exterior ball condition from Definition 1. Let $g \in H^1(\Omega) \cap C(\overline{\Omega})$ be a given function and let $h \in H^1(\Omega)$ be the weak solution to the problem*

$$\Delta h = 0 \quad \textit{in} \quad \Omega\ , \qquad h = g \quad \textit{su} \quad \partial\Omega.$$

*Then, the function*

$$\widetilde{h} : \overline{\Omega} \to \mathbb{R}\ , \qquad \widetilde{h}(x) := \begin{cases} h(x) & \textit{if} \quad x \in \Omega, \\ g(x) & \textit{if} \quad x \in \partial\Omega, \end{cases}$$

*is continuous on $\overline{\Omega}$.*

*Dimostrazione.* Suppose that $x_n$ is a sequence of points in $\overline{\Omega}$ converging to some $x_0 \in \overline{\Omega}$. We will show that

$$\lim_{n\to+\infty} \widetilde{h}(x_n) = \widetilde{h}(x_0).$$

Since

$$h : \Omega \to \mathbb{R} \qquad \text{and} \qquad g : \partial\Omega \to \mathbb{R}$$

are both continuous, we only need to consider the case

$$x_0 \in \partial\Omega \qquad \text{and} \qquad x_n \in \Omega \quad \text{for every} \quad n \geq 1.$$

Suppose by contradiction that $h(x_n)$ do not converge to $g(x_0)$. Up to extracting a subsequence (and up to replacing $h$ and $g$ with $-h$ and $-g$), there is $\varepsilon > 0$ such that

$$\lim_{n\to+\infty} h(x_n) \geq g(x_0) + \varepsilon.$$

By hypothesis, we know that there is a ball $B_R(y_0)$ such that

$$\overline{B}_R(y_0) \cap \overline{\Omega} = \{x_0\}.$$

We aim to construct a function

$$\varphi : \mathbb{R}^d \to \mathbb{R}$$

with the following properties:

- $\varphi$ is continuous on $\mathbb{R}^d$ and Lipschitz continuous on $\overline{\Omega}$;
- $\varphi(x_0) = \frac{\varepsilon}{2} + g(x_0)$;
- $\varphi$ is harmonic in $\mathbb{R}^d \setminus \overline{B}_R(z_0)$;
- $\varphi(x) \geq g(x)$ on $\partial\Omega$.

Once we have such a function $\varphi$, by the maximum principle we have that $h \leq \varphi$ in $\Omega$ so that

$$g(x_0) + \varepsilon \leq \lim_{n\to+\infty} h(x_n) \leq \lim_{n\to+\infty} \varphi(x_n) = \varphi(x_0) = g(x_0) + \frac{\varepsilon}{2},$$

which leads to a contradiction. In the rest of the proof we will show that such a function $\varphi$ exists.

**Construction of $\varphi$.** We start by noticing that, since $g$ is continuous at $x_0$, there is a radius $\delta > 0$ such that

$$|g(y) - g(x_0)| \leq \frac{\varepsilon}{2} \qquad \text{for every} \qquad y \in B_\delta(x_0) \cap \partial\Omega.$$

We next choose a radius

$$r := \min\left\{\frac{\delta}{4}, R\right\},$$

and we take the ball

$$B_r(z_0) \quad \text{with center} \quad x_0 + \frac{r}{R}(y_0 - x_0),$$

which is contained in $B_R(y_0)$ and tangent to $\partial B_R(y_0)$ at $x_0$.

We consider two cases.

**Case 1.** The dimension of the space is $d = 2$. Then, we consider the function

$$\psi(x) := \begin{cases} \ln\left(\dfrac{|x - z_0|}{r}\right) & \text{if} \quad |x - z_0| \geq r, \\ 0 & \text{if} \quad |x - z_0| \leq r. \end{cases}$$

In this case we have that

$$\psi(x) \geq \ln 2 \quad \text{for all} \quad x \in \mathbb{R}^2 \setminus B_{2r}(z_0).$$

**Case 2.** The dimension of the space is $d > 2$. in this case, we define the function $\psi$ as

$$\psi(x) := \begin{cases} \frac{1}{r^{d-2}} - \frac{1}{|x-z_0|^{d-2}} & \text{if} \quad |x - z_0| \geq r, \\ 0 & \text{if} \quad |x - z_0| \leq r. \end{cases}$$

In this case we have that

$$\psi(x) \geq \frac{1}{2}\frac{1}{r^{d-2}} \quad \text{for all} \quad x \in \mathbb{R}^d \setminus B_{2r}(z_0).$$

In both cases we have that there is a constant $\kappa > 0$ (depending on $r$ and $d$) such that

$$\psi(x) \geq \kappa \quad \text{for all} \quad x \in \mathbb{R}^d \setminus B_{2r}(z_0).$$

Moreover, in both cases the function $\psi$ is continuous on $\mathbb{R}^d$ and harmonic in $\mathbb{R}^d \setminus \overline{B}_r(z_0)$.

We now define the function $\varphi$ as

$$\varphi(x) = g(x_0) + \frac{\varepsilon}{2} + \frac{2\|g\|_{L^\infty(\partial\Omega)}}{\kappa}\psi(x).$$

By the choice of the radius $\delta$ we have that

$$\varphi(x) \geq g(x_0) + \frac{\varepsilon}{2} \geq g(x) \quad \text{for all} \quad x \in B_\delta(x_0).$$

On the other hand, by the choice of $r \leq \delta/4$ we have that

$$B_{2r}(z_0) \subset B_\delta(x_0),$$

so for all $x \in \mathbb{R}^d \setminus B_\delta(x_0)$ we have:

$$\begin{aligned}\varphi(x) &\geq g(x_0) + \frac{2\|g\|_{L^\infty(\partial\Omega)}}{\kappa}\psi(x)\\ &\geq g(x_0) + 2\|g\|_{L^\infty(\partial\Omega)}\\ &\geq \|g\|_{L^\infty(\partial\Omega)}\\ &\geq g(x).\end{aligned}$$

This implies that

$$\varphi(x) \geq g(x) \quad \text{on} \quad \partial\Omega,$$

and concludes the proof. □

As an immediate consequence we obtain the following

**Corollario 6** (Continuity up to the boundary)**.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ that satisfies the exterior ball condition from Definition 1. Let $g : \partial\Omega \to \mathbb{R}$ be a given Lipschitz continuous function and let $h \in H^1(\Omega)$ be the weak solution to the problem*

$$\Delta h = 0 \quad in \quad \Omega\ , \qquad h = g \quad su \quad \partial\Omega.$$

*Then, the function*

$$\widetilde{h} : \overline{\Omega} \to \mathbb{R}\ , \qquad \widetilde{h}(x) := \begin{cases} h(x) & if \quad x \in \Omega,\\ g(x) & if \quad x \in \partial\Omega,\end{cases}$$

*is continuous on $\overline{\Omega}$.*

*Dimostrazione.* It is sufficient to notice that any Lipschitz continuous function $g : \partial\Omega \to \mathbb{R}$ can be extended to a Lipschitz continuous $\widetilde{g} : \mathbb{R}^d \to \mathbb{R}$. Since the Lipschitz extension $\widetilde{g}$ lies is in both $H^1(\Omega)$ and $C(\overline{\Omega})$, we can apply Theorem 5. □