Chapter 2.2
The ball

In the first part of this book, the unit disk Δ played quite a distinguished role: it was the model space of the whole theory. It is natural then to conjecture that the euclidean unit ball B^n of $\mathbf{C}^n$ will hold a similar position in this part of the book. Well; this is partly true and partly false. On one side, the unit ball is the simplest example of strongly pseudoconvex domain, and the theory on it is a good representative of the generic situation (this is the spirit of Rudin [1980]). On the other side, every hyperbolic Riemann surface was obtained by the unit disk following a fixed procedure, whereas nothing of this kind is even vaguely true in several variables. The unit ball is more an ideal model to follow than an effective tool for proving general theorems. Furthermore, the existence of a unique universal covering space for all hyperbolic Riemann surfaces allowed us to avoid too many topological considerations; in several variables, the absence of such a device will force us to focus on domains without topological obstructions, like convex domains.

Nevertheless, the unit ball is both an instructive and useful example, and we decided to devote a whole chapter to it. We shall deal with the main themes of this book — iteration theory, angular derivatives and common fixed points —, and the rest of this work will essentially be a description of how to extend the theorems proved for the ball to more general domains.

We shall start with a thorough discussion of $\mathrm{Aut}(B^n)$; then, after talking about Schwarz's lemma in B^n and horospheres (the multidimensional version of the horocycles), we shall drive toward angular derivatives, passing through a discussion of Lindelöf's theorem in B^n and of several extensions of the non-tangential limit. Next, iteration theory: we shall obtain a complete analogue of the Wolff-Denjoy theorem, anticipating the general discussion of chapter 2.4. Finally, common fixed points: we shall both generalize Shields' theorem and make up a fixed point for every compact group of automorphisms of B^n.

Summing up, this chapter will give you a fair idea of what we shall do in the rest of the book. How we shall do it, however, will be revealed only starting from the next chapter...

2.2.1 *The automorphism group*

So we begin the study of our main example, the *unit ball* of $\mathbf{C}^n$

$$B^n = \{z \in \mathbf{C}^n \mid \|z\| < 1\},$$

where $\|\cdot\|$ is the usual euclidean norm $\|z\| = \big(\sum_{j=1}^n |z_j|^2\big)^{1/2}$, induced by the canonical hermitian product $(z, w) = \sum_{j=1}^n z_j \overline{w_j}$. This section is devoted to the construction and an accurate examination of the automorphism group of B^n. As usual, the focus is on the fixed point sets.

First of all, we want to explicitely describe $\mathrm{Aut}(B^n)$. For reasons we shall discuss in the next section, this time Schwarz's lemma is not the natural tool. The right replacement is Corollary 2.1.23, whereby the isotropy group of the origin is $\mathbf{U}(n)$, the group of $n\times n$ unitary matrices. So to compute $\mathrm{Aut}(B^n)$ we need only a transitive subset of automorphisms.

Let $a \in \mathbf{C}^n$, $a \neq 0$, and define P_a, $Q_a\colon \mathbf{C}^n \to \mathbf{C}^n$ by

$$P_a(z) = \frac{(z,a)}{(a,a)}a \qquad \text{and} \qquad Q_a(z) = z - P_a(z).$$

P_a is the orthogonal projection on the subspace generated by a, and Q_a is the projection on the orthogonal complement. In particular, both P_a and Q_a send B^n into itself.

If $a \in B^n$, put $s_a = (1 - \|a\|^2)^{1/2}$ and define $\gamma_a\colon B^n \to \mathbf{C}^n$ by

$$\gamma_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - (z,a)}. \tag{2.2.1}$$

Note that if $n = 1$ then γ_a is an automorphism of Δ; see (1.1.3).

Lemma 2.2.1: *For every $a \in B^n$, $a \neq 0$, we have:*

(i) *$\gamma_a(0) = a$ and $\gamma_a(a) = 0$;*

(ii) *$d(\gamma_a)_0 = -s_a^2 P_a - s_a Q_a$ and $d(\gamma_a)_a = -P_a/s_a^2 - Q_a/s_a$;*

(iii) *for all z, $w \in \overline{B^n}$ we have*

$$1 - \big(\gamma_a(z), \gamma_a(w)\big) = \frac{\big(1-(a,a)\big)\big(1-(z,w)\big)}{\big(1-(z,a)\big)\big(1-(a,w)\big)}; \tag{2.2.2}$$

(iv) *for all $z \in \overline{B^n}$ we have*

$$1 - \|\gamma_a(z)\|^2 = \frac{(1-\|a\|^2)(1-\|z\|^2)}{|1-(z,a)|^2}; \tag{2.2.3}$$

(v) *γ_a is an involution, that is $\gamma_a \circ \gamma_a = \mathrm{id}_{B^n}$;*

(vi) *γ_a extends to a homeomorphism of $\overline{B^n}$ onto $\overline{B^n}$, and is an automorphism of B^n.*

Proof: (i) Obvious.

