The residual index and the dynamics of holomorphic maps tangent to the identity

Marco Abate[1]

Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail: abate@mat.uniroma2.it

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Abstract. Let f be a (germ of) holomorphic self-map of $\mathbb{C}^2$ such that the origin is an isolated fixed point, and such that $df_O = \mathrm{id}$. Let $\nu(f)$ be the degree of the first non-vanishing term in the homogeneous expansion of $f - \mathrm{id}$. We generalize to $\mathbb{C}^2$ the classical Leau-Fatou Flower Theorem proving that there exist $\nu(f)-1$ holomorphic curves f-invariant, with the origin in their boundary, and attracted by O under the action of f.

0. Introduction

One of the most famous theorems in one-dimensional holomorphic dynamics is

Theorem 0.1: (Leau-Fatou Flower Theorem [L, F]) *Let $g(\zeta) = \zeta + a_k\zeta^k + O(\zeta^{k+1})$, with $k \ge 2$ and $a_k \ne 0$, be a holomorphic function fixing the origin. Then there are $k-1$ disjoint domains $D_1, \ldots, D_{k-1}$ with the origin in their boundary, invariant under g (that is, $g(D_j) \subset D_j$) and such that $(g|_{D_j})^n \to 0$ as $n \to \infty$, for $j = 1, \ldots, k-1$, where g^n denotes the composition of g with itself n times.*

Any such domain is called a *parabolic domain* for f at the origin, and they are (together with attracting basins, Siegel disks and Hermann rings) among the building blocks of Fatou sets of rational functions (see, e.g., [CG] for a modern exposition).

A natural problem in higher dimensional holomorphic dynamics is to find a generalization of this result, where the function g is replaced by a germ f of self-map of $\mathbb{C}^n$ fixing the origin and *tangent to the identity*, that is such that $df_O = \mathrm{id}$. After preliminary results in $\mathbb{C}^2$ obtained by Ueda [U] and Weickert [W], a very important step in this direction has been made by Hakim [H1, 2] (inspired by previous works by Ecalle [E]).

To describe her results, we need a couple of definitions. Let f be a germ of holomorphic self-map of $\mathbb{C}^n$ fixing the origin and tangent to the identity. A *parabolic curve* for f at the origin is a injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^n$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous at the origin, and $\varphi(0) = O$;
(iii) $\varphi(\Delta)$ is invariant under f, and $(f|_{\varphi(\Delta)})^n \to O$ as $n \to \infty$.

Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbb{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^n \setminus \{O\}$ onto $\mathbb{P}^{n-1}$) we say that φ is *tangent to* $[v]$ at the origin.

Writing $f = (f_1, \ldots, f_n)$, let $f_j = z_j + P_{j,\nu_j} + P_{j,\nu_j+1} + \cdots$ be the homogeneous expansion of f in series of homogeneous polynomial, where $\deg P_{j,k} = k$ (or $P_{j,k} \equiv 0$), and $P_{j,\nu_j} \not\equiv 0$. The *order* $\nu(f)$ is defined by $\nu(f) = \min\{\nu_1, \ldots, \nu_n\}$. A *characteristic direction* for f is a vector $[v] = [v_1 : \cdots : v_n] \in \mathbb{P}^{n-1}$ such that there is $\lambda \in \mathbb{C}$ so that $P_{j,\nu(f)}(v_1, \ldots, v_n) = \lambda v_j$ for $j = 1, \ldots, n$. If $\lambda \ne 0$ we shall say that $[v]$ is *non-degenerate;* otherwise it is *degenerate*.

Then Hakim's result is:

Theorem 0.2: (Hakim [H1, 2]) *Let f be a (germ of) holomorphic self-map of $\mathbb{C}^n$ fixing the origin and tangent to the identity. Then for every non-degenerate characteristic direction $[v]$ of f there are $\nu(f)-1$ parabolic curves tangent to $[v]$ at the origin.*

This is a very good generalization of Theorem 0.1, but applies only to generic maps: if f has *no* non-degenerate characteristic directions, this theorem gives no informations about the dynamics of f.

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A similar situation occurred for continuous holomorphic dynamics. It was known since the end of the last century, thanks, e.g., to Poincaré [P], that a generic holomorphic vector field with an isolated singularity at the origin in $\mathbb{C}^n$ admits invariant submanifolds passing through the singularity; but it remained unknown for more than one hundred years, even replacing "submanifold" by "complex analytic subvariety", whether this was true for *any* holomorphic vector field with an isolated singularity. At last, in 1982 Camacho and Sad proved the following

Theorem 0.3: (Camacho and Sad [CS]) *Let F be a (germ of) holomorphic vector field with an isolated singularity at $O \in \mathbb{C}^2$. Then there exists a complex analytic subvariety invariant by F passing through the origin.*

See [T] for a different proof of part of this result. It should also be mentioned that Theorem 0.3 is not true in $\mathbb{C}^3$: Gómez-Mont and Luengo [GL] found a family of holomorphic vector fields with an isolated singularity at the origin in $\mathbb{C}^3$ and no invariant complex analytic subvariety passing through the singularity.

Our main result is an exact discrete analogue of Theorem 0.3 (and thus a complete generalization of the Leau-Fatou Flower Theorem), namely:

Theorem 0.4: *Let f be a (germ of) holomorphic self-map of $\mathbb{C}^2$ tangent to the identity and such that the origin is an isolated fixed point. Then there exist (at least) $\nu(f)-1$ parabolic curves for f at the origin.*

We shall also be able to prove the existence of parabolic curves for germs with $df_O = J_2$, the canonical Jordan matrix associated to the eigenvalue 1; see Corollary 3.4.

The proof of Theorem 0.3 was based on three main ingredients: Poincaré's (and others') results on generic vector fields; a canonical reduction (developed by Briot and Bouquet [BB], Dumortier [D], Seidenberg [S] and Ven den Essen [V]) via blow-ups of the singularity to simpler, irreducible cases (see [MM] for a good account); and an index, introduced by Camacho and Sad, associated to a singularity of the vector field on an invariant 1-dimensional submanifold.

In our situation, Theorem 0.2 (or better, a simplified version we shall discuss in Section 3) is the natural replacement of Poincaré's results; the bulk of this paper is devoted to the construction of the remaining two ingredients in the discrete case.

In Section 1 we define a *residual index* $\iota_p(f,S) \in \mathbb{C}$, where f is a holomorphic self-map of a complex 2-manifold which is the identity on a compact 1-dimensional submanifold S, and $p \in S$. It turns out that this index is either not defined anywhere on S (and we say that f is *degenerate* along S) or everywhere defined. Furthermore, though the definition and the context are definitely different, it formally behaves exactly as Camacho-Sad's index. In particular we recover an Index Theorem:

Theorem 0.5: *Let S be a 1-dimensional compact submanifold of a complex 2-manifold M, and let f be a germ about S of a holomorphic self-map of M such that $f|_S = \mathrm{id}_S$. Assume that df acts as the identity on the normal bundle ν_S of S in M, and that f is non-degenerate along S. Then*

$$\sum_{p \in S} \iota_p(f,S) = c_1(\nu_S),$$

where $c_1(\nu_S)$ is the first Chern class of ν_S.

We also have a similar result, without assumptions on the action of df on the normal bundle, if M is the total space of a holomorphic line bundle over S: see Theorem 1.2.

Section 2 is devoted to the proof of a Reduction Theorem. Let f be tangent to the identity at the origin in $\mathbb{C}^2$, and write $f = (z + \ell g^o, w + \ell h^o)$, for suitable functions ℓ, g^o and h^o, with g^o and h^o relatively prime. The first main observation is that, loosely speaking (see Proposition 2.1 for a precise statement), the origin is dinamically relevant only if g^o and h^o vanish there — we shall say that O is *singular* for f. Applying this observation to the blow-up of f we get the notion of *singular directions*, which turn out to be the dynamically correct generalization of non-degenerate characteristic directions. Then the first step of the reduction consists in showing that after a finite number of blow-ups we can lift f to a map whose singularities are *dicritical* (roughly speaking, this means that all tangent directions are singular) or such that the linear part of (g^o, h^o) is not vanishing.

In the latter case, it is easy to check that the eigenvalues $\{\lambda_1, \lambda_2\}$ of the linear part of (g^o, h^o) are independent of the coordinates. Then the second step of the reduction is to show that after a finite number of blow-ups we can control the eigenvalues: to be precise, after a finite number of blow-ups we can assume that at each singular point which is not dicritical we have either $\lambda_1\lambda_2 \neq 0$, λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbb{N}$, or $\lambda_1 \neq 0$, $\lambda_2 = 0$. The third and last step of the reduction, yielding the Reduction Theorem 2.10, consists in showing that possibly after some other blow-ups we can control the residual indeces of the blown-up map at all singularities.

Finally, in Section 3 we prove a simplified version of Theorem 0.2 which is enough for our aims. In this way we have recovered all the ingredients needed to follow Camacho-Sad's argument, and we obtain at last Theorem 0.4 (see Theorem 3.2 and Corollary 3.3).

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1. The Residual Index Theorem

We begin by fixing a number of notations and definitions that we shall freely use in the paper. $\mathcal{O}_n$ will denote the ring of germs of holomorphic functions defined in a neighbourhood of the origin O of $\mathbb{C}^n$. Any $g \in \mathcal{O}_n$ has a *homogeneous expansion* as infinite sum of homogeneous polynomials, $g = P_0 + P_1 + \cdots$, with $\deg P_j = j$ (or $P_j \equiv 0$); the least $j \geq 0$ such that P_j is not identically zero is the *order* $\nu(g)$ of g.

If S is a subset of a complex 2-dimensional manifold M, we denote by $\mathrm{End}(M, S)$ the set of germs about S of holomorphic self-maps of M sending S into itself. If S is a 1-dimensional submanifold of M, a chart (U, φ) of M about $p \in S$ is *adapted* to S if $U \cap S = \varphi^{-1}(\{(z, w) \mid w = 0\})$; in particular, $(U \cap S, \varphi_1|_{U\cap S})$ is a chart of S about p.

Let $f \in \mathrm{End}(\mathbb{C}^2, O)$. We shall always write $f = (f_1, f_2)$; furthermore, $f_1 = P_1 + P_2 + \cdots$ and $f_2 = Q_1 + Q_2 + \cdots$ will be the homogeneous expansions of f_1 and f_2 (in most cases, $P_1(z, w) = z$ and $Q_1(z, w) = w$). We shall consistently write $f_1 = P_1 + g$ and $f_2 = Q_1 + h$; furthermore, by definition, the *order* of f is $\nu(f) = \min\{\nu(g), \nu(h)\}$. We shall always assume $\nu(f) < +\infty$, that is $f \neq \mathrm{id}_{\mathbb{C}^2}$.

Borrowing a word from continuous dynamics, we shall say that the origin is *dicritical* if we have $wP_{\nu(f)}(z, w) \equiv zQ_{\nu(f)}(z, w)$. Following Hakim [H1, 2], we shall say that $[u_0 : v_0] \in \mathbb{P}^1$ is a *characteristic direction* for f at the origin if there exists $\lambda \in \mathbb{C}$ such that $P_{\nu(f)}(u_0, v_0) = \lambda u_0$ and $Q_{\nu(f)}(u_0, v_0) = \lambda v_0$; it is *non-degenerate* if $\lambda \neq 0$, and *degenerate* otherwise.

We now recall some basic definitions and results on blowing up maps, referring to [A] for details. Let M be a complex 2-manifold, and $p \in M$. The *blow-up* of M at p is the set $\tilde{M} = (M \setminus \{p\}) \cup \mathbb{P}(T_pM)$, endowed with the manifold structure we shall presently describe, together with the projection $\pi: \tilde{M} \to M$ given by $\pi|_{M\setminus\{p\}} = \mathrm{id}_{M\setminus\{p\}}$ and $\pi|_{\mathbb{P}(T_pM)} \equiv p$. The set $S = \mathbb{P}(T_pM) = \pi^{-1}(p)$ is the *exceptional divisor* of the blow-up.

