A harmonic function with Lipschitz boundary datum

For every $\ell > 0$ we define the rectangle

$$\mathcal{R}_\ell := \big(-\ell, \ell\big) \times \big(0, \ell\big).$$

Consider the functions

$$\phi : \mathcal{R}_1 \to \mathbb{R}\ , \quad \phi(x, y) = |x|.$$

and $h : \mathcal{R}_1 \to \mathbb{R}$, solution to

$$\Delta h = 0 \quad \text{in} \quad \mathcal{R}_1\ , \qquad h = \phi \quad \text{on} \quad \partial\mathcal{R}_1.$$

Proposizione 1. *The function $h : \mathcal{R}_1 \to \mathbb{R}$ is not Lipschitz continuous in $(0, 0)$.*

Proof. We claim that:

$$h(x, y) - \phi(x, y) \geq \varepsilon y \qquad \text{in} \qquad \mathcal{R}_{1/2}\,. \tag{1}$$

We next define the functions

$$h_n : \mathcal{R}_1 \to \mathbb{R}\ , \qquad h_n(x, y) = 2^n h\left(\frac{x}{2^n}, \frac{y}{2^n}\right),$$

and we notice that for every n,

$$\Delta h_n = 0 \quad \text{in} \quad \mathcal{R}_1.$$

By the definition of h_1 and the estimate (1), we get that

$$h_1(x, y) = 2h\left(\frac{x}{2}, \frac{y}{2}\right) \geq 2\Big(\phi\left(\frac{x}{2}, \frac{y}{2}\right) + \varepsilon\frac{y}{2}\Big) = \phi(x, y) + \varepsilon y,$$

for every $(x, y) \in \mathcal{R}_1$. Thus, by the maximum principle,

$$h_1(x, y) \geq h(x, y) + \varepsilon y \geq \phi(x, y) + 2\varepsilon y \quad \text{in} \quad \mathcal{R}_{1/2}.$$

Then,

$$h_2(x, y) = 2h_1\left(\frac{x}{2}, \frac{y}{2}\right) \geq 2\Big(\phi\left(\frac{x}{2}, \frac{y}{2}\right) + 2\varepsilon\frac{y}{2}\Big) \geq \phi(x, y) + 2\varepsilon y \quad \text{in} \quad \mathcal{R}_1.$$

Arguing by induction, we have that

$$h_n(x, y) \geq \phi(x, y) + n\varepsilon y \quad \text{in} \quad \partial\mathcal{R}_1.$$

In particular,

$$h_n(0, 1) \geq n\varepsilon.$$

But then,

$$h\left(0, \frac{1}{2^n}\right) \geq \frac{n\varepsilon}{2^n},$$

which proves that h is not Lipschitz continuous in $(0, 0)$. □