(ii) For any $z \in \overline{B^n}$ (2.2.1) can be rewritten in the form

$$\begin{aligned} \gamma_a(z) &= \big[1 + (z,a) + (z,a)^2 + \cdots\big]\big[a - (P_a + s_a Q_a)(z)\big] \\ &= \gamma_a(0) + (z,a)a - (P_a + s_a Q_a)(z) + O(\|z\|^2) \\ &= \gamma_a(0) - (s_a^2 P_a + s_a Q_a)(z) + O(\|z\|^2), \end{aligned}$$

and we get the first formula. The second one follows from

$$\gamma_a(a+h) = \frac{-P_a(h) - s_a Q_a(h)}{s_a^2 - (h,a)}.$$

(iii) Since P_a and Q_a are self-adjoint projections, we have

$$\begin{aligned}1-\big(\gamma_a(z),\gamma_a(w)\big) &= 1-\frac{\big(a-P_a(z),a-P_a(w)\big)+s_a^2\big(Q_a(z),Q_a(w)\big)}{\big(1-(z,a)\big)\big(1-(a,w)\big)}\\ &= 1-\frac{\big(a-P_a(z),a-w\big)+\big(1-(a,a)\big)\big(z-P_a(z),w\big)}{\big(1-(z,a)\big)\big(1-(a,w)\big)}\\ &= \frac{\big(1-(a,a)\big)\big(1-(z,w)\big)}{\big(1-(z,a)\big)\big(1-(a,w)\big)}.\end{aligned}$$

(iv) By (iii), with $z=w$.

(v) Let $\psi=\gamma_a\circ\gamma_a$. Then $\psi(0)=0$ and $d\psi_0 = d\gamma_a(a)\cdot d\gamma_a(0)=P_a+Q_a=\mathrm{id}$, for $P_a^2=P_a$, $Q_a^2=Q_a$ and $P_aQ_a=Q_aP_a=0$. By Theorem 2.1.21, $\psi=\mathrm{id}_{B^n}$.

(vi) γ_a sends B^n into B^n and ∂B^n into ∂B^n, by (iv). It follows from (v) that γ_a is invertible, and we are done, **q.e.d.**

Hence the automorphism group is given by

Corollary 2.2.2: *Every $\gamma\in\mathrm{Aut}(B^n)$ is of the form*

$$\gamma = U\gamma_a,$$

where $a=\gamma^{-1}(0)$ and $U\in\mathbf{U}(n)$. In particular, $\mathrm{Aut}(B^n)$ *acts transitively on B^n, and every element of* $\mathrm{Aut}(B^n)$ *extends continuously to a homeomorphism of $\overline{B^n}$ onto itself.*

Proof: The map $\gamma\circ\gamma_a$ is an automorphism of B^n that fixes the origin; hence, by Corollary 2.1.23, it is linear. Therefore it should be unitary, and the assertion follows, **q.e.d.**

We shall see later that $\mathrm{Aut}(B^n)$ acts doubly transitively on ∂B^n.

A consequence of (2.2.2) and Corollary 2.2.2 is that for every $\gamma\in\mathrm{Aut}(B^n)$ we have

$$\forall z,\,w\in\overline{B^n}\qquad 1-\big(\gamma(z),\gamma(w)\big)=\frac{\big(1-(a,a)\big)\big(1-(z,w)\big)}{\big(1-(z,a)\big)\big(1-(a,w)\big)},\tag{2.2.4}$$

where $a=\gamma^{-1}(0)$.

Another consequence that we shall need in the next chapter is:

Corollary 2.2.3: *Take $\gamma\in\mathrm{Aut}(B^n)$ and set $a=\gamma^{-1}(0)$. Then*

$$\forall z\in B^n\qquad \big|\det{(d\gamma)_z}\big|^2=\left(\frac{1-\|a\|^2}{|1-(z,a)|^2}\right)^{n+1}.$$