Fix a chart $\varphi = (z_1, z_2): U \to \mathbb{C}^2$ of M centered at p. Set $U_j = (U \setminus \{z_j = 0\}) \cup \big(S \setminus \mathrm{Ker}(dz_j|_p)\big)$, and let $\chi_j: U_j \to \mathbb{C}^n$ be given by

$$\chi_j(q)_h = \begin{cases} z_j(q) & \text{if } j = h \text{ and } q \in U \setminus \{z_j = 0\}, \\ z_h(q)/z_j(q) & \text{if } j \neq h \text{ and } q \in U \setminus \{z_j = 0\}, \\ d(z_h)_p(q)/d(z_j)_p(q) & \text{if } j \neq h \text{ and } q \in S \setminus \mathrm{Ker}(dz_j|_p), \\ 0 & \text{if } j = h \text{ and } q \in S \setminus \mathrm{Ker}(dz_j|_p). \end{cases} \tag{1.1}$$

Then the charts (U_j, χ_j), together with an atlas of $M \setminus \{p\}$, endow $\tilde{M}$ with a structure of 2-dimensional complex manifold such that the projection π is holomorphic everywhere and given by

$$[\varphi \circ \pi \circ \chi_j^{-1}(w)]_h = \begin{cases} w_j & \text{if } j = h, \\ w_j w_h & \text{if } j \neq h. \end{cases} \tag{1.2}$$

Let $f \in \mathrm{End}(M, p)$ be such that df_p is invertible. Then ([A]) there exists a unique map $\tilde{f} \in \mathrm{End}(\tilde{M}, S)$, the *blow-up* of f at p, such that $\pi \circ \tilde{f} = f \circ \pi$. The action of $\tilde{f}$ on S is induced by the action of df_p on $\mathbb{P}(T_pM)$; in particular, if $df_p = \mathrm{id}$ then $\tilde{f}|_S = \mathrm{id}_S$.

We can finally start working. Let S be a 1-dimensional submanifold of a complex 2-manifold M. If $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β) are two adapted charts with $U_\alpha\cap U_\beta\ne\varnothing$, then $\varphi_{\beta\alpha}=(\varphi^1_{\beta\alpha},\varphi^2_{\beta\alpha})=\varphi_\beta\circ\varphi_\alpha^{-1}$ has the form

$$\begin{cases} z_\beta=\varphi^1_{\beta\alpha}(z_\alpha,w_\alpha)=\psi_{\beta\alpha}(z_\alpha)+w_\alpha\,\theta^1_{\beta\alpha}(z_\alpha,w_\alpha),\\ w_\beta=\varphi^2_{\beta\alpha}(z_\alpha,w_\alpha)=w_\alpha\,\theta^2_{\beta\alpha}(z_\alpha,w_\alpha).\end{cases}\tag{1.3}$$

Notice that $\{\xi_{\alpha\beta}(z_\beta):=\theta^2_{\alpha\beta}(z_\beta,0)\}$ is the cocycle representing the normal bundle $\nu_S=(TM|_S)/TS$.

Remark 1.1: As a particular case we can consider the total space M of a line bundle E over S (identifying S with the zero section of E). Using as charts only trivializations of the bundle, (1.3) simplifies to

$$\begin{cases} \varphi^1_{\beta\alpha}(z_\alpha,w_\alpha)=\psi_{\beta\alpha}(z_\alpha),\\ \varphi^2_{\beta\alpha}(z_\alpha,w_\alpha)=w_\alpha\,\xi_{\beta\alpha}(z_\alpha).\end{cases}$$

In this case ν_S is canonically isomorphic to E, and thus the notation is consistent.

Now let $f\in\mathrm{End}(M,S)$ be a (germ about S of) holomorphic self-map of M such that $f|_S\equiv\mathrm{id}_S$. Setting $f_\alpha=\varphi_\alpha\circ f\circ\varphi_\alpha^{-1}$, we can write

$$\begin{cases} f_{1,\alpha}(z_\alpha,w_\alpha)=z_\alpha+w_\alpha^{\mu_\alpha+1}\,g^\bullet_\alpha(z_\alpha,w_\alpha),\\ f_{2,\alpha}(z_\alpha,w_\alpha)=b_\alpha(z_\alpha)\,w_\alpha+w_\alpha^{\nu_\alpha+2}h^\bullet_\alpha(z_\alpha,w_\alpha).\end{cases}\tag{1.4}$$

for suitable $g^\bullet_\alpha$, $h^\bullet_\alpha\in\mathcal{O}_2$, $b_\alpha\in\mathcal{O}_1$ and μ_α, $\nu_\alpha\in\mathbb{N}\cup\{\infty\}$, where $\mu_\alpha=\infty$ (respectively, $\nu_\alpha=\infty$) means $g^\bullet_\alpha\equiv0$ (respectively, $h^\bullet_\alpha\equiv0$), and w_α does not divide either $g^\bullet_\alpha$ or $h^\bullet_\alpha$.

Lemma 1.1: *If S is compact, then the function b_α is constant and independent of the adapted chart chosen.*

Proof: Since $f|_S=\mathrm{id}_S$, the normal bundle ν_S is invariant under the action of the differential of f; in particular, being ν_S of rank 1, there should exist a holomorphic function $\lambda\colon S\to\mathbb{C}$ such that $df_p(v)=\lambda(p)v$ for all $p\in S$ and $v\in(\nu_S)_p$. But S is compact; therefore λ is necessarily constant. Finally, an easy computation in local coordinates shows that $\lambda(p)=b_\alpha\big(\varphi_\alpha(p)\big)$, and we are done. □

Denoting by $b=b(f)\in\mathbb{C}$ this constant, we introduce the (locally defined) meromorphic function

$$k_\alpha(z_\alpha)=\lim_{w_\alpha\to0}\frac{f_{2,\alpha}(z_\alpha,w_\alpha)-b\,w_\alpha}{w_\alpha\big(f_{1,\alpha}(z_\alpha,w_\alpha)-z_\alpha\big)}=\begin{cases}0 & \text{if } \mu_\alpha<\nu_\alpha;\\ \dfrac{h^\bullet_\alpha}{g^\bullet_\alpha}(z_\alpha,0) & \text{if } \mu_\alpha=\nu_\alpha;\\ \infty & \text{if } \mu_\alpha>\nu_\alpha.\end{cases}$$

We shall say that $p\in S$ is a *strictly fixed point* if $\varphi_\alpha(p)$ is a pole of k_α. If $k_\alpha\equiv\infty$, we shall say that f is *degenerate* along S.

Remark 1.2: We shall momentarily show that these definitions are well-posed (i.e., they do not depend on the adapted chart chosen); for the time being let us justify the name. The first non-linear term in the power series expansion of f_α at the origin is $\big(g^\bullet_\alpha(0,0)(w_\alpha)^{\mu_\alpha+1},h^\bullet_\alpha(0,0)(w_\alpha)^{\nu_\alpha+2}\big)$; thus if $\mu_\alpha\le\nu_\alpha$ the first non-linear term is of order $\mu_\alpha+1$ *unless* the origin is a strictly fixed point. Therefore, in a loose sense, strictly fixed points are "more fixed" than other points of S.

We need to know the behavior of $g^\bullet_\alpha$, $h^\bullet_\alpha$ and k_α under change of coordinates. Since $f_\beta=\varphi_{\beta\alpha}\circ f_\alpha\circ\varphi_{\alpha\beta}$, recalling (1.3) we get

$$\begin{cases} z_\beta+(w_\beta)^{\mu_\beta+1}\,g^\bullet_\beta=\psi_{\beta\alpha}(f^1_\alpha\circ\varphi_{\alpha\beta})+(f^2_\alpha\circ\varphi_{\alpha\beta})\cdot\theta^1_{\beta\alpha}(f_\alpha\circ\varphi_{\alpha\beta}),\\ b\,w_\beta+(w_\beta)^{\nu_\beta+2}h^\bullet_\beta=(f^2_\alpha\circ\varphi_{\alpha\beta})\cdot\theta^2_{\beta\alpha}(f_\alpha\circ\varphi_{\alpha\beta}).\end{cases}\tag{1.5}$$

Plugging (1.4) in the second equation we find

$$(w_\beta)^{\nu_\beta}h^\bullet_\beta=\frac{1}{[\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)]^2}\left[b\,\frac{\theta^2_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}{w_\alpha}+\theta^2_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)\cdot(w_\alpha)^{\nu_\alpha}h^\bullet_\alpha(z_\alpha,w_\alpha)\right].$$

Expressing f_α as (z_α, w_α) plus a remainder we can write

$$\begin{aligned}\frac{\theta^2_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}{w_\alpha} &= \frac{\partial\theta^2_{\beta\alpha}}{\partial z_\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\mu_\alpha}g^\bullet_\alpha \\ &\quad+\frac{\partial\theta^2_{\beta\alpha}}{\partial w_\alpha}(z_\alpha,w_\alpha)\big((b-1)+(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha\big)+o\left(\frac{\|f_\alpha-(z_\alpha,w_\alpha)\|}{w_\alpha}\right).\end{aligned}$$

In particular, if $b=1$ or if $\partial\theta^2_{\beta\alpha}/\partial w_\alpha\equiv 0$ (e.g., in the line bundle situation) we get

$$\frac{\theta^2_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}{w_\alpha}=\frac{\partial\theta^2_{\beta\alpha}}{\partial z_\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\mu_\alpha}g^\bullet_\alpha+\frac{\partial\theta^2_{\beta\alpha}}{\partial w_\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha+o\big((w_\alpha)^{\min\{\mu_\alpha,\nu_\alpha+1\}}\big).$$

On the other hand,

$$\theta^2_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)\cdot(w_\alpha)^{\nu_\alpha}h^\bullet_\alpha(z_\alpha,w_\alpha)=\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\nu_\alpha}h^\bullet_\alpha(z_\alpha,w_\alpha)+o\big((w_\alpha)^{\nu_\alpha}\big).$$

Assuming $b=1$ or $\partial\theta^2_{\beta\alpha}/\partial w_\alpha\equiv 0$ we can then distinguish three cases:

(a) $\mu_\alpha>\nu_\alpha$. In this case we get

$$(w_\beta)^{\nu_\beta}h^\bullet_\beta(z_\beta,w_\beta)=\frac{1}{\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}(w_\alpha)^{\nu_\alpha}h^\bullet_\alpha(z_\alpha,w_\alpha)+o\big((w_\alpha)^{\nu_\alpha}\big);$$

in particular, $\nu_\beta=\nu_\alpha$, because $w_\beta=\xi_{\beta\alpha}(z_\alpha)w_\alpha+o(w_\alpha)$.

(b) $\mu_\alpha=\nu_\alpha$. In this case we get

$$(w_\beta)^{\nu_\beta}h^\bullet_\beta(z_\beta,w_\beta)=\frac{(w_\alpha)^{\nu_\alpha}}{\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}\left[h^\bullet_\alpha(z_\alpha,w_\alpha)+\frac{b}{\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}\frac{\partial\theta^2_{\beta\alpha}}{\partial z_\alpha}(z_\alpha,w_\alpha)g^\bullet_\alpha(z_\alpha,w_\alpha)\right]+o\big((w_\alpha)^{\nu_\alpha}\big);$$

in particular, $\nu_\beta\ge\nu_\alpha$.

(c) $\mu_\alpha<\nu_\alpha$. In this case we get

$$(w_\beta)^{\nu_\beta}h^\bullet_\beta(z_\beta)=\frac{b}{[\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)]^2}(w_\alpha)^{\mu_\alpha}\frac{\partial\theta^2_{\beta\alpha}}{\partial z_\alpha}(z_\alpha,w_\alpha)g^\bullet_\alpha(z_\alpha,w_\alpha)+o\big((w_\alpha)^{\mu_\alpha}\big);$$

in particular, $\nu_\beta\ge\mu_\alpha$.

Let us now study $g^\bullet_\alpha$. The first equation in (1.5) yields

$$\begin{aligned}(w_\beta)^{\mu_\beta}g^\bullet_\beta=\frac{1}{\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}\Bigg[&\frac{\psi_{\beta\alpha}\big(z_\alpha+(w_\alpha)^{\mu_\alpha+1}g^\bullet_\alpha(z_\alpha,w_\alpha)\big)-\psi_{\beta\alpha}(z_\alpha)}{w_\alpha}\\ &+\big(b+(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha(z_\alpha,w_\alpha)\big)\cdot\theta^1_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)\Bigg].\end{aligned}$$

Arguing as before we find

$$\frac{\psi_{\beta\alpha}\big(z_\alpha+(w_\alpha)^{\mu_\alpha+1}g^\bullet_\alpha(z_\alpha,w_\alpha)\big)-\psi_{\beta\alpha}(z_\alpha)}{w_\alpha}=\psi'_{\beta\alpha}(z_\alpha)(w_\alpha)^{\mu_\alpha}g^\bullet_\alpha(z_\alpha,w_\alpha)+o\big((w_\alpha)^{\mu_\alpha}\big),$$

and

$$\begin{aligned}&\big(b+(w_\alpha)^{\mu_\alpha+1}h^\bullet_\alpha(z_\alpha,w_\alpha)\big)\cdot\theta^1_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)\\ &\qquad=[\theta^1_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big)-\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)]+\big(b-1+(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha(z_\alpha,w_\alpha)\big)\theta^1_{\beta\alpha}\big(f_\alpha(z_\alpha,w_\alpha)\big).\end{aligned}$$

In particular, if $b=1$ or if $\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)\equiv 0$ (e.g., in the line bundle situation) we get

$$\begin{aligned}&\bigl(b+(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha(z_\alpha,w_\alpha)\bigr)\cdot\theta^1_{\beta\alpha}\bigl(f_\alpha(z_\alpha,w_\alpha)\bigr)-\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)\\&\quad=\frac{\partial\theta^1_{\beta\alpha}}{\partial z_\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\mu_\alpha+1}g^\bullet_\alpha(z_\alpha,w_\alpha)+\theta^1_{\beta\alpha}(z_\alpha,w_\alpha)(w_\alpha)^{\nu_\alpha+1}h^\bullet_\alpha(z_\alpha,w_\alpha)+o\bigl((w_\alpha)^{\min\{\mu_\alpha,\nu_\alpha\}+1}\bigr).\end{aligned}$$

This time we have two possibilities:

(a) $\mu_\alpha>\nu_\alpha$: in this case we have $\mu_\beta\ge\nu_\alpha+1$.

