Poisson's formula

In this section we will show that for every continuous function

$$\phi : \partial B_R \to \mathbb{R}$$

on the $(d-1)$-dimensional sphere of radius R in $\mathbb{R}^d$, there is a continuous function

$$u : \overline{B}_R \to \mathbb{R}\,,$$

which is C^∞ in the interior of the ball B_R and solves the problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad B_R\,; \\ u = \phi & \text{on} \quad \partial B_R\,. \end{cases}$$

Before proving the main theorem (Theorem 4) we will need several preliminary lemmas.

Lemma 1 (Two identities for the Green function)**.** *Let $Y \in \mathbb{R}^d$ be fixed and let*

$$G(X) := \frac{1}{|X-Y|^{d-2}}\,.$$

Then:

(1) G is harmonic in $\mathbb{R}^d \setminus \{Y\}$;
(2) for every $r > |Y|$ we have

$$\int_{\partial B_r} X \cdot \nabla G(X)\, d\mathcal{H}^{d-1}(X) = -(d-2)d\omega_d r\ ;$$

(3) for every $r > |Y|$ we have

$$\int_{\partial B_r} G(X)\, d\mathcal{H}^{d-1}(X) = d\omega_d r\ ;$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. We compute

$$\nabla G(X) = -(d-2)\frac{X-Y}{|X-Y|^d}\,,$$

and

$$\begin{aligned}\Delta G(X) &= -(d-2)\mathrm{div}_X\left(\frac{X-Y}{|X-Y|^d}\right)\\ &= -(d-2)\left(\frac{\mathrm{div}_X(X-Y)}{|X-Y|^d} + (X-Y)\cdot\nabla_X\left(\frac{1}{|X-Y|^d}\right)\right)\\ &= -(d-2)\left(\frac{d}{|X-Y|^d} + (X-Y)\cdot\left(\frac{-d(X-Y)}{|X-Y|^{d+2}}\right)\right) = 0.\end{aligned}$$

Next, we set

$$\phi(r) := \frac{1}{r^{d-1}}\int_{\partial B_r} G(X)\, d\mathcal{H}^{d-1}(X) = \int_{\partial B_1} G(rX)\, d\mathcal{H}^{d-1}(X)$$

and we compute

$$\begin{aligned}\phi'(r) &= \frac{d}{dr}\int_{\partial B_1} G(rX)\, d\mathcal{H}^{d-1}(X)\\ &= \int_{\partial B_1} X\cdot\nabla G(rX)\, d\mathcal{H}^{d-1}(X)\\ &= \frac{1}{r^{d-1}}\int_{\partial B_r} \frac{X}{r}\cdot\nabla G(X)\, d\mathcal{H}^{d-1}(X).\end{aligned}$$

Now, by the divergence theorem, we have that for every $\varepsilon > 0$

$$\begin{aligned}\int_{\partial B_r} \frac{X}{r} \cdot \nabla G(X)\, d\mathcal{H}^{d-1}(X) &= \int_{\partial B_\varepsilon(Y)} \frac{X-Y}{\varepsilon} \cdot \nabla G(X)\, d\mathcal{H}^{d-1}(X)\\ &= -(d-2)\int_{\partial B_\varepsilon(Y)} \frac{X-Y}{\varepsilon} \cdot \frac{X-Y}{|X-Y|^d}\, d\mathcal{H}^{d-1}(X) = -(d-2)d\omega_d.\end{aligned}$$

Thus, we have that

$$\phi'(r) = -\frac{(d-2)d\omega_d}{r^{d-1}}.$$

Integrating between r and R, we get

$$\phi(R) - \phi(r) = \int_r^R \phi'(s)\, ds = \int_r^R \frac{-(d-2)d\omega_d}{s^{d-1}}\, ds = \frac{d\omega_d}{R^{d-2}} - \frac{d\omega_d}{r^{d-2}}.$$

Since

$$\lim_{R\to+\infty} \phi(R) = 0,$$

we get that

$$\phi(r) = \frac{d\omega_d}{r^{d-2}},$$

which concludes the proof. □

Lemma 2 (Harmonicity of the Poisson's kernel)**.** *For any fixed vector $Y \in \mathbb{R}^d$, the function*

$$F(X) := \frac{|Y|^2 - |X|^2}{|X-Y|^d}\ ,$$

is harmonic in $\mathbb{R}^d \setminus \{Y\}$.