Proof: Fix $z\in B^n$, and set $w=\gamma(z)$. Then 0 is a fixed point of $\gamma_w\circ\gamma\circ\gamma_z$ and so $\gamma=\gamma_wU\gamma_z$ for some $U\in\mathbf{U}(n)$. In particular,

$$(d\gamma)_z=(d\gamma_w)_0U(d\gamma_z)_z.$$

By Lemma 2.2.1.(ii), $(d\gamma_w)_0$ has a one-dimensional eigenspace of eigenvalue $-s_w^2$, and a $(n-1)$-dimensional eigenspace of eigenvalue $-s_w$, so that $\det(d\gamma_w)_0 = (-1)^n s_w^{n+1}$. Analogously, $\det(d\gamma_z)_z = (-1)^n s_z^{-n-1}$; hence

$$|\det(d\gamma)_z|^2 = \left(\frac{1-\|\gamma(z)\|^2}{1-\|z\|^2}\right)^{n+1},$$

and the assertion follows from (2.2.4), **q.e.d.**

Exactly as for $n=1$, there is a different realization of B^n which is often useful, for instance to study $\mathrm{Aut}(B^n)$. It is the *Siegel upper half-space* $H^n \subset \mathbf{C}^n$ defined by

$$H^n = \bigl\{w \in \mathbf{C}^n \bigm| \mathrm{Im}\, w_1 > \|w'\|^2\bigr\},$$

where for every $w = (w_1, w_2, \ldots, w_n) \in \mathbf{C}^n$ we set $w' = (w_2, \ldots, w_n)$.

The *Cayley transform* $\Psi\colon B^n \to H^n$ given by

$$\Psi(z) = \left(i\,\frac{1+z_1}{1-z_1}, \frac{iz_2}{1-z_1}, \ldots, \frac{iz_n}{1-z_1}\right)$$

is, exactly as in the one dimensional case, a biholomorphism between B^n and H^n, with inverse given by

$$\Psi^{-1}(w) = \left(\frac{w_1-i}{w_1+i}, \frac{2w_2}{w_1+i}, \ldots, \frac{2w_n}{w_1+i}\right).$$

We notice incidentally that $\Psi = (\Psi_1, \ldots, \Psi_n)$ satisfies

$$\forall z \in B^n \qquad \mathrm{Im}\, \Psi_1(z) - \sum_{j=2}^{n} |\Psi_j(z)|^2 = \frac{1-\|z\|^2}{|1-z_1|^2}. \tag{2.2.5}$$

The boundary of H^n in $\mathbf{C}^n$ is given by $\partial H^n = \bigl\{w \in \mathbf{C}^n \bigm| \mathrm{Im}\, w_1 = \|w'\|^2\bigr\}$. We shall denote by $\overline{H^n}$ the one-point compactification of $H^n \cup \partial H^n$; it is easily seen that Ψ extends to a homeomorphism between $\overline{B^n}$ and $\overline{H^n}$ sending $(1, 0, \ldots, 0)$ in the point at infinity of $\overline{H^n}$.

There are two subgroups of $\mathrm{Aut}(H^n)$ which are worth mentioning. The first one is composed by the *(non-isotropic) dilatations* δ_t, where $t > 0$, given by

$$\delta_t(w) = (t^2 w_1, tw').$$

Every δ_t fixes only 0 and ∞; hence $\Psi^{-1} \circ \delta_t \circ \Psi$ is an automorphism of B^n fixing no points of B^n and exactly two points of ∂B^n. Moreover, $(\delta_t)^{-1} = \delta_{1/t}$.

The second subgroup consists of the *translations* h_a, where $a \in \partial H^n$, given by

$$h_a(w) = \bigl(w_1 + a_1 + 2i(w', a'), w' + a'\bigr). \tag{2.2.6}$$

Every h_a fixes only ∞; hence $\Psi^{-1} \circ h_a \circ \Psi$ is an automorphism of B^n fixing no points of B^n and exactly one point of ∂B^n. Moreover, $(h_a)^{-1} = h_{\tilde a}$, where $\tilde a = (-\overline{a_1}, -a')$.

We can use dilations and translations to describe $\mathrm{Aut}(H^n)$:

Proposition 2.2.4: *Every automorphism γ of H^n is of the form*

$$\gamma = \delta_t \circ h_a \circ \mu_U,$$

for suitable $U \in \mathbf{U}(n)$, $t > 0$ and $a \in \partial H^n$, where $\mu_U(w) = \Psi\big(U\Psi^{-1}(w)\big)$ is an automorphism of H^n fixing $(i, 0, \ldots, 0)$.