(b) $\mu_\alpha\le\nu_\alpha$: in this case we find

$$(w_\beta)^{\mu_\beta}g^\bullet_\beta(z_\beta,w_\beta)=\frac{1}{\theta^2_{\beta\alpha}(z_\alpha,w_\alpha)}\psi'_{\beta\alpha}(z_\alpha)(w_\alpha)^{\mu_\alpha}g^\bullet_\alpha(z_\alpha,w_\alpha)+o\bigl((w_\alpha)^{\mu_\alpha}\bigr),\tag{1.6}$$

and thus $\mu_\beta=\mu_\alpha$.

In particular, we have shown that $\mu_\alpha>\nu_\alpha$ iff $\mu_\beta>\nu_\beta$; this means that *the degeneracy of f is independent of the coordinates,* as claimed.

Finally, assume that f is not degenerate, and that $b(f)=1$ or we are in the line bundle situation. Then

$$k_\beta(z_\beta)=\frac{1}{\psi'_{\beta\alpha}(z_\alpha)}\left[k_\alpha(z_\alpha)+b\,\frac{\xi'_{\beta\alpha}(z_\alpha)}{\xi_{\beta\alpha}(z_\alpha)}\right].$$

In particular, k_β and k_α have the same poles, and thus the definition of strictly fixed points is independent of the coordinates. Furthermore, if we set

$$\eta_\alpha=k_\alpha\,dz_\alpha,$$

we have

$$\eta_\beta=\eta_\alpha+d(b\log\xi_{\beta\alpha}).\tag{1.7}$$

So the family of meromorphic forms $\{\eta_\alpha\}$ behaves exactly as the forms by the same name defined in [CS] — and thus we can draw the same consequences. First of all, since $d(b\log\xi_{\beta\alpha})$ is a *holomorphic* $(1,0)$-form, the residue of η_α at a point is independent of the coordinates. We shall then call *residual index* of f at p *along* S the number

$$\iota_p(f,S)=\operatorname{Res}\bigl(\eta_\alpha;\varphi_\alpha(p)\bigr);$$

it might be non-zero only at the strictly fixed points of f.

Secondly, arguing as in the Appendix of [CS] we get

Theorem 1.2: (Residual Index Theorem) *Let S be a 1-dimensional compact submanifold of a complex 2-manifold M, and take $f\in\operatorname{End}(M,S)$ such that $f|_S=\mathrm{id}_S$. Assume that $b(f)=1$ or that M is the total space of a line bundle E over S. Assume moreover that f is non-degenerate along S. Then*

$$\sum_{p\in S}\iota_p(f,S)=b(f)\,c_1(\nu_S),$$

where $c_1(\nu_S)$ is the first Chern class of ν_S (which is equal to $c_1(E)$ in the line bundle situation).

In the sequel we shall need to know how the residual index changes under blow-ups. So take $p\in S$, and let $\tilde M_p$ be the blow-up of M at p, and $\tilde S$ the proper transform of S. If df_p is invertible (that is, if $b(f)\ne 0$), then f lifts to a (germ of) holomorphic map $\tilde f\in\operatorname{End}(\tilde M_p,\tilde S)$; furthermore, since $\tilde f$ on the exceptional divisor is induced by the differential of f, we still have $\tilde f|_{\tilde S}=\mathrm{id}_{\tilde S}$.

Proposition 1.3: *Let $q \in \tilde{M}_p$ be the intersection between $\tilde{S}$ and the exceptional divisor. Assume $b(f) \neq 0$. Then:*
(i) $b(\tilde{f}) = b(f)$;
(ii) $\tilde{f}$ *is non-degenerate along* $\tilde{S}$ *iff* f *is non-degenerate along* S;
(iii) $\iota_q(\tilde{f}, \tilde{S}) = \iota_p(f, S) - b(f)$.

Proof: Choose an adapted chart (U, φ) in M centered at p. Then the corresponding chart (U_1, χ_1) in $\tilde{M}_p$ centered at q is such that the relation between the coordinates of f and $\tilde{f}$ is given by

$$\begin{cases} f_1(z, zw) = \tilde{f}_1(z, w), \\ f_2(z, zw) = \tilde{f}_1(z, w)\tilde{f}_2(z, w). \end{cases}$$

Putting (1.4) into the first equation we immediately find $\tilde{\mu} = \mu$ and

$$\tilde{g}^\bullet(z, w) = z^{\mu+1} g^\bullet(z, zw). \tag{1.8}$$

Applying (1.4) to the second equation and dividing by zw we get

$$\big(1 + z^\mu w^{\mu+1} g^\bullet(z, zw)\big)\big(b(\tilde{f}) + w^{\tilde{\nu}+1}\tilde{h}^\bullet(z, w)\big) = b(f) + (zw)^{\nu+1} h^\bullet(z, zw), \tag{1.9}$$

and thus (i) follows setting $w = 0$.

Now, (1.8) and (1.9) yield

$$w^{\tilde{\nu}}\tilde{h}^\bullet(z, w) = \frac{z^{\nu+1} w^\nu h^\bullet(z, zw) - b(f) z^\mu w^\mu g^\bullet(z, zw)}{1 + z^\mu w^{\mu+1} g^\bullet(z, zw)};$$

in particular, (ii) holds. Furthermore,

$$\tilde{k}(z) = k(z) - \frac{b(f)}{z},$$

and we are done. □

As already remarked, S will often be the exceptional divisor of a blow-up; it turns out that in this case we have an important relationship between dicriticality downstairs and degeneracy upstairs.

Proposition 1.4: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be such that $df_O = \mathrm{id}$. Let M be the blow-up of $\mathbb{C}^2$ at the origin, $S \subset M$ the exceptional divisor, and $\tilde{f} \in \mathrm{End}(M, S)$ the blow-up of f. Then:*
(i) *non degenerate characteristic directions for f are strictly fixed points for $\tilde{f}$;*
(ii) *strictly fixed points for $\tilde{f}$ are characteristic directions for f;*
(iii) *$\tilde{f}$ is degenerate along S iff the origin is dicritical for f.*

Proof: First of all notice that $[z_0 : 1] \in \mathbb{P}^1 = S$ is a characteristic direction iff $P_{\nu(f)}(z_0, 1) - z_0 Q_{\nu(f)}(z_0, 1) = 0$, and it is degenerate iff $Q_{\nu(f)}(z_0, 1) = 0$.

In the canonical chart of M containing $[z_0 : 1]$, the blow-up $\tilde{f}$ is given by

$$\begin{cases} \tilde{f}_1(z, w) = z + w^{\nu(f)-1} \dfrac{\big(P_{\nu(f)}(z, 1) - zQ_{\nu(f)}(z, 1)\big) + w\big(P_{\nu(f)+1}(z, 1) - zQ_{\nu(f)+1}(z, 1)\big) + \cdots}{1 + w^{\nu(h)-1}Q_{\nu(h)}(z, 1) + w^{\nu(h)}Q_{\nu(h)+1}(z, 1) + \cdots}, \\ \tilde{f}_2(z, w) = w + w^{\nu(h)}Q_{\nu(h)}(z, 1) + w^{\nu(h)+1}Q_{\nu(h)+1}(z, 1) + \cdots. \end{cases}$$

Clearly, $\tilde{\nu} = \nu(h) - 2$ and $\tilde{h}^\bullet(z, w) = Q_{\nu(h)}(z, 1) + wQ_{\nu(h)+1}(z, 1) + \cdots$. On the other hand, $\tilde{\mu} = \nu(f) - 2$ if the origin is not dicritical; $\tilde{\mu} > \nu(f) - 2$ if the origin is dicritical. In particular, since dicriticality implies $\nu(g) = \nu(h) = \nu(f)$, if the origin is dicritical we have $\tilde{\mu} > \tilde{\nu}$, and $\tilde{f}$ is degenerate; conversely, if the origin is not dicritical we have $\tilde{\mu} \leq \tilde{\nu}$, and $\tilde{f}$ is not degenerate.

If the origin is dicritical, all directions are characteristic, and all points of S are strictly fixed; therefore the Proposition is proved in this case. If the origin is not dicritical, there are two possibilities to consider. If $\nu(h) > \nu(f)$, there are no strictly fixed points in this chart, but (being $Q_{\nu(f)} \equiv 0$) all possible characteristic directions are degenerate. On the other hand, if $\nu(h) = \nu(f)$ we get

$$k(z) = \frac{Q_{\nu(f)}(z, 1)}{P_{\nu(f)}(z, 1) - zQ_{\nu(f)}(z, 1)}. \tag{1.10}$$

So if z_0 is a strictly fixed point then $P_{\nu(f)}(z_0, 1) - z_0 Q_{\nu(f)}(z_0, 1) = 0$, and thus $[z_0 : 1]$ is a characteristic direction. Conversely, if $[z_0 : 1]$ is a non-degenerate characteristic direction, then $Q_{\nu(f)}(z_0, 1) \neq 0$, $P_{\nu(f)}(z_0, 1) - z_0 Q_{\nu(f)}(z_0, 1) = 0$, and so z_0 is a strictly fixed point. □

Remark 1.3: Hakim [H1, 2] associated to every non-degenerate characteristic direction $[z_0 : 1]$ the number $R'(z_0)/Q_{\nu(f)}(z_0,1)$, where $R(z) = P_{\nu(f)}(z,1) - zQ_{\nu(f)}(z,1)$. It turns out that, when not zero, this number is exactly the reciprocal of the residual index of $\tilde f$ at $[z_0 : 1]$. In fact, in this case we should have $\nu(h) = \nu(f)$ and $R'(z_0) \neq 0$; therefore $k(z)$ is given by (1.10), $R(z) = R'(z_0)(z - z_0) + o(z - z_0)$, and so $\iota_{[z_0:1]}(\tilde f, S) = Q_{\nu(f)}(z_0,1)/R'(z_0)$.

2. The Reduction Theorem

Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be such that $df_O = \mathrm{id}$. The aim of this section is to show that a finite sequence of blow-ups can substantially simplify the local expression of f — at the expense of changing the geometry of the underlying space.

But first we need another set of definitions and notations. We shall consistently write $f_1 = z + g$ and $f_2 = w+h$, as before. We denote by $\ell \in \mathcal{O}_2$ the greatest common divisor (g,h) of g and h (which is defined up to units in $\mathcal{O}_2$), and write $g = \ell g^o$ and $h = \ell h^o$. The homogeneous expansion of g^o (respectively, of h^o, ℓ) will be $g^o = P_0^o + P_1^o + \cdots$ (respectively, $h^o = Q_0^o + Q_1^o + \cdots$, $\ell = R_0 + R_1 + \cdots$), and we shall denote by $\kappa = \nu(\ell)$ the order of ℓ. The *pure order* of f is $\nu_o(f) = \min\{\nu(g^o), \nu(h^o)\}$. Obviously, $\nu_o(f) + \kappa = \nu(f) \geq 2$.

It is clear that $\ell(z,w) = 0$ is a (not necessarily reduced) local equation of the germ at the origin of the fixed point set $\mathrm{Fix}(f)$ of f. If $\mathrm{Fix}(f)$ has (at least) two smooth (local) components intersecting transversally at the origin, we shall say that the origin is a *corner.*

We shall say that the origin is a *singular point* for f if the pure order of f is at least 1 (and we shall prove in a moment that the pure order — as well as being dicritical — is independent of the coordinates). Notice that if the origin is dicritical then $wP^o_{\nu_o(f)}(z,w) \equiv zQ^o_{\nu_o(f)}(z,w)$, and thus it is necessarily singular.