Proof. We compute

$$\begin{aligned}F(X) &= \frac{|Y|^2 - |X|^2}{|Y-X|^d}\\ &= -\frac{|X|^2 - |Y|^2}{|Y-X|^d}\\ &= -\frac{(X-Y)\cdot(X+Y)}{|Y-X|^d}\\ &= -\frac{1}{|X-Y|^{d-2}} + \frac{(X-Y)\cdot 2Y}{|Y-X|^d}\\ &= -\frac{1}{|X-Y|^{d-2}} + \frac{2}{d-2} Y \cdot \nabla_X \left(\frac{1}{|Y-X|^{d-2}}\right),\end{aligned}$$

Since the function

$$X \mapsto \frac{1}{|X-Y|^{d-2}}$$

and its partial derivatives are harmonic in $\mathbb{R}^d \setminus \{Y\}$, we get the claim. □

Lemma 3 (The integral of the Poisson's kernel over the sphere)**.** *Let B_R be the ball of radius R in $\mathbb{R}^d$ and let $\omega_d := |B_1|$. Then, for every $Y \in B_R$ it holds*

$$\frac{R^2 - |Y|^2}{d\omega_d R} \int_{\partial B_R} \frac{d\mathcal{H}^{d-1}(X)}{|Y-X|^d} = 1\,.$$

Proof. **Proof in dimension $d = 2$.** We suppose that $Y = (Rx, 0)$ for some $x < 1$. Then

$$\begin{aligned}\frac{R^2 - |Y|^2}{Rd\omega_d} \int_{\partial B_R} \frac{d\mathcal{H}^{d-1}(X)}{|X-Y|^d} &= \frac{1-x^2}{2\pi} \int_0^{2\pi} \frac{d\theta}{(x-\cos\theta)^2 + \sin^2\theta}\\ &= \frac{1-x^2}{\pi} \int_0^{\pi} \frac{d\theta}{1 + x^2 - 2x\cos\theta}.\end{aligned}$$

Applying the change of variables

$$t = \tan\frac{\theta}{2}$$

we have

$$\begin{aligned}\frac{1-x^2}{\pi}\int_0^{\pi}\frac{d\theta}{1+x^2-2x\cos\theta} &= \frac{1-x^2}{\pi}\int_0^{+\infty}\frac{1}{1+x^2-2x\frac{1-t^2}{1+t^2}}\frac{2\,dt}{1+t^2}\\ &= \frac{1-x^2}{\pi}\int_0^{+\infty}\frac{2\,dt}{(1+x)^2t^2+(1-x)^2}\\ &= \frac{1-x^2}{\pi}\frac{2}{(1-x)^2}\int_0^{+\infty}\frac{dt}{\left(\frac{1+x}{1-x}t\right)^2+1}\\ &= \frac{1-x^2}{\pi}\frac{2}{(1-x)^2}\frac{1-x}{1+x}\left[\arctan\left(\frac{1+x}{1-x}t\right)\right]_{t=0}^{+\infty} = 1.\end{aligned}$$

Proof in dimension $d \geq 3$. As above we compute

$$\begin{aligned}\frac{|X|^2-|Y|^2}{|X-Y|^d} &= -\frac{|Y|^2-|X|^2}{|Y-X|^d}\\ &= -\frac{(Y-X)\cdot(Y+X)}{|Y-X|^d}\\ &= -\frac{1}{|Y-X|^{d-2}}+\frac{(X-Y)\cdot 2X}{|Y-X|^d}\\ &= -\frac{1}{|X-Y|^{d-2}}-\frac{2}{d-2}X\cdot\nabla_X\left(\frac{1}{|X-Y|^{d-2}}\right)\\ &= -G(X)-\frac{2}{d-2}X\cdot\nabla G(X)\,,\end{aligned}$$

where

$$G(X) := \frac{1}{|X-Y|^{d-2}}.$$

By the lemma above, we get that

$$\begin{aligned}\frac{R^2-|Y|^2}{d\omega_d}\int_{\partial B_R}\frac{d\mathcal{H}^{d-1}(X)}{|Y-X|^d} &= \frac{1}{d\omega_d}\int_{\partial B_R}\frac{|X|^2-|Y|^2}{|Y-X|^d}d\mathcal{H}^{d-1}(X)\\ &= -\frac{1}{d\omega_d}\int_{\partial B_R}G(X)d\mathcal{H}^{d-1}(X)\\ &\quad -\frac{1}{d\omega_d}\int_{\partial B_R}\frac{2}{d-2}X\cdot\nabla G(X)d\mathcal{H}^{d-1}(X)\\ &= -\frac{1}{d\omega_d}d\omega_d R+\frac{1}{d\omega_d}\frac{2}{d-2}(d-2)d\omega_d R = R,\end{aligned}$$

which concludes the proof. □

Teorema 4 (Poisson's formula). *Let $B_R \subset \mathbb{R}^d$ and let $\phi \in C(\partial B_R)$. We define the function $u : B_R \to \mathbb{R}$ as*

$$u(X) = \frac{R^2-|X|^2}{d\omega_d R}\int_{\partial B_R}\frac{\phi(Y)}{|X-Y|^d}\,d\mathcal{H}^{d-1}(Y), \quad \textit{for every} \quad X \in B_R.$$

Then, the following holds:

(i) $u \in C^2(B_R)$ and $\Delta u = 0$ in B_R.
(ii) for every $X_0 \in \partial B_R$ we have that $\lim_{X\to X_0} u(X) = \phi(X_0)$.