Proof: Let $b = (b_1, \ldots, b_n) = \gamma(i, 0, \ldots, 0)$. Since $b \in H^n$, we have $\operatorname{Im} b_1 > \|b'\|^2$; set $t = \big(\operatorname{Im} b_1 - \|b'\|^2\big)^{1/2}$. Now define $a = (a_1, \ldots, a_n) \in \partial H^n$ by

$$\begin{aligned} a_1 &= \frac{1}{t^2}\big[\operatorname{Re} b_1 + i\|b'\|^2\big], \\ a' &= b'/t, \end{aligned}$$

and set $\tilde{a} = (-\overline{a_1}, -a')$. Then $h_{\tilde{a}} \circ \delta_{1/t}(b) = (i, 0, \ldots, 0)$; hence, by Corollary 2.2.2, $h_{\tilde{a}} \circ \delta_{1/t} \circ \gamma = \mu_U$ for a suitable $U \in \mathbf{U}(n)$, and $\gamma = \delta_t \circ h_a \circ \mu_U$, **q.e.d.**

A first corollary is the promised

Corollary 2.2.5: $\operatorname{Aut}(B^n)$ *acts doubly transitively on ∂B^n.*

Proof: Since $\operatorname{Aut}(B^n)$ contains $\mathbf{U}(n)$, it suffices to show that for every $x \in \partial B^n$, $x \neq e_1$, there is $\gamma \in \operatorname{Aut}(B^n)$ such that $\gamma(e_1) = e_1$ and $\gamma(-e_1) = x$, where $e_1 = (1, 0, \ldots, 0)$. Using the Cayley transform, we can rephrase the problem in the following terms: given $a \in \partial H^n$ find $\gamma \in \operatorname{Aut}(H^n)$ such that $\gamma(\infty) = \infty$ and $\gamma(0) = a$. But then it suffices to take $\gamma = h_a$, **q.e.d.**

Now we want to study the fixed point set of an automorphism of B^n. We saw that an automorphism of B^n can have no fixed points in B^n. On the other hand, if $U \in \mathbf{U}(n) \subset \operatorname{Aut}(B^n)$ its fixed point set is the intersection of B^n with a complex linear subspace of $\mathbf{C}^n$. To describe the general case, we need a couple of definitions.

An *affine subspace* of $\mathbf{C}^n$ is the translation of a linear subspace; an *affine subset* of B^n is the intersection of B^n with an affine subspace of $\mathbf{C}^n$. If L is a linear subspace of $\mathbf{C}^n$ and $c_0 \in B^n$, then the affine subset $E = (c_0 + L) \cap B^n$ is said *parallel* to L and *passing through* c_0.

The idea is that the fixed point set in B^n of an automorphism of B^n is either empty or an affine subset of B^n. To prove this assertion, we need the following

Lemma 2.2.6: *Every automorphism of B^n sends affine subsets into affine subsets.*

Proof: Let $\gamma \in \operatorname{Aut}(B^n)$, and set $a = \gamma^{-1}(0)$. Let E be an affine subset of B^n, and write $E = (c_0 + L) \cap B^n$, where $c_0 \in B^n$ and L is a linear subspace of $\mathbf{C}^n$. Fix a basis $\{v_1, \ldots, v_k\}$ of L, and set $c_j = v_j + c_0 \in E$, for $j = 1, \ldots, k$; we can choose the basis so that $c_1, \ldots, c_k \in B$. Then it is easy to see that

$$E = \left\{ \sum_{j=0}^{k} \lambda_j c_j \;\middle|\; \lambda_0, \ldots, \lambda_k \in \mathbf{C},\ \sum_{j=0}^{k} \lambda_j = 1 \right\} \cap B^n.$$

Take $z = \sum\limits_{j=0}^{k} \lambda_j c_j \in E$. Then

$$\gamma(z) = \sum_{j=0}^{k} \mu_j \gamma(c_j),$$

where

$$\mu_j = \lambda_j \, \frac{1-(c_j,a)}{1-(z,a)}, \qquad (j = 0, \ldots, k).$$

Then $\sum\limits_{j=0}^{k} \mu_j = 1$ and $\gamma(E) = \big(\gamma(c_0) + L_1\big) \cap B^n$, where L_1 is the linear subspace of $\mathbf{C}^n$ generated by $\big\{\gamma(c_1) - \gamma(c_0), \ldots, \gamma(c_k) - \gamma(c_0)\big\}$, **q.e.d.**

Then

Theorem 2.2.7: *Let $\gamma \in \mathrm{Aut}(B^n)$. Then the fixed point set of γ in B^n is either empty or an affine subset. Conversely, every affine subset of B^n is the fixed point set of some automorphism of B^n.*

Proof: Suppose $\gamma(z_0) = z_0$ for some $z_0 \in B$. Then $\gamma_{z_0} \circ \gamma \circ \gamma_{z_0}$ is an unitary transformation, and its fixed point set E (in B^n) is the intersection of B^n with a linear subspace of $\mathbf{C}^n$. Therefore the fixed point set of γ is $\gamma_{z_0}(E)$ and hence affine, by Lemma 2.2.6.