There is a dynamical reason for singleing out singular points:

Proposition 2.1: *Let S be a compact 1-dimensional submanifold of a 2-dimensional complex manifold M, and $f \in \mathrm{End}(M,S)$ such that $f|_S = \mathrm{id}_S$. Assume that $b(f) = 1$ and that f is not degenerate along S. Let $p_0 \in S$ be not singular and not a corner. Then no infinite orbit of f can stay arbitrarily close to p_0, that is there exists a neighbourhood U of p_0 such that for all $q \in U$ either the orbit of q lands on S or $f^{n_0}(q) \notin U$ for some $n_0 \in \mathbb{N}$. In particular, no infinite orbit is converging to p_0.*

Proof: We shall work in a chart adapted to S and centered in p_0. Since p_0 is not a corner, we have $\ell(z,w) = w^\sigma$ for a suitable $\sigma \geq 1$; then we can write

$$\begin{cases} z_1 := f_1(z,w) = z + w^\sigma\bigl(a_0 + A_1(z,w)\bigr), \\ w_1 := f_2(z,w) = w + w^\sigma\bigl(b_0 + B_1(z,w)\bigr), \end{cases}$$

with $\nu(A_1)$, $\nu(B_1) \geq 1$. Since f is not degenerate along S, we must have $b_0 = 0$ and $B_1 = wB_0$; since p_0 is not singular, we must have $a_0 \neq 0$ — and after a linear change of coordinates we can actually assume $a_0 = 1$.

We then make the following change of variables:

$$\begin{cases} Z = z, \\ W = w\,(1 + A_1(z,w))^{1/\sigma}. \end{cases}$$

Then

$$\begin{cases} Z_1 = Z + W^\sigma, \\ W_1 = W + W^{\sigma+1}\tilde B_0(Z,W). \end{cases}$$

In particular,

$$\frac{1}{W_1^\sigma} = \frac{1}{W^\sigma} + a(Z) + Wb(Z,W),$$

for suitable holomorphic functions $a(Z)$, $b(Z,W)$. Thus we can find $d > 0$ such that $\bigl|(1/W_1^\sigma) - (1/W^\sigma)\bigr| \leq d$ if (Z,W) belongs to a compact set of the form $\{|Z| \leq r, |W| \leq \rho\}$. Following [U], we now choose $0 < r_0 < (2d)^{-1}\log 2$, and set $U = \{|Z| < r_0, |W| < \rho\}$; we claim that no point in $U \setminus S$ can have an orbit completely contained in $U \setminus S$.

Suppose, by contradiction, that $(Z_0, W_0) \in U \setminus S$ is such that $(Z_n, W_n) = f^n(Z_0, W_0) \in U \setminus S$ for all $n \geq 0$. In particular, $W_n \neq 0$ for all $n \geq 0$ and so $\bigl|(1/W_n^\sigma) - (1/W_0^\sigma)\bigr| \leq nd$. Hence

$$\left|\left(\frac{W_0}{W_n}\right)^\sigma - 1\right| \leq nd|W_0|^\sigma$$

for all $n \geq 0$. This implies that if $nd|W_0|^\sigma < 1$ then $(W_n/W_0)^\sigma$ is in the disk which has the segment $[(1+nd|W_0|^\sigma)^{-1}, (1-nd|W_0|^\sigma)^{-1}]$ as diameter, and thus

$$\mathrm{Re}\left(\frac{W_n}{W_0}\right)^\sigma \geq \frac{1}{1+nd|W_0|^\sigma}.$$

Let $n_0 \geq 1$ be the integer such that $(n_0-1)d|W_0|^\sigma < 1 \leq n_0 d|W_0|^\sigma$. Then

$$\mathrm{Re}\left(\frac{W_j}{W_0}\right)^\sigma \geq \frac{1}{(n_0+j)d|W_0|^\sigma}$$

for $0 \leq j \leq n_0 - 1$. But this implies

$$\begin{aligned}|Z_{n_0} - Z_0| &= \left|\sum_{j=0}^{n_0-1} W_j^\sigma\right| = |W_0|^\sigma \left|\sum_{j=0}^{n_0-1}\left(\frac{W_j}{W_0}\right)^\sigma\right| \\ &\geq |W_0|^\sigma \sum_{j=0}^{n_0-1} \mathrm{Re}\left(\frac{W_j}{W_0}\right)^\sigma \\ &\geq \sum_{j=0}^{n-1} \frac{1}{(n_0+j)d} \geq \frac{\log 2}{d} > 2r_0,\end{aligned}$$

and so $(Z_{n_0}, W_{n_0}) \notin U$, contradiction. □

Since we shall show in Remark 2.1 that all the corners we obtain blowing up are singular, and we shall never blow-up a dicritical point, the upshot of this Proposition is that the only interesting dynamics is concentrated nearby singular points.

The *singular cone* of f is given by

$$C_f = \{[u:v] \in \mathbb{P}^1 \mid vP^o_{\nu_o(f)}(u,v) - uQ^o_{\nu_o(f)}(u,v) = 0\} \subset \mathbb{P}^1;$$

clearly, $C_f = \mathbb{P}^1$ iff the origin is dicritical, and it is otherwise a finite set containing $\nu_o(f)+1$ points (counted with multiplicities). Any $[u_0 : v_0] \in C_f$ is said a *singular direction* for f at the origin. The *multiplicity* of a singular direction is the multiplicity as root of $vP^o_{\nu_o(f)} - uQ^o_{\nu_o(f)}$. Since $P_{\nu(f)} = R_\kappa P^o_{\nu_o(f)}$ and $Q_{\nu(f)} = R_\kappa Q^o_{\nu_o(f)}$, it is clear that non-degenerate characteristic directions are singular directions, and that singular directions are characteristic directions.

Now let $\pi\colon M \to \mathbb{C}^2$ be the blow-up of the origin, and $S \subset M$ the exceptional divisor. Let $s_j(z,w) = 0$ be the equation of S, and π_j the expression of π, in the canonical chart U_j; see (1.1) and (1.2). For any $g \in \mathcal{O}_2$ and $j = 1, 2$ we then set

$$\hat{g}_{(j)} = \frac{g \circ \pi_j}{s_j^{\nu(g)}}.$$

When the context indicates clearly (or it does not matter) in which chart we are working in, we shall drop the index j and simply write $\hat{g}$.

Lemma 2.2: *Let $f\in\mathrm{End}(\mathbb{C}^2,O)$, and denote by $\tilde f\in\mathrm{End}(M,S)$ its blow-up. Fix a canonical chart (U_j,χ_j) on M. Then:*

(i) *we have $(\hat g,\hat h)=\widehat{(g,h)}$;*
(ii) *we have $(\tilde g,\tilde h)=s_j^{\nu(f)-1}\widehat{(g,h)}$ if O is not dicritical, and $(\tilde g,\tilde h)=s_j^{\nu(f)}\widehat{(g,h)}$ if O is dicritical.*

Proof: For the sake of definiteness, we shall work in the chart (U_2,χ_2).

(i) The operation $\hat{}$ preserves the multiplication; therefore it suffices to prove that $(g,h)=1$ implies $(\hat g,\hat h)=1$.

Up to a linear change of coordinates, we can assume ([GR, p.13]) that g and h are regular with respect to x, i.e., that $g(x,0)$ vanishes of order $\nu(g)$ and $h(x,0)$ vanishes of order $\nu(h)$. Then, up to units, by the Weierstrass Preparation Theorem we can assume that they are Weierstrass polynomials with respect to x whose order coincides with the degree. In particular, they belong to the subring $\mathcal{W}\subset\mathcal{O}_2$ of germs of the form

$$p(x,y)=a_0(y)x^\nu+a_1(y)x^{\nu-1}+\cdots+a_\nu(y),$$

where $a_0,\ldots,a_\nu\in\mathcal{O}_1$ satisfy $\nu(a_j)\ge j$. Now

$$\hat p(z,w)=a_0(w)z^\nu+\frac{a_1(w)}{w}z^{\nu-1}+\cdots+\frac{a_\nu(w)}{w^\nu},$$

and thus $\hat{}\colon\mathcal{W}\to\mathcal{O}_1[z]$ is bijective. This implies that $(\hat g,\hat h)=1$ in $\mathcal{O}_1[z]$; it remains to prove that $(\hat g,\hat h)=1$ in $\mathcal{O}_2$.

So let p_1, $p_2\in\mathcal{O}_1[z]$ such that $(p_1,p_2)=1$ in $\mathcal{O}_1[z]$. Up to a linear change of coordinates of the form $(z,w)=(\alpha Z+W,W)$ — which is an automorphism of $\mathcal{O}_1[z]$ —, we can assume that both p_j's are regular with respect to z. Suppose, by contradiction, that there is $\ell\in\mathcal{O}_2$, not a unit, such that $p_j=\ell q_j$. Being p_j regular with respect to z, both ℓ and q_j must be so. Then we can write $\ell=u_0r_0$ and $q_j=u_jr_j$, where u_0, $u_j\in\mathcal{O}_2$ are units, and r_0, $r_j\in\mathcal{O}_1[z]$ are Weierstrass polynomials. Therefore $p_j=(u_0u_j)(r_0r_j)$; the Weierstrass Division Theorem then implies $u_0u_j\in\mathcal{O}_1[z]$. But this means that r_0 divides both p_1 and p_2 in $\mathcal{O}_1[z]$, against the assumption.

(ii) We have

$$\begin{aligned}\tilde g(z,w)&=w^{\nu(f)-1}\hat\ell(z,w)\frac{w^{\nu(g)-\nu(f)}\hat g^o(z,w)-zw^{\nu(h)-\nu(f)}\hat h^o(z,w)}{1+w^{\nu(h)-1}\hat h(z,w)},\\ \tilde h(z,w)&=w^{\nu(h)}\hat\ell(z,w)\hat h^o(z,w);\end{aligned}\tag{2.1}$$

in particular, $w^{\nu(f)-1}\hat\ell$ divides $(\tilde g,\tilde h)$. Assume that $s\in\mathcal{O}_2$ divides $w^{\nu(h)-\nu(f)+1}\hat h^o$; since, by construction, $(w,\hat h^o)=1$, we should have either $s|w$ or $s|\hat h^o$. In the latter case, if s divides $\tilde g/(w^{\nu(f)-1}\hat\ell)$ it must also divide $\hat g^o$, against (i). So (up to units) we have $s=w^r$, with $0\le r\le\nu(h)-\nu(f)+1$; but w divides $\tilde g/(w^{\nu(f)-1}\hat\ell)$ iff the origin is dicritical, as claimed. □

Corollary 2.3: *Let $f\in\mathrm{End}(\mathbb{C}^2,O)$ be tangent to the identity, and assume that the origin is not dicritical. Let $\tilde f\in\mathrm{End}(M,S)$ denote the blow-up of f at the origin. Then:*

(i) *the singular directions of f are exactly the singular points of $\tilde f$ in S;*
(ii) *strictly fixed points of $\tilde f$ are singular directions of f.*

Proof: (i) Up to a linear change of coordinates, it suffices to prove that $[0:1]\in S$ is singular for $\tilde f$ iff it belongs to C_f. But by the previous Lemma and (2.1) we have $\tilde h^o(z,w)=w^{\nu(h)-\nu(f)+1}\hat h^o(z,w)$ and

$$\tilde g^o(z,0)=P^o_{\nu_o(f)}(z,1)-zQ^o_{\nu_o(f)}(z,1),$$

and the assertion follows.

(ii) We shall prove the slightly more general assertion that if $f\in\mathrm{End}(M,S)$ with $f|_S=\mathrm{id}_S$ is non-degenerate along S then every strictly fixed point is singular. Fix a chart adapted to S and centered in $p\in S$. Write $g(z,w)=w^\sigma\ell_1(z,w)g^o(z,w)$ and $h(z,w)=w^\sigma\ell_1(z,w)h^o(z,w)$, so that $(g,h)=w^\sigma\ell_1$ (and w does not divide ℓ_1). By Lemma 1.1 we know that either $\sigma\ge2$ or w divides h^o (or both). If $\sigma\ge2$ but w does not divide h^o, then f is degenerate; therefore $h^o=wh^1$. But then w cannot divide g^o, $\nu(h^o)\ge1$ and $k(z)=h^1(z,0)/g^o(z,0)$; therefore if p is a strictly fixed point we must have $g^o(0,0)=0$, that is $\nu(g^o)\ge1$ and p is singular. □

In particular, if O is non dicritical we have the following inclusions:

$$\begin{aligned}\{\text{Non-degenerate characteristic directions for } f\} &\subseteq \{\text{Strictly fixed points for } \tilde f\}\\ &\subseteq \{\text{Singular points for } \tilde f\} = C_f\\ &\subseteq \{\text{Characteristic directions for } f\};\end{aligned}$$

all inclusions might be proper. If O is dicritical, then strictly fixed points, C_f and characteristic directions all agree with $\mathbb{P}^1$.