Proof. The proof of (i) follows by Lemma 2. In order to prove (ii), we fix $\varepsilon > 0$ and we aim to show that

$$\left|\lim_{X\to X_0} u(X) - \phi(X_0)\right| \leq \varepsilon.$$

Let $\delta > 0$ be chosen in such a way that for all $Y \in \partial B_R$ satisfying $|Y - X_0| < \delta$ we have $|\phi(Y) - \phi(X_0)| < \varepsilon$. By the definition of u and by the fact that (by Lemma 3)

$$\frac{R^2-|X|^2}{R d\omega_d}\int_{\partial B_R}\frac{1}{|X-Y|^d}\,d\mathcal{H}^{d-1}(Y) = 1,$$

we have that for every $X \in B_R$ it holds

$$u(X) - \phi(X_0) = \frac{R^2 - |X|^2}{Rd\omega_d} \int_{\partial B_R} \frac{\phi(Y)}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) - \frac{R^2 - |X|^2}{Rd\omega_d} \int_{\partial B_R} \frac{\phi(X_0)}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y)$$
$$= \frac{R^2 - |X|^2}{Rd\omega_d} \int_{\partial B_R} \frac{\big(\phi(Y) - \phi(X_0)\big)}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y).$$

Then, for every $X \in B_R \cap B_{\delta/2}(X_0)$, we have

$$\begin{aligned}
|u(X) - \phi(X_0)| &\le \frac{R^2 - |X|^2}{Rd\omega_d} \int_{\partial B_R} \frac{|\phi(Y) - \phi(X_0)|}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) \\
&\le \frac{R^2 - |X|^2}{Rd\omega_d} \int_{(\partial B_R)\setminus B_\delta(X_0)} \frac{|\phi(Y) - \phi(X_0)|}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) \\
&\qquad + \frac{R^2 - |X|^2}{Rd\omega_d} \int_{B_\delta(X_0)\cap \partial B_R} \frac{|\phi(Y) - \phi(X_0)|}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) \\
&\le \frac{R^2 - |X|^2}{Rd\omega_d} \int_{(\partial B_R)\setminus B_\delta(X_0)} \frac{|\phi(Y) - \phi(X_0)|}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) \\
&\qquad + \frac{R^2 - |X|^2}{Rd\omega_d} \int_{\partial B_R} \frac{\varepsilon}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y) \\
&\le \frac{R^2 - |X|^2}{Rd\omega_d} \int_{(\partial B_R)\setminus B_\delta(X_0)} \frac{2\|\phi\|_{L^\infty(\partial B_1)}}{(\delta/2)^d} \, d\mathcal{H}^{d-1}(Y) + \varepsilon \\
&\le \big(R^2 - |X|^2\big)\, R^{d-2} \frac{2\|\phi\|_{L^\infty(\partial B_R)}}{(\delta/2)^d} + \varepsilon.
\end{aligned}$$

Passing to the limit as $X \to X_0$, we obtain

$$\limsup_{X\to X_0} |u(X) - \phi(X_0)| \le \varepsilon + \limsup_{X\to X_0} \big(R^2 - |X|^2\big)\, R^{d-2} \frac{2\|\phi\|_{L^\infty(\partial B_1)}}{(\delta/2)^d} = \varepsilon.$$

Since ε is arbitrary, we get that

$$\limsup_{X\to X_0} |u(X) - \phi(X_0)| = 0,$$

which concludes the proof. □

More generally, we have the following Poisson formula for more general traces.

Teorema 5. *Let $B_R \subset \mathbb{R}^d$ and let $\phi \in L^1(\partial B_R)$. We define the function $u : B_R \to \mathbb{R}$ as*

$$u(X) = \frac{R^2 - |X|^2}{d\omega_d R} \int_{\partial B_R} \frac{\phi(Y)}{|X-Y|^d} \, d\mathcal{H}^{d-1}(Y), \quad \text{for every} \quad X \in B_R.$$

Then, the following holds:

(i) $u \in C^2(B_R)$ and $\Delta u = 0$ in B_R ;

(ii) if $X_0 \in \partial B_R$ is a Lebesgue point for ϕ (on ∂B_R) such that

$$\limsup_{\substack{X \to X_0 \\ X \in \partial B_R}} |\phi(X) - \phi(X_0)| = 0,$$

then

$$\limsup_{\substack{X \to X_0 \\ X \in B_R}} |u(X) - \phi(X_0)| = 0.$$