For the converse, let E be an affine subset of B^n, and take $z_0 \in E$. Then $\gamma_{z_0}(E)$ is the fixed point set (in B^n) of an unitary transformation U, and E is the fixed point set of $\gamma = \gamma_{z_0} U \gamma_{z_0}$, **q.e.d.**

It may happen that an automorphism γ of B^n has no fixed points in B^n; on the other hand, every $\gamma \in \mathrm{Aut}(B^n)$ has a fixed point in $\overline{B^n}$. This is a consequence of *Brouwer's theorem:*

Theorem 2.2.8: *Let $f\colon K \to K$ be a continuous map of a compact convex subset K of $\mathbf{R}^N$ into itself. Then f has a fixed point.*

A proof can be found in Massey [1967], or in Spanier [1966], for instance.

It turns out that an automorphism without fixed points in B^n can have at most two fixed points in ∂B^n:

Proposition 2.2.9: *Let $\gamma \in \mathrm{Aut}(B^n)$ be without fixed points in B^n. Then γ has at least one and at most two fixed points in ∂B^n.*

Proof: By Brouwer's theorem, γ has at least one fixed point in ∂B^n. Assume that there exist three distinct fixed points z_1, z_2, $z_3 \in \partial B^n$ of γ. (2.2.4) yields

$$1 - (z_j, z_k) = \frac{\big(1-(a,a)\big)\big(1-(z_j,z_k)\big)}{\big(1-(z_j,a)\big)\big(1-(a,z_k)\big)},$$

for j, $k = 1, 2, 3$, where $a = \gamma^{-1}(0)$. If $j \neq k$, then $(z_j, z_k) \neq 1$ and hence

$$1 - (z_j, a) = \frac{1 - (a, a)}{1 - (a, z_k)}.$$

With $k = 3$ and $j = 1$ or 2 this implies that

$$(z_1, a) = (z_2, a). \tag{2.2.7}$$

Put $z = \frac{1}{2}(z_1 + z_2) \in B^n$. Then (2.2.7) implies

$$\gamma_a(z) = \frac{1}{2}\big(\gamma_a(z_1) + \gamma_a(z_2)\big).$$

Hence, if we write $\gamma = U\gamma_a$ for a suitable $U \in \mathbf{U}(n)$, we obtain

$$\gamma(z) = U\gamma_a(z) = \frac{1}{2}\big(\gamma(z_1) + \gamma(z_2)\big) = z,$$

contradiction, **q.e.d.**

We end this section characterizing the automorphisms of B^n with just one or two fixed points in ∂B^n and none in B^n. For sake of simplicity, we transfer the problem to H^n, and we ask for a description of the automorphisms of H^n keeping fixed either only ∞ or only 0 and ∞ (there is no loss in generality thanks to Corollary 2.2.5).

Proposition 2.2.10: *Let $\gamma \in \mathrm{Aut}(H^n)$. Then ∞ is the only fixed point of γ iff*

$$\forall w \in H^n \qquad \gamma(w) = h_a(w_1, U'w') \tag{2.2.8}$$

for some $U' \in \mathbf{U}(n-1)$ and $a \in \partial H^n$, where U' and a are such that for every solution $w' \in \mathbf{C}^{n-1}$ of $(I_{n-1} - U')w' = a'$ we have $\mathrm{Re}\, a_1 \neq 2\,\mathrm{Im}(w', a')$.

Proof: Write $\gamma = \delta_t \circ h_a \circ \mu_U$ for some $t > 0$, $a \in \partial H^n$ and $U \in \mathbf{U}(n)$. Since ∞ is a fixed point for both δ_t and h_a, it follows that $\mu_U(\infty) = \infty$. In particular, setting $e_1 = (1, 0, \dots, 0)$, $Ue_1 = e_1$ and so $\mu_U(w) = (w_1, U'w')$ for a suitable $U' \in \mathbf{U}(n-1)$.