Remark 2.1: Assume that $f \in \mathrm{End}(M,S)$ with $f|_S = \mathrm{id}_S$ and $b(f) = 1$ is non degenerate along S, and that $p_0 \in S$ is not dicritical. Fix a chart adapted to S centered in p_0, and write

$$f = (z + w^{\mu+1} g^\bullet, w + w^{\nu+2} h^\bullet) = (z + w^\sigma \ell_1 g^o, w + w^\sigma \ell_1 h^o).$$

Arguing as in the proof of Corollary 2.3 we see that w must divide h^o. Now let $\tilde f$ be the blow-up of f, and q_0 the intersection between the proper transform of S and the exceptional divisor; in the canonical coordinates, $q_0 = [1:0]$, and it is a corner. Using Lemma 2.2.(ii) and (2.1), we see that $\nu(\tilde g^o) \ge 1$ always, and that $\nu(\tilde h^o) \ge 1$ if $\nu(h) > \nu(f)$. If $\nu(h) = \nu(f)$, we have $\tilde h^o(0,0) = \hat h^o(0,0) = Q^o_{\nu(h^o)}(1,0)$; but $w|h^o$ forces $Q^o_{\nu(h^o)}(1,0) = 0$, and thus $\nu(\tilde h^o) \ge 1$ in this case too. Summing up, we have proved that *if f is non-degenerate along S, and $p_0 \in S$ is not dicritical, then the corner over p_0 in the blow-up is always singular for $\tilde f$.*

Following ideas used by Ven den Essen [V] in the continuous case, we now introduce another technical tool fundamental for the proof of the Reduction Theorem. If g, $h \in \mathcal{O}_2$, we denote by $I(g,h;O) \in \mathbb{N} \cup \{\infty\}$ the intersection multiplicity of g and h at the origin (see [Fu, GH, C] for several equivalent definitions). It has the following properties:

(o) $I(g,w;O)$ is the multiplicity of 0 as root of $g(z,0) = 0$;
(i) $I(g,h;O) = I(h,g;O)$;
(ii) $I(g,h;O) = 0$ iff $\min\{\nu(g),\nu(h)\} = 0$;
(iii) $I(g,h;O) = \infty$ iff $(g,h) \ne 1$, that is iff the origin is not isolated in $g^{-1}(0) \cap h^{-1}(0)$;
(iv) $I(g_1 \cdot g_2, h; O) = I(g_1,h;O) + I(g_2,h;O)$;
(v) if $M \in GL(2,\mathcal{O}_2)$ and $\begin{vmatrix} g_1 \\ h_1 \end{vmatrix} = M \begin{vmatrix} g \\ h \end{vmatrix}$, then $I(g_1,h_1;O) = I(g,h;O)$;
(vi) if χ is a germ of biholomorphism of $\mathbb{C}^2$ with $\chi(O) = O$, then $I(g\circ\chi, h\circ\chi; O) = I(g,h;O)$;
(vii) let $\pi\colon M \to \mathbb{C}^2$ be the blow-up of the origin, and S the exceptional divisor. Then [GH, pp. 475-476]

$$I(g,h;O) = \nu(g)\nu(h) + \sum_{p\in S} I(\hat g, \hat h; p),$$

where to compute $\hat g$ and $\hat h$ nearby p we choose a canonical chart containing p (it does not matter which one if p belongs to both).

The *pure intersection index* of $f = (z+g, w+h)$ at the origin is then, by definition, $I_O(f) = I(g^o, h^o; O)$. The main properties of the pure intersection index are contained in the following lemmas:

Lemma 2.4: *The order, the pure order, the pure intersection index and the dicriticality are invariant under change of coordinates.*

Proof: Given a germ χ of biholomorphism of $\mathbb{C}^2$ fixing the origin, set $(z,w) = \chi(\hat z, \hat w)$ and $\hat f = \chi^{-1} \circ f \circ \chi$. As already remarked, $\ell = 0$ is an equation of the set of non-trivial irreducible components of $\mathrm{Fix}(f)$ at the origin. Since χ^{-1} sends this set onto the corresponding set for $\hat f$, whose equation is $\hat\ell = 0$, we must have $\hat\ell^p = (\ell\circ\chi)^q$ for some p, $q \in \mathbb{N}^*$. But now

$$\begin{aligned}\hat\ell \cdot \begin{vmatrix} \hat g^o \\ \hat h^o \end{vmatrix} = \hat f - \begin{vmatrix} \hat z \\ \hat w \end{vmatrix} &= \chi^{-1}\left(\chi(\hat z,\hat w) + (\ell\circ\chi)\cdot \begin{vmatrix} g^o\circ\chi \\ h^o\circ\chi \end{vmatrix}\right) - \chi^{-1}\bigl(\chi(\hat z,\hat w)\bigr)\\ &= (\ell\circ\chi)(\mathrm{Jac}(\chi^{-1})\circ\chi)\cdot \begin{vmatrix} g^o\circ\chi \\ h^o\circ\chi \end{vmatrix} + (\ell\circ\chi)^2(\mathrm{Hess}(\chi^{-1})\circ\chi)\left(\begin{vmatrix} g^o\circ\chi \\ h^o\circ\chi \end{vmatrix}\right) + \cdots,\end{aligned} \tag{2.2}$$

which forces $\hat\ell=\ell\circ\chi$ and $\nu(\hat\ell)=\nu(\ell)$. It moreover follows that

$$\begin{vmatrix}\hat P^o_{\nu_0(\hat f)}\\ \hat Q^o_{\nu_0(\hat f)}\end{vmatrix}=A_1^{-1}\cdot\begin{vmatrix}P^o_{\nu_0(f)}\circ A_1\\ Q^o_{\nu_0(f)}\circ A_1\end{vmatrix}, \tag{2.3}$$

where A_1 is the linear part of χ; thus $\nu_0(\hat f)=\nu_0(f)$, $\nu(\hat f)=\nu(f)$, and O is dicritical for $\hat f$ if it is so for f.

Furthermore, (2.2) implies that

$$\begin{vmatrix}\hat g^o\\ \hat h^o\end{vmatrix}=M(\hat z,\hat w)\cdot\begin{vmatrix}g^o\circ\chi\\ h^o\circ\chi\end{vmatrix},$$

where $M(\hat z,\hat w)$ is a suitable matrix with $M(0,0)=A_1^{-1}$. So $M\in GL(2,\mathcal{O}_2)$, and $I_O(\hat f)=I_O(f)$ follows from properties (v) and (vi) of the intersection multiplicity. □

Lemma 2.5: *Assume that the origin is non dicritical, and let $\tilde f$ be the blow-up of f. Then*

$$I_O(f)=\nu_o(f)^2-\nu_o(f)-1+\sum_{p\in S}I_p(\tilde f),$$

where S is the exceptional divisor.

Proof: First of all, by property (ii), $I_p(\tilde f)\ne 0$ iff p is a singular point of $\tilde f$, and hence $I_p(\tilde f)\ne 0$ iff $p\in C_f$.

Up to a linear change of coordinates, we can assume $\nu(g^o)=\nu(h^o)=\nu_o(f)$ and $[1:0]\notin C_f$. Set $R(u,v)=vP^o_{\nu_o(f)}(u,v)-uQ^o_{\nu_o(f)}(u,v)$, so that $C_f=\{R=0\}$. For $p_0=[s_0:1]\in C_f$, let $\mu_{p_0}\in\mathbb{N}$ denote its multiplicity. Clearly,

$$\sum_{p_0\in S}\mu_{p_0}=\nu_o(f)+1.$$

Now,

$$\frac{\hat g^o(s,w)-s\hat h^o(s,w)}{1+w^{\nu(f)-1}\hat g(s,w)}=R(s,1)+O(w);$$

therefore property (o) yields

$$I\left(\frac{\hat g^o-s\hat h^o}{1+w^{\nu(f)-1}\hat g},w;p_0\right)=\mu_{p_0}.$$

Then the properties of the intersection multiplicity, Lemma 2.2 and (2.1) yield

$$\begin{aligned}I_{p_0}(\tilde f)&=I\left(\frac{\hat g^o-s\hat h^o}{1+w^{\nu(f)-1}\hat g},w\hat h^o;p_0\right)=I\left(\frac{\hat g^o-s\hat h^o}{1+w^{\nu(f)-1}\hat g},w;p_0\right)+I\left(\frac{\hat g^o-s\hat h^o}{1+w^{\nu(f)-1}\hat g},\hat h^o;p_0\right)\\&=\mu_{p_0}+I\big(\hat g^o,\hat h^o;p_0\big).\end{aligned}$$

Thanks to Lemma 2.2.(i), the latter number is always finite. Therefore property (vii) yields

$$I_O(f)=I(g^o,h^o;O)=\nu_o(f)^2+\sum_{p_0\in S}I\big(\hat g^o,\hat h^o;p_0\big)=\nu_o(f)^2-\nu_0(f)-1+\sum_{p_0\in S}I_{p_0}(\tilde f).$$

□

We are then able to prove a first reduction theorem:

Theorem 2.6: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be tangent to the identity. Assume that O is an isolated singular point of f. Then there exists a complex 2-manifold M, a holomorphic projection $\pi: M \to \mathbb{C}^2$, and a holomorphic map $\tilde{f} \in \mathrm{End}(M, S)$, where $S = \pi^{-1}(O)$, satisfying the following properties:*

(i) $\pi|_{M\setminus S}: M \setminus S \to \mathbb{C}^2 \setminus \{O\}$ *is a biholomorphism;*
(ii) *S is either a point or the union of a finite number of projective lines intersecting each other transversally and at most in one point;*
(iii) $\pi \circ \tilde{f} = f \circ \pi$;
(iv) $\tilde{f}|_S = \mathrm{id}_S$;
(v) *the singular points of $\tilde{f}$ on S are isolated, and dicritical or of pure order* 1.

Proof: We construct the manifold M and the map $\tilde{f}$ using a sequence of blow-ups proceeding by induction on the pure intersection index of f at the origin. If $I_O(f) = 1$ then, by Lemma 2.5, either O is dicritical or has pure order 1 (because there is always at least one singular direction), and we are done.

Assume then $I_O(f) > 1$. Again, if O is dicritical or has pure order 1 we are done. Otherwise, we blow it up. By Lemma 2.5, all the singularities of the blow-up of f in the exceptional divisor must have pure intersection index strictly less than $I_O(f)$; therefore the inductive asssumption ensures us that after a finite number of blow-ups we remain only with singularities which are dicritical or of pure order one, as desired. □

The next step consists in a further reduction of the singularities of pure order one — but we need one more definition. Assume that the origin is a singularity of pure order one; by (2.3), once ℓ is chosen the eigenvalues of the linear map $\left|\begin{matrix} P_1^o \\ Q_1^o \end{matrix}\right|$ are independent of the coordinates; we shall call them the *eigenvalues* of the singularity. Since ℓ is defined up to units of $\mathcal{O}_2$, they are uniquely determined up to a non-zero scalar multiple.

Lemma 2.7: *Let O be a non-dicritical singularity. Then every singular direction $p_0 \in C_f$ of multiplicity one is a singularity of the blow-up of f of pure order one and with at least one non-zero eigenvalue.*

Proof: Up to a linear change of coordinates, we can assume $p_0 = [0 : 1]$. This means that

$$R(u,v) := vP^o_{\nu_o(f)}(u,v) - uQ^o_{\nu_o(f)}(u,v) = u\prod_{j=1}^k (\alpha_j u + \beta_j v)^{\mu_j},$$

with $\beta_1 \cdots \beta_k \neq 0$. Since

$$\begin{cases} \tilde{g}^o(z,w) = R(z,1) + O(w), \\ \tilde{h}^o(z,w) = w^{\nu(h)-\nu(f)+1} Q^o_{\nu_o(f)}(z,1) + O(w^{\nu(h)-\nu(f)+2}), \end{cases}$$

it follows immediately that $\nu(\tilde{g}^o) = 1$ and that $\prod_j \beta_j^{\mu_j} \neq 0$ is an eigenvalue of p_0. □

Let O be singular. We shall say that O is *irreducible* if:

(a) $\nu_o(f) = 1$, $\nu(\ell) \geq 1$, and
(b) the eigenvalues λ_1, λ_2 of O satisfy either:
 ($\star_1$) $\lambda_1, \lambda_2 \neq 0$ and λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbb{N}$; or
 ($\star_2$) $\lambda_1 \neq 0$, $\lambda_2 = 0$.

The second reduction theorem says that every non dicritical singularity of pure order one can be reduced to an irreducible singularity:

Theorem 2.8: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be tangent to the identity. Assume that O is an isolated singular point of f. Then there exists a complex 2-manifold M, a holomorphic projection $\pi: M \to \mathbb{C}^2$, and a holomorphic map $\tilde{f} \in \mathrm{End}(M, S)$, where $S = \pi^{-1}(O)$, satisfying the following properties:*

(i) $\pi|_{M\setminus S}: M \setminus S \to \mathbb{C}^2 \setminus \{O\}$ *is a biholomorphism;*
(ii) *S is either a point or the union of a finite number of projective lines intersecting each other transversally and at most in one point;*
(iii) $\pi \circ \tilde{f} = f \circ \pi$;

(iv) $\tilde f|_S = \mathrm{id}_S$;
(v) *the singular points of* $\tilde f$ *on* S *are isolated, and dicritical or irreducible.*

Proof: By Theorem 2.6, we can assume that all singularities are dicritical or of pure order one; to get the assertion we must show that by blowing up we may reduce all non-dicritical reducible singularities of pure order one to irreducible ones.

Assume p is such a singularity, and choose coordinates centered in p and adapted to the exceptional divisor (it can always be done because, by construction, the worst singularities in the exceptional divisor are normal crossings). We have three cases to consider:

(a) 0 *is the only eigenvalue.* Since the singularity has pure order one, up to a linear change of coordinates we can assume $g^o(x,y) = A_2(x,y)$ and $h^o(x,y) = x + B_2(x,y)$, with $\nu(A_2)$, $\nu(B_2) \ge 2$. In particular, $C_f = \{[0:1]\}$, and $\mu_{[0:1]} = 2$; moreover, necessarily $\kappa = \nu(\ell) \ge 1$ (because the singularity belongs to the exceptional divisor). Blowing-up we get

$$\begin{cases} \tilde g^o(z,w) = \dfrac{wA_0(z,w) - z\bigl(z + wB_0(z,w)\bigr)}{1 + w^\kappa \hat\ell(z,w)\bigl(z + wB_0(z,w)\bigr)} = -z^2 + wC_0(z,w), \\ \tilde h^o(z,w) = w\bigl(z + wB_0(z,w)\bigr), \end{cases}$$

where $A_0(z,w) = A_2(zw,w)/w^2$, $B_0(z,w) = B_2(zw,w)/w^2$ and $\nu(C_0) \ge 0$. Notice that the pure order of $\tilde f$ at $[0:1]$ can well be greater than one; we claim that blowing-up we can reduce all singularities to pure order one with at least one non-zero eigenvalue.

We shall prove the claim in the following more general situation:

$$\begin{cases} g^o(z,w) = -nz^2 + wA_0(z,w), \\ h^o(z,w) = w\bigl(z + wB_0(z,w)\bigr), \end{cases}$$

where $n \in \mathbb{N}^*$.

(a.1) $A_0(0,0) = a_0 \ne 0$. In this case we have only one singular direction, $[1:0]$, of multiplicity two. Blowing up again we find

$$\begin{cases} \tilde g^o(z,t) = -nz^2 + a_0 zt + O(z^2 t), \\ \tilde h^o(z,t) = (n+1)zt - a_0 t^2 + O(zt^2, z^2 t); \end{cases}$$

in particular, the equation of $C_{\tilde f}$ is $zt[2a_0 t - (2n+1)z] = 0$. This means that all singular directions have multiplicity one; by Lemma 2.7, another blow-up provides then singularities of pure order one with at least one non-zero eigenvalue — and thus we are outside case (a).

(a.2) $A_0(0,0) = 0$. Let us write $A_0(z,w) = a_1 z + a_2 w + A_2(z,w)$ and $B_0(z,w) = b_0 + B_1(z,w)$, with $\nu(A_2) \ge 2$ and $\nu(B_1) \ge 1$. Then

$$C_f = \{v\bigl(-(n+1)u^2 + (a_1 - b_0)uv + a_2 v^2\bigr) = 0\}.$$

Write $-(n+1)u^2 + (a_1 - b_0)uv + a_2 v^2 = -(n+1)(u - c_1 v)(u - c_2 v)$. We have two subcases:

(a.2.i) $c_1 \ne c_2$. Then we have three singular directions of multiplicity one; by Lemma 2.7 blowing up we end up with three singularities of pure order one and outside case (a).

(a.2.ii) $c_1 = c_2$. Lemma 2.7 already says that after blowing up $[1:0]$ will become a singularity of pure order one and not in case (a); we should check what happens to $[c_1:1]$. Since $c_1 = (a_1 - b_0)/2(n+1)$ and $-a_2 = (a_1 - b_0)^2/4(n+1)$, blowing up and then setting $s' = s - c_1$ we get

$$\begin{cases} \tilde g^o(s',w) = -(n+1)(s')^2 + w\tilde A_0(s',w), \\ \tilde h^o(s',w) = w\bigl(c_1 + b_0 + s' + w\tilde B_0(s',w)\bigr). \end{cases}$$

If $c_1 + b_0 \ne 0$ we have a singularity of pure order one with a non-zero eigenvalue, and we are outside case (a). If $c_1 + b_0 = 0$, and $\tilde A_0(0,0) \ne 0$, we are back in case (a.1). Finally, if $c_1 + b_0 = \tilde A_0(0,0) = 0$, we have a singularity of pure order two and of the same kind we are studying; but we already know that after a finite number of blow-ups every singularity must become of pure order one, and thus we shall eventually be outside of this case.

(b) $\lambda \neq 0$ *is the only eigenvalue.* Since the singularity has pure order one and it is non-dicritical, up to a linear change of coordinates we can assume $g^o(x,y) = \lambda x + y + A_2(x,y)$ and $h^o(x,y) = \lambda y + B_2(x,y)$, with $\nu(A_2)$, $\nu(B_2) \geq 2$. Then there is only one singular direction, $[1:0]$, of multiplicity two. Blowing up we find

$$\begin{cases} \tilde{g}^o(z,w) = \lambda z + zw + z^2\tilde{A}_0(z,w), \\ \tilde{h}^o(z,w) = -w^2 + z\tilde{B}_0(z,w); \end{cases}$$

therefore we are in case $(\star_2)$.

(c) *There are two distinct eigenvalues* $\lambda_1 \neq \lambda_2$. If one eigenvalue is zero we are in case $(\star_2)$; if both are non-zero and λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbb{N}$, we are in case $(\star_1)$. Suppose instead that $\lambda_1/\lambda_2 \in \mathbb{N}$ or $\lambda_2/\lambda_1 \in \mathbb{N}$, with $\lambda_1\lambda_2 \neq 0$; up to a linear change of coordinates, we can assume that $g^o(x,y) = \lambda x + A_2(x,y)$ and $h^o(x,y) = n\lambda y + B_2(x,y)$, with $\nu(A_2), \nu(B_2) \geq 2$, $\lambda \neq 0$ and $n \in \mathbb{N}$, $n \geq 2$; in particular, $C_f = \{[1:0],[0:1]\}$.

After blowing-up, in $[0:1]$ we have

$$\begin{cases} \tilde{g}^o(z,w) = (1-n)\lambda z + w\tilde{A}_0(z,w), \\ \tilde{h}^o(z,w) = n\lambda w + w^2\tilde{B}_0(z,w); \end{cases}$$

since $(1-n)/n < 0$, we are then in case $(\star_1)$.

The situation is slightly more complicated in $[1:0]$. Blowing up we get

$$\begin{cases} \tilde{g}^o(z,w) = \lambda z + z^2\tilde{A}_0(z,w), \\ \tilde{h}^o(z,w) = (n-1)\lambda w + z\tilde{B}_0(z,w). \end{cases}$$

This means that if $n > 2$ with $n-2$ blow-ups we can reduce n to 2 (producing $n-2$ singularities of type $(\star_1)$ along the way). If $n = 2$, we end up either with a dicritical singularity (if $\tilde{B}_0(0,0) = 0$) or in case (b) — and yet another blow-up lands us in case $(\star_2)$. □

This is not enough; we need to control the residual indeces with respect to the various branches of the exceptional divisor. The case $(\star_1)$ is relatively easy — but quite important:

Proposition 2.9: *Let p be an irreducible singularity of type $(\star_1)$ produced by Theorem 2.8. Then:*

(i) *If S denotes a branch of the exceptional divisor containing p, then $\iota_p(\tilde{f},S) \notin \mathbb{N}$.*
(ii) *If p is a corner, and S_1, S_2 are the branches of the exceptional divisor meeting transversally at p, then*

$$\iota_p(\tilde{f},S_1) \cdot \iota_p(\tilde{f},S_2) = 1.$$

Proof: (i) Choose a chart centered in p and adapted to S. Then we can write

$$\begin{cases} \tilde{g}(z,w) = w^\mu \ell_1(z,w)\big(a_{11}z + a_{12}w + A_2(z,w)\big), \\ \tilde{h}(z,w) = w^\mu \ell_1(z,w)\big(a_{21}z + a_{22}w + B_2(z,w)\big), \end{cases} \tag{2.4}$$

with $\mu \geq 1$, $(w,\ell_1) = 1$, $\nu(A_2), \nu(B_2) \geq 2$, and $\mathrm{sp}\big((a_{ij})\big) = \{\lambda_1, \lambda_2\}$. Since $\tilde{f}$ is not degenerate along S (we have never blown-up a dicritical singularity), we must have $a_{21} = 0$ and $B_2 = wB_1$ with $\nu(B_1) \geq 1$. In particular, then, a_{11}, a_{22} are the eigenvalues of the singularity, and $\iota_O(f,S) = a_{22}/a_{11} \notin \mathbb{N}$.

(ii) Choose a chart centered in p and adapted to S_1 and S_2. Then in (2.4) we must have $\ell_1 = z^\sigma \ell_2$, with $\sigma \geq 1$ and $(z,\ell_2) = 1$. Repeating the previous argument for both S_1 and S_2 we find $\iota_O(f,S_1) = a_{22}/a_{11}$ and $\iota_O(f,S_2) = a_{11}/a_{22}$, and we are done. □

We are now ready to prove the final version of the Reduction Theorem:

Theorem 2.10: (Reduction Theorem) *Let $f\in\mathrm{End}(\mathbb{C}^2,O)$ be tangent to the identity. Assume that O is an isolated singular point of f. Then there exists a complex 2-manifold M, a holomorphic projection $\pi\colon M\to\mathbb{C}^2$, and a holomorphic map $\tilde f\in\mathrm{End}(M,S)$, where $S=\pi^{-1}(O)$, satisfying the following properties:*

(i) $\pi|_{M\setminus S}\colon M\setminus S\to\mathbb{C}^2\setminus\{O\}$ *is a biholomorphism;*
(ii) *S is either a point or the union of a finite number of projective lines intersecting each other transversally and at most in one point;*
(iii) $\pi\circ\tilde f=f\circ\pi$*;*
(iv) $\tilde f|_S=\mathrm{id}_S$*;*
(v) *the singular points of $\tilde f$ on S are isolated, and dicritical or irreducible;*
(vi) *if $p\in S$ is a non-dicritical irreducible singular point of type $(\star_2)$, then the residual index of $\tilde f$ at p along at least one of the branches of S containing p is zero.*

Proof: Let $p\in S$ be a non-dicritical irreducible singularity of type $(\star_2)$, and choose a chart centered at p and adapted to a branch S_1 of S containing p. Then we can again write

$$\begin{cases}\tilde g(z,w)=w^\mu\ell_1(z,w)\bigl(a_{11}z+a_{12}w+A_2(z,w)\bigr),\\ \tilde h(z,w)=w^\mu\ell_1(z,w)\bigl(a_{21}z+a_{22}w+B_2(z,w)\bigr),\end{cases}$$

with $\mu\ge 1$, $(w,\ell_1)=1$, $\nu(A_2),\nu(B_2)\ge 2$, and $\mathrm{sp}\bigl((a_{ij})\bigr)=\{\lambda,0\}$ with $\lambda\ne 0$. As in the proof of Proposition 2.9, the non degeneracy of f implies $a_{21}=0$ and $B_2=wB_1$ with $\nu(B_1)\ge 1$; then we must have either $a_{11}=0$ or $a_{22}=0$ (but not both). We can tell apart these two cases in an intrinsic way as follows: the matrix (a_{ij}) has two distinct eigendirections, one for each eigenvalue. Then $a_{22}=0$ means that the branch S_1 we singled out is tangent to the eigendirection associated to the non-zero eigenvalue, whereas $a_{11}=0$ means that S_1 is transversal to it.