Now we claim that $t = 1$. Indeed, assume by contradiction that $t \neq 1$. Then $\det(U' - t^{-1}I_{n-1}) \neq 0$; hence there is $v' \in \mathbf{C}^{n-1}$ such that $t(U'v' + a') = v'$. Set $v_1 = \alpha + i\|v'\|^2$, with $\alpha \in \mathbf{R}$ to be chosen; then $v = (v_1, v') \in \partial H^n$, and we have

$$\gamma(v_1, v) = \big(t^2[\alpha + i\|v'\|^2 + a_1 + 2i(v', a')], v'\big).$$

Now, $\gamma \in \mathrm{Aut}(H^n)$; hence $\gamma(v) \in \partial H^n$ and

$$\mathrm{Im}\big[t^2\big(\alpha + i\|v'\|^2 + a_1 + 2i(v', a')\big)\big] = \|v'\|^2 = \mathrm{Im}\, v_1.$$

On the other hand,

$$\mathrm{Re}\big[t^2\big(\alpha + i\|v'\|^2 + a_1 + 2i(v', a')\big)\big] = t^2\alpha + t^2\,\mathrm{Re}\big(a_1 + 2i(v', a')\big);$$

since $t \neq 1$, we can choose $\alpha \in \mathbf{R}$ so that

$$t^2\alpha + t^2 \operatorname{Re}\bigl(a_1 + 2i(v', a')\bigr) = \alpha,$$

and thus $\gamma(v) = v$, contradiction.

It remains to show that an automorphism of the form (2.2.8) has a fixed point different from ∞ iff the equation $(I_{n-1} - U')w' = a'$ has a solution $w' \in \mathbf{C}^{n-1}$ such that $\operatorname{Re} a_1 = 2\operatorname{Im}(w', a')$. Indeed, $h_a(w_1, U'w') = (w_1, w')$ iff

$$\begin{cases} w' = U'w' + a', \\ a_1 + 2i(U'w', a') = 0. \end{cases}$$

Recalling that $\|a'\|^2 = \operatorname{Im} a_1$, if we plug the first equation in the second one we find that $h_a(w_1, U'w') = (w_1, w')$ iff

$$\begin{cases} w' = U'w' + a', \\ \operatorname{Re} a_1 = 2\operatorname{Im}(w', a'), \end{cases}$$

q.e.d.

Proposition 2.2.11: *Let $\gamma \in \operatorname{Aut}(H^n)$. Then 0 and ∞ are the only fixed points of γ iff*

$$\forall w \in H^n \qquad \gamma(w) = \delta_t(w_1, U'w') \tag{2.2.9}$$

for some $U' \in \mathbf{U}(n-1)$ and $t > 0$, with $t \neq 1$.

Proof: This time it is obvious that every γ of the form (2.2.9) fixes only 0 and ∞ iff $t \neq 1$. For the converse, write $\gamma = \delta_t \circ h_a \circ \mu_U$, and assume 0 and ∞ are the only fixed points of γ. Since both δ_t and h_a have ∞ as fixed point, it follows as before that $\mu_U(w) = (w_1, U'w')$ for a suitable $U' \in \mathbf{U}(n-1)$. Finally, since both δ_t and μ_U have 0 as fixed point, it follows that $a = 0$, **q.e.d.**

2.2.2 *Schwarz's lemma and horospheres*

We start this section by introducing our staunch fellow, Schwarz's lemma, in its grown-up version:

Theorem 2.2.12: *Let $f\colon B^n \to B^n$ be holomorphic and such that $f(0) = 0$. Then*

$$\forall z \in B^n \qquad \|f(z)\| \leq \|z\|, \tag{2.2.10}$$

and

$$\forall v \in \mathbf{C}^n \qquad \|df_0(v)\| \leq \|v\|. \tag{2.2.11}$$

Proof: Fix x, $y \in \partial B^n$ and define $\varphi\colon \Delta \to \Delta$ by $\varphi(\zeta) = \bigl(f(\zeta x), y\bigr)$. Then the one variable Schwarz lemma yields $|\varphi'(0)| \leq 1$ and $|\varphi(\zeta)| \leq |\zeta|$ for all $\zeta \in \Delta$. Since x is an arbitrary element of ∂B^n this implies

$$\bigl|\bigl(df_0(v), y\bigr)\bigr| \leq \|v\|$$

for all $v \in \mathbf{C}^n$ and

$$\bigl|\bigl(f(z), y\bigr)\bigr| \leq \|z\|$$

for all $z \in B^n$. Since also y is an arbitrary element of ∂B^n, the assertion follows, **q.e.d.**

It should be emphasized that the equality at one point in (2.2.10) or in (2.2.11) do *not* imply either the linearity of f or its invertibility (this is the reason we could not use Schwarz's lemma in the computation of the automorphism group of B^n). For instance, define $f\colon B^2 \to B^2$ by

$$f(z,w) = (z + \tfrac{1}{2}w^2, 0).$$

Then $f(z,0) = (z,0)$ for all $z \in \Delta$ and $df_0(\lambda,0) = (\lambda,0)$ for all $\lambda \in \mathbf{C}$, but f is not linear, and not even surjective or injective. The problem is similar to the one we discussed talking about holomorphic retractions in the previous chapter: the behavior along a direction cannot (at least not very strongly) control the behavior along orthogonal directions, and so there are many new possibilities. Anyway, as for holomorphic retractions the image was not exceedingly wild, so also in our case we have some sort of regularity:

Lemma 2.2.13: *Let $f \in \mathrm{Hol}(B^n, B^n)$ be such that $f(0) = 0$, and take $z \in B^n$, $z \neq 0$. Then*
(i) $\|f(z)\| = \|z\|$ *iff* $\|df_0(z)\| = \|z\|$;
(ii) $f(z) = z$ *iff* $df_0(z) = z$.

Proof: Let $z_0 \in B^n$, $z_0 \neq 0$, be such that $\|f(z_0)\| = \|z_0\|$. Take $U \in \mathbf{U}(n)$ so that $Uf(z_0) = z_0$, set $x_0 = z_0/\|z_0\|$, and define $\varphi \in \mathrm{Hol}(\Delta,\Delta)$ by

$$\varphi(\zeta) = \big(Uf(\zeta x_0), x_0\big). \tag{2.2.12}$$

Then $\varphi(\|z_0\|) = \|z_0\|$; by the classical Schwarz lemma, $\varphi = \mathrm{id}_\Delta$ and, in particular, $\varphi'(0) = 1$. Then

$$\|z_0\|^2 = \big(U df_0(z_0), z_0\big) \leq \|U df_0(z_0)\| \cdot \|z_0\| \leq \|z_0\|^2$$

by (2.2.11), and this is possible iff $U df_0(z_0) = z_0$, that is iff $\|df_0(z_0)\| = \|z_0\|$. In particular, if $f(z_0) = z_0$ we can take $U = I_n$ and thus infer $df_0(z_0) = z_0$.

Conversely, assume $\|df_0(z_0)\| = \|z_0\|$. Again, take $U \in \mathbf{U}(n)$ so that $U df_0(z_0) = z_0$ — with $U = I_n$ in case (ii) —, set $x_0 = z_0/\|z_0\|$ and define $\varphi \in \mathrm{Hol}(\Delta,\Delta)$ as in (2.2.12). Then $\varphi'(0) = 1$ and, again by Schwarz's lemma, $\varphi = \mathrm{id}_\Delta$. Therefore $\varphi(\|z_0\|) = \|z_0\|$, and

$$\|z_0\|^2 = \big(U f_0(z_0), z_0\big) \leq \|U f_0(z_0)\| \cdot \|z_0\| \leq \|z_0\|^2$$

by (2.2.10), and again this is possible iff $Uf(z_0) = z_0$, **q.e.d.**

Proposition 2.2.14: *Let $f \in \mathrm{Hol}(B^n, B^n)$ be such that $f(0) = 0$. Assume $\|f(z_0)\| = \|z_0\|$ for some $z_0 \in B^n$ with $z_0 \neq 0$ — or $\|df_0(v_0)\| = \|v_0\|$ for some $v_0 \in \mathbf{C}^n$ with $v_0 \neq 0$. Then there is a linear subspace V of $\mathbf{C}^n$ containing z_0 — respectively v_0 — such that $f|_{V\cap B^n} = df_0|_{V\cap B^n}$ is the restriction of a suitable $U \in \mathbf{U}(n)$.*

Proof: By Lemma 2.2.13, the set of $z \in B^n$ such that $\|f(z)\| = \|z\|$ coincides with the set of $z \in B^n$ such that $\|df_0(z)\| = \|z\|$. Set

$$V = \{v \in \mathbf{C}^n \mid \|df_0(v)\| = \|v\|\}.$$

If $\zeta \in \mathbf{C}$ and v_1, $v_2 \in V$ we have

$$\|\zeta v_1 + v_2\|^2 = |\zeta|^2\|v_1\|^2 + \|v_2\|^2 + 2\,\mathrm{Re}(\zeta v_1, v_2),$$
$$\|df_0(\zeta v_1 + v_2)\|^2 = |\zeta|^2\|v_1\|^2 + \|v_2\|^2 + 2\,\mathrm{Re}\big(df_0(\zeta v_1), df_0(v_2)\big).$$

Since, by (2.2.11), $\|df_0(\zeta v_1 + v_2)\| \le \|\zeta v_1 + v_2\|$, it follows that

$$\forall \zeta \in \mathbf{C} \qquad \mathrm{Re}\big[\zeta\big(df_0(v_1), df_0(v_2)\big)\big] \le \mathrm{Re}[\zeta(v_1, v_2)],$$

which is possible iff $\big(df_0(v_1), df_0(v_2)\big) = (v_1, v_2)$. In particular,

$$\forall v_1,\, v_2 \in V \qquad \|df_0(v_1 + v_2)\| = \|v_1 + v_2\|.$$

So V is a linear subspace of $\mathbf{C}^n$, and $df_0|_V\colon V \to df_0(V)$ is an isometry. Therefore there is $U \in \mathbf{U}(n)$ such that $df_0|_V = U|_V$, and it remains to show that $f|_{V\cap B^n} = df_0|_{V\cap B^n}$.