If S_1 is tangent to the eigendirection associated to the non-zero eigenvalue, that is $a_{22}=0$, we have

$$k(z)=\lim_{w\to 0}\frac{w^{\mu-1}\ell_1(z,w)B_1(z,w)}{w^{\mu-1}\ell_1(z,w)\bigl(a_{11}z+a_{12}w+A_2(z,w)\bigr)}=\frac{O(z)}{a_{11}z+O(z^2)}=\frac{O(1)}{a_{11}+O(z)},$$

and thus $\iota_p(\tilde f,S_1)=0$.

On the other hand, if S_1 is transversal to the eigendirection associated to the non-zero eigenvalue, that is $a_{11}=0$, then a quick computation yields $k(z)=\bigl(a_{22}+B_1(z,0)\bigr)/A_2(z,0)$, and thus in general we cannot say anything on the residual index.

Now, if p is a corner we can choose coordinates adapted to both the branches of S intersecting at p; this means that $\ell_1=z^\sigma\ell_2$ for a suitable $\sigma\ge 1$, and thus there is always at least one branch of S tangent to the eigendirection associated to the non-zero eigenvalue — and thus a branch such that the residual index is zero.

Finally, assume that p is not a corner, and that S_1 is transversal to the eigendirection associated to the non-zero eigenvalue; in particular, z does not divide ℓ_1. We have $C_f=\{[1:0],[-a_{12}/a_{22}:1]\}$. Blowing-up, near $[1:0]$ we find

$$\begin{cases}\tilde{\tilde g}(z,t)=z^\mu t^\mu\ell_1(z,zt)z\bigl(a_{12}t+A_1(z,t)\bigr),\\ \tilde{\tilde h}(z,t)=z^\mu t^\mu\ell_1(z,zt)\dfrac{a_{22}t+tB_1(z,zt)-a_{12}t^2-tA_1(z,t)}{1+z^\mu t^\mu\ell_1(z,zt)\bigl(a_{12}t+A_1(z,t)\bigr)},\end{cases}$$

where $A_1(z,t)=A_2(z,zt)/z$, and so we get a corner of type $(\star_2)$.

Finally, near $v_1=[-a_{12}/a_{22}:1]$ we have

$$\begin{cases}\tilde{\tilde g}(s,w)=w^\mu\ell_1(sw,w)\dfrac{a_{12}-a_{22}s+wA_0(s,w)-sB_1(sw,w)}{1+w^\mu\ell_1(sw,w)\bigl(a_{22}+B_1(sw,w)\bigr)},\\ \tilde{\tilde h}(s,w)=w^\mu\ell_1(sw,w)w\bigl(a_{22}+B_1(sw,w)\bigr),\end{cases}$$

where $wA_0(s,w)=A_2(sw,w)/w$. Setting $s=s'+a_{12}/a_{22}$, it is easy to see that v_1 is a $(\star_1)$ point with eigenvalues $\{-a_{22},a_{22}\}$, and not a corner. □

3. Dynamics

The next step in the continuous case would be to show the existence of invariant submanifolds passing through the singularity in the dicritical or in the $(\star_1)$ case — and this is the point where the continuous theory and the discrete theory actually differ. Indeed, although it is possible to find a formal power series expression for a holomorphic curve passing through the singularity and invariant under f, it turns out that this power series in general is *not* converging.

An example of this phenomenon is given by the following apparently tame $(\star_1)$ singularity:

$$\begin{cases} f_1(z,w) = z\exp(-\alpha w) + w^3 = z + w\bigl(-\alpha z + w^2 + O(zw)\bigr), \\ f_2(z,w) = \dfrac{w}{1+w} = w + w\bigl(-w + O(w^2)\bigr), \end{cases}$$

with α, $1/\alpha \notin \mathbb{N}^*$. Then it is not difficult to show that there is a power series $\eta(\zeta)$ such that the formal curve $\zeta \mapsto \bigl(\zeta\eta(\zeta), \zeta\bigr)$ is f-invariant; on the other hand, arguing as in [H1, p. 409] we see that any injective f-invariant holomorphic curve passing through the origin must be contained in $\mathrm{Fix}(f) = \{w = 0\}$, and thus the power series $\eta(\zeta)$ cannot be converging.

It turns out that the problem lies in assuming that the origin is inside the invariant curve. The correct replacement is the following: a *parabolic curve* for f is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^2$ such that:

(i) Δ is a simply connected domain with $0 \in \partial\Delta$;
(ii) φ is holomorphic, injective, continuous at the origin and such that $\varphi(0) = O$;
(iii) $f\bigl(\varphi(\Delta)\bigr) \subset \varphi(\Delta)$;
(iv) $f^n\bigl(\varphi(\zeta)\bigr) \to O$ as $n \to +\infty$ for any $\zeta \in \Delta$.

REMARK 3.1: It is not restrictive to assume that a parabolic curve is continuous up to the boundary. Indeed, by (ii) and (iii) the map $\Phi = \varphi^{-1} \circ f \circ \varphi$ is a holomorphic self-map of Δ; by (iv) the iterates of Φ converge to the origin. By Wolff's lemma, this implies that the horocycles centered at the origin are invariant under Φ; therefore the restriction of φ to any horocycle satisfies (i)–(iv) and it is continuous up to the boundary.

So we need to prove the existence of parabolic curves for dicritical or $(\star_1)$ singularities:

Theorem 3.1: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be tangent to the identity. Then:*

(i) *if O is a singularity of type $(\star_1)$ such that $\mathrm{Fix}(f)$ is smooth at the origin, then there exist $\nu(f) - 1$ parabolic curves for f;*
(ii) *if O is dicritical then there exist infinitely many parabolic curves for f.*

REMARK 3.2: If p is a $(\star_1)$ singularity, not a corner, obtained in the Reduction Theorem 2.10 starting from an isolated fixed point then p satisfies the conditions of Theorem 3.1.(i).

Proof: In case (i), after possibly a change of coordinates we can write

$$\begin{cases} f_1(z,w) = z + \ell(z,w)[\lambda_1 z + A_2(z,w)], \\ f_2(z,w) = w + \ell(z,w)[\lambda_2 w + B_2(z,w)], \end{cases}$$

with $\nu(A_2)$, $\nu(B_2) \geq 2$, $\lambda_1\lambda_2 \neq 0$, λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbb{N}$, and $\ell(z,w) = (az + bw)^\kappa$ with $\kappa = \nu(f) - 1 \geq 1$ and $a \neq 0$. Blowing up and focusing our attention to the chart containing $[1:0]$ we get

$$\begin{cases} \tilde f_1(z,y) = z + \lambda_1 a^\kappa z^{\kappa+1} + O(z^{\kappa+2}, z^{\kappa+1}y), \\ \tilde f_2(z,y) = y[1 + (\lambda_2 - \lambda_1)a^\kappa z^\kappa + O(z^{\kappa+1}, z^\kappa y)] + O(z^{\kappa+1}). \end{cases}$$

Setting $x = \alpha z$, where $\alpha^\kappa = -\lambda_1 a^\kappa$, we reduce to

$$\begin{cases} \tilde f_1(x,y) = x - x^{\kappa+1} + O(x^{\kappa+2}, x^{\kappa+1}y), \\ \tilde f_2(x,y) = y\,\bigl[1 - (\lambda_2/\lambda_1 - 1)\,x^\kappa + O(x^{\kappa+1}, x^\kappa y)\bigr] + O(x^{\kappa+1}). \end{cases} \tag{3.1}$$

Notice that a parabolic curve for $\tilde f$ cannot intersect the exceptional divisor, since all points of the curve are attracted to the origin. Therefore the push-forward of a parabolic curve for $\tilde f$ is a parabolic curve for f (tangent to $[1:0]$ at the origin), and (i) will follow if we prove the existence of κ parabolic curves at the origin for $\tilde f$.

In case (ii) we can write

$$\begin{cases} f_1(z,w) = z + \ell(z,w)[P_\mu(z,w) + A_{\mu+1}(z,w)], \\ f_2(z,w) = w + \ell(z,w)[Q_\mu(z,w) + B_{\mu+1}(z,w)], \end{cases}$$

with $\mu \ge 1$, $\nu(A_{\mu+1})$, $\nu(B_{\mu+1}) \ge \mu+1$, and $zQ_\mu - wP_\mu \equiv 0$. Writing $\ell(z,w) = R_\kappa(z,w) + C_{\kappa+1}(z,w)$ with $\kappa+\mu = \nu(f) \ge 2$ and $\nu(C_{\kappa+1}) \ge \kappa+1$, we are interested to the directions $[u_0 : v_0] \in \mathbb{P}^1$ such that

$$R_\kappa(u_0,v_0)P_\mu(u_0,v_0) \ne 0. \tag{3.2}$$

Up to a linear change of coordinates we can assume $[u_0 : v_0] = [1:0]$, but what we are going to say applies to the other directions too.

Blowing up and focusing our attention to the chart containing $[1:0]$ we get

$$\begin{cases} \tilde f_1(z,y) = z + R_\kappa(1,y)P_\mu(1,y)z^\nu + O(z^{\nu+1}), \\ \tilde f_2(z,y) = y[1 + O(z^\nu)] + O(z^\nu), \end{cases}$$

where $\nu = \nu(f)$. Setting $x = \alpha z$, with $\alpha^{\nu-1} = -R_\kappa(1,0)P_\mu(1,0)$, we reduce to

$$\begin{cases} \tilde f_1(x,y) = x - x^\nu + O(x^{\nu+1}, x^\nu y), \\ \tilde f_2(x,y) = y[1 + O(x^\nu)] + O(x^\nu). \end{cases} \tag{3.3}$$

Again, the push-forward of any parabolic curve for $\tilde f$ will be a parabolic curve for f tangent to $[1:0]$; therefore if we prove the existence of parabolic curves for $\tilde f$ we have proved (ii), because we can repeat the argument for the infinite number of directions satisfying (3.2).

Summing up, we must prove the existence of r parabolic curves at the origin for a map of the form

$$\begin{cases} f_1(z,w) = z - z^{r+1} + O(z^{r+2}, z^{r+1}w), \\ f_2(z,w) = w\bigl(1 - \lambda z^r + O(z^{r+1}, z^r w)\bigr) + z^{r+1}\psi_r(z), \end{cases} \tag{3.4}$$

where $r \ge 1$, $\lambda \notin \mathbb{N}^*$ and $\psi_r \in \mathcal{O}_1$. This is a consequence of the general results of [H1], adapted as in [H2] if $r > 1$. We describe here a slightly simplified approach, which is enough for our aims.

First of all, since $\lambda \ne 1$, a linear change of coordinates allows to replace $\psi_r(z)$ in (3.4) by $z\psi_{r+1}(z)$. Then blowing up and checking nearby $[1:0]$ we see that $\tilde f$ is still of the form (3.4) but with $\lambda - 1$ instead of λ. This means that after a finite number of blow-ups and linear change of coordinates we can assume $\operatorname{Re}\lambda < 0$ and $\psi_r = z\psi_{r+1}$ in (3.4). Furthermore, the change of variables $Z = z$, $W = w + \bigl(\psi_{r+1}(0)/(\lambda-2)\bigr)z^2$ allows to replace $z\psi_{r+1}$ by $z^2\psi_{r+2}$, and thus we have

$$\begin{cases} z_1 := f_1(z,w) = z - z^{r+1} + O(z^{r+2}, z^{r+1}w), \\ w_1 := f_2(z,w) = w\bigl(1 - \lambda z^r + O(z^{r+1}, z^r w)\bigr) + z^{r+3}\psi_{r+2}(z). \end{cases} \tag{3.5}$$

Now set $D_{\delta,r} = \{\zeta \in \mathbb{C} \mid |\zeta^r - \delta| < \delta\}$. This set has r connected (and simply connected) components, all of them with the origin in the boundary. Put $\mathcal{E}(\delta) = \{u \in \mathrm{Hol}(D_{\delta,r}, \mathbb{C}^2) \mid u(\zeta) = \zeta^2 u^o(\zeta),\ \|u^o\|_\infty < \infty\}$; it is a Banach space with the norm $\|u\|_{\mathcal{E}(\delta)} = \|u^o\|_\infty$. For $u \in \mathcal{E}(\delta)$ put $f^u(\zeta) = f_1\bigl(\zeta, u(\zeta)\bigr)$. The classical Fatou theory for maps of the form $f(\zeta) = \zeta - \zeta^{r+1} + O(\zeta^{r+2})$ shows that there exists a $\delta_0 = \delta_0(\|u^o\|_\infty) > 0$ such that if $0 < \delta < \delta_0$ then f^u sends every component of $D_{\delta,r}$ into itself, and $|(f^u)^n(\zeta)| = O(1/n^{1/r})$.

Assume we have found $u \in \mathcal{E}(\delta)$ such that

$$u\Bigl(f_1\bigl(\zeta, u(\zeta)\bigr)\Bigr) = f_2\bigl(\zeta, u(\zeta)\bigr) \tag{3.6}$$

for all $\zeta \in D_{\delta,r}$; then the restriction of $\varphi(\zeta) = \bigl(\zeta, u(\zeta)\bigr)$ to any component of $D_{\delta,r}$ is a parabolic curve for f.

So we must find a solution of (3.6). If f is given by (3.5), and z, z_1 belongs to the same component of $D_{\delta,r}$, we can define

$$H(z,w) = w - \frac{z^\lambda}{z_1^\lambda}\, w_1 = O(z^{r+1}w, z^r w^2, z^{r+3}); \tag{3.7}$$

then for $u \in \mathcal{E}(\delta)$ we set

$$Tu(\zeta_0) = \zeta_0^\lambda \sum_{n=0}^\infty \zeta_n^{-\lambda} H\bigl(\zeta_n, u(\zeta_n)\bigr),$$

where $\zeta_n = (f^u)^n(\zeta_0)$. If we choose u and δ so that $\|u^o\| \le c_0$ and $\delta \le \delta_0(c_0)$, then $H\bigl(\zeta_n, u(\zeta_n)\bigr)$ is well-defined for any $\zeta_0 \in D_{\delta,r}$; furthermore, since $\operatorname{Re}\lambda < 0$, the series is normally convergent in $D_{\delta,r}$, and $Tu \in \mathcal{E}(\delta)$. Furthermore, it is not difficult to see that u is a fixed point of T iff it satisfies (3.6); therefore we are left to finding a fixed point for T.

Take $\delta < \delta_0(1)$ and $u \in \mathcal{E}(\delta)$ with $\|u^o\|_\infty \le 1$. Then $\zeta_1 = \zeta_0 - \zeta_0^{r+1} - \zeta_0^{r+2}\psi_u(\zeta_0)$, where $\|\psi_u\|_\infty$ is bounded independently of u. Then

$$\frac{1}{\zeta_1^r} = \frac{1}{\zeta_0^r} + r + \zeta_0\theta_u(\zeta_0), \tag{3.8}$$

with again $\|\theta_u\|_\infty$ bounded independently of u. Summing up from 1 to n we get

$$\frac{1}{\zeta_n^r} = \frac{1}{\zeta_0^r}(1 + nr\zeta_0^r)\left[1 + \frac{\zeta_0^r}{1+nr\zeta_0^r}\sum_{j=0}^{n-1}\zeta_j\theta_u(\zeta_j)\right].$$

Since $\zeta_j = O(1/j^{1/r})$, the quantity in square brackets is uniformly (with respect to u and ζ_0) close to 1 if δ is small enough; therefore we have

$$|\zeta_n|^r \le 2\frac{|\zeta_0|^r}{|1+nr\zeta_0^r|} \tag{3.9}$$

as soon as δ is small enough. From this it follows the existence for any $s > r$ of a constant $C_s = C_s(\delta) \ge 1$ such that

$$\sum_{n=0}^\infty |\zeta_n|^s \le C_s|\zeta_0|^{s-r}. \tag{3.10}$$

Now assume that $\|u^o\|_\infty \le 1$ and $|u'(\zeta)| \le |\zeta|$ for all $\zeta \in D_{\delta,r}$; in particular, $\|(u^o)'\|_\infty \le 3$, and $\|\theta_u'\|_\infty$ too is bounded indipendently of u. Let $K = \|(\zeta\theta_u)'\|_\infty$. Differentiating (3.8) with respect to ζ_0 we get

$$\frac{d\zeta_1}{d\zeta_0} = \frac{\zeta_1^{r+1}}{\zeta_0^{r+1}}\left[1 - \frac{\zeta_0^{r+1}}{r}\bigl(\theta_u(\zeta_0) + \zeta_0\theta_u'(\zeta_0)\bigr)\right].$$

In particular if $\delta < r/(2KC_{r+1})$ we have

$$\left|\frac{d\zeta_1}{d\zeta_0}\right| \le 2\frac{|\zeta_1|^{r+1}}{|\zeta_0|^{r+1}}\,.$$

We can now argue by induction. Assume that

$$\left|\frac{d\zeta_j}{d\zeta_0}\right| \le 2\frac{|\zeta_j|^{r+1}}{|\zeta_0|^{r+1}} \tag{3.11}$$

for $j = 1, \ldots, n-1$. Since

$$\frac{1}{\zeta_n^r} = \frac{1}{\zeta_0^r} + nr + \sum_{j=0}^{n-1}\zeta_j\theta_u(\zeta_j),$$

differentiating with respect to ζ_0 we get

$$\frac{d\zeta_n}{d\zeta_0} = \frac{\zeta_n^{r+1}}{\zeta_0^{r+1}}\left[1 - \frac{\zeta_0^{r+1}}{r}\sum_{j=0}^{n-1}\hat\theta'_u(\zeta_j)\frac{d\zeta_j}{d\zeta_o}\right],$$

where $\hat\theta_u(\zeta) = \zeta\theta_u(\zeta)$. Now, (3.10) and (3.11) yield

$$\left|\sum_{j=0}^{n-1}\hat\theta'_u(\zeta_j)\frac{d\zeta_j}{d\zeta_o}\right| \le \frac{2KC_{r+1}}{|\zeta_0|^r};$$

therefore again if $\delta < r/(2KC_{r+1})$ we get (3.11) for $j = n$ too.

Fix then δ small, and take $u \in \mathcal{E}(\delta)$ with $\|u^o\|_\infty \le 1$. Then (3.7) and (3.10) yield

$$|Tu(\zeta)| \le K_1|\zeta|^3,$$

for a suitable $K_1 \ge 1$; in particular, if $\delta < 1/K_1$ we get $\|(Tu)^o\|_\infty \le 1$. Analogously, if moreover u satisfies $|u'(\zeta)| \le |\zeta|$, then (3.7), (3.10) and (3.11) yield

$$\left|\frac{dTu}{d\zeta}(\zeta)\right| \le K_2|\zeta|^2;$$

therefore if $\delta < 1/K_2$ we get $|(Tu)'(\zeta)| \le |\zeta|$. This means that we can choose $\delta > 0$ so small that T sends into itself the convex closed set

$$\mathcal{F}(\delta) = \{u \in \mathcal{E}(\delta) \mid \|u^o\|_\infty \le 1,\ |u'(\zeta)| \le |\zeta|\}.$$

Then it suffices to show that, for δ small enough, T is a contraction on $\mathcal{F}(\delta)$. And this can be done as in [H1, Proposition 4.8], using (3.10) as before. □

We have finally collected all we need to prove our main theorem:

Theorem 3.2: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be tangent to the identity, and assume that the origin is an isolated fixed point. Then there exist (at least) $\nu(f) - 1$ parabolic curves at the origin for f.*

Proof: First of all we apply the Reduction Theorem 2.10, and replace f by $\tilde f \in \mathrm{End}(M, S)$. As already remarked in the proof of Theorem 3.1, the push-forward of any parabolic curve for $\tilde f$ will be a parabolic curve for f. Notice furthermore that, by Lemma 2.2.(ii) and (2.1), the order of $\tilde f$ at any singular point is at least $\nu(f)$.

If $\tilde f$ has a dicritical singularity, by Theorem 3.1 we are done. If $p \in S$ is a singularity which is not a corner and of type $(\star_1)$, we are done again. The only possibility left is that no singularity is dicritical, and that if $p \in S$ is a singularity which is not a corner, it is necessarily of type $(\star_2)$, and thus its residual index with respect to S is zero. Therefore to conclude we must prove that *if no singularity of $\tilde f$ is dicritical, then there is at least a singularity $p \in S$ which is not a corner and with residual index different from zero.*

Assume then, by contradiction, that the singularities of $\tilde f$ are only non-dicritical corners or of type $(\star_2)$. We have proven the following formal properties of the residual index:

(i) the sum of the residual indeces over a branch S_1 of S is equal to the first Chern class of the normal bundle of S_1 in M (Theorem 1.2);

(ii) if p is a singularity belonging to a branch S_1, then the residual index of the corner over p along the proper transform of S_1 is one less the residual index of p along S_1 (Proposition 1.3.(iii));

(iii) the product of the residual indeces along the two branches of a corner of type $(\star_1)$ is one (Proposition 2.9);

(iv) at least one of the residual indeces along the two branches of a corner of type $(\star_2)$ is zero (Theorem 2.10.(vi)).

Exactly as in [CS], the main consequence of these properties (under the to-be-proven-contradictory assumption) is that the residual indeces $\lambda_1, \dots, \lambda_k$ of the singular directions $p_1, \dots, p_k$ of the original map f with respect to the exceptional divisor of the first blow-up of the origin are completely determined by the geometrical combinatorics of the resolution S (where by geometrical combinatorics we mean the relative positions of the branches of S, the Chern classes of their normal bundles, and the placement of the $(\star_2)$ corners). In particular, arguing as in [CS, Proposition 3.3] we see that $\lambda_1, \dots, \lambda_k$ must be non-negative rational numbers; but $\lambda_1 + \cdots + \lambda_k = -1$ by Theorem 1.2, and this is a contradiction. □

Actually, the last part of the argument yields a generalization of Theorem 0.2 to singular directions whose residual index is not a non-negative rational number:

Corollary 3.3: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be tangent to the identity, and assume that the origin is an isolated fixed point. Let $[v] \in \mathbb{P}^1$ be a singular direction of f such that $\iota_{[v]}(\tilde{f}, \mathbb{P}^1) \notin \mathbb{Q}^+$ (where here $\mathbb{P}^1$ is the exceptional divisor of the blow-up of the origin, and $\tilde{f}$ is the blow-up of f). Then there are $\nu(f) - 1$ parabolic curves tangent to $[v]$ at the origin.*

Proof: The point is that if applying the Reduction Theorem 2.10 to $\tilde{f}$ at $[v]$ we end up only with non-dicritical corners or singularities of type $(\star_2)$, then the argument quoted at the end of the previous proof forces $\iota_{[v]}(\tilde{f}, \mathbb{P}^1) \in \mathbb{Q}^+$. Therefore we must obtain a dicritical singularity or a non-corner of type $(\star_1)$ — and thus at least $\nu(\tilde{f}) - 1$ parabolic curves for $\tilde{f}$ at $[v]$. Blowing down we then get at least $\nu(f) - 1$ parabolic curves for f at the origin, as claimed. □

As a final application, we can prove the existence of parabolic curves even when df_O is not diagonalizable:

Corollary 3.4: *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be such that $df_O = J_2$, the canonical Jordan matrix associated to the eigenvalue 1, and assume that the origin is an isolated fixed point. Then there is at least one parabolic curve tangent to $[1:0]$ for f at the origin.*

Proof: Write $f = (f_1, f_2)$ and

$$\begin{aligned} f_1(z,w) &= z + w + a^1_{11} z^2 + 2a^1_{12} zw + a^2_{22} w^2 + \cdots, \\ f_2(z,w) &= w + a^2_{11} z^2 + 2a^2_{12} zw + a^2_{22} w^2 + a^2_{111} z^3 + \cdots \end{aligned}$$

In [A] we proved the existence of (at least) one parabolic curve for f at the origin but for the case $a^2_{11} = 0$, $a^1_{11} + a^2_{12} = 0$ and $(a^1_{11} - a^2_{12})^2 + 2a^2_{111} = 0$. In this case, blowing up the origin and looking at a neighbourhood of $[1:0]$ (the only fixed point of the blow-up $\tilde{f}$ of f) we find that $\tilde{f}$ is of the form

$$\tilde{f}(z_1, z_2) = \big(z_1 + \alpha z_1^2 + z_1 z_2 + O(\|z\|^3), z_2 - 2\alpha^2 z_1^2 - 3\alpha z_1 z_2 - z_2^2 + O(\|z\|^3)\big),$$

for some $\alpha \in \mathbb{C}$. This map has two singular directions, $[1 : -\alpha]$ and $[0 : 1]$. The latter is tangent to the exceptional divisor (which is a parabolic curve for $\tilde{f}$) and thus it should be discarded, because it is killed blowing down. But the former, even though as characteristic direction is degenerate, gives rise to a honest parabolic curve. In fact, the residual index of $\tilde{f}$ at $[1 : -\alpha]$ with respect to the exceptional divisor is $-1/2$, and so we can apply Corollary 3.3. This parabolic curve is not tangent to the exceptional divisor, and thus it can be blown down, producing the parabolic curve tangent to $[1 : 0]$ we were looking for. □

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