Take $z_0 \in V \cap B^n$. Then, setting $g = f - df_0$, for every $\zeta \in \Delta$ we have

$$|\zeta|^2\|z_0\|^2 = \|f(\zeta z_0)\|^2 = |\zeta|^2\|z_0\|^2 + 2\,\mathrm{Re}\big(g(\zeta z_0), df_0(\zeta z_0)\big) + \|g(\zeta z_0)\|^2. \tag{2.2.13}$$

Therefore $\zeta \mapsto \mathrm{Re}\big(g(\zeta z_0), df_0(\zeta z_0)\big)$ is a non-positive harmonic function in Δ vanishing at 0; so it is identically zero and, by (2.2.13), $f(z_0) = df_0(z_0)$, **q.e.d.**

So the equality in (2.2.10) or in (2.2.11) implies at least a partial linearity of f; it is better than nothing.

Another consequence of Lemma 2.2.13 is the description of the fixed point sets of holomorphic maps $f\colon B^n \to B^n$, and thus of the holomorphic retracts of B^n:

Corollary 2.2.15: *Let $f \in \mathrm{Hol}(B^n, B^n)$. Then the fixed point set of f is either empty or an affine subset of B^n.*

Proof: This follows from Lemma 2.2.13, Lemma 2.2.6 and Corollary 2.2.2, **q.e.d.**

Corollary 2.2.16: *The holomorphic retracts of B^n are exactly the affine subsets of B^n.*

Proof: One direction is Corollary 2.2.15. Conversely, by Lemma 2.2.6 every affine subset of B^n is, up to an automorphism, the image of B^n under an orthogonal projection of $\mathbf{C}^n$ onto a linear subspace, **q.e.d.**

In particular, then, every fixed point set in B^n is a holomorphic retract; in chapter 2.5 we shall generalize this fact to bounded convex domains.

Now, come back to Schwarz's lemma. First of all, using $\mathrm{Aut}(B^n)$ we can get an invariant version of Theorem 2.2.12:

Proposition 2.2.17: *Let $f \in \mathrm{Hol}(B^n, B^n)$. Then for every z, $w \in B^n$ we have*

$$\frac{\bigl|1-\bigl(f(z),f(w)\bigr)\bigr|^2}{(1-\|f(z)\|^2)(1-\|f(w)\|^2)} \le \frac{|1-(z,w)|^2}{(1-\|z\|^2)(1-\|w\|^2)}, \tag{2.2.14}$$

and for every $z \in B^n$ and $v \in \mathbf{C}^n$ we have

$$\begin{aligned}\frac{1}{(1-\|f(z)\|^2)^2}\Bigl[\bigl|\bigl(df_z(v),f(z)\bigr)\bigr|^2 + (1-\|f(z)\|^2)\|df_z(v)\|^2\Bigr]&\\ \le \frac{1}{(1-\|z\|^2)^2}\bigl[|(v,z)|^2+(1-\|z\|^2)\|v\|^2\bigr].&\end{aligned} \tag{2.2.15}$$

In particular, if f is an automorphism of B^n then both (2.2.14) *and* (2.2.15) *are always equalities.*

Proof: (2.2.14) is just (2.2.10) applied to $\gamma_{f(w)}\circ f\circ\gamma_w$ and computed in $\gamma_w(z)$. Analogously, (2.2.15) is (2.2.11) applied to $d(\gamma_{f(z)}\circ f\circ\gamma_z)_0$ and computed in $d(\gamma_z)_z(v)$, **q.e.d.**

Exactly as happened in section 1.1.1, (2.2.15) invites us to introduce a differential metric and a distance on B^n.

Let ds^2 denote the euclidean differential metric at 0. Then the *Bergmann metric* $d\kappa^2$ on B^n is given setting $d\kappa_0^2 = ds^2$ and, for all $a \in B^n$ and u, $v \in \mathbf{C}^n$

$$d\kappa_a^2(u,v) = \bigl[(\gamma_a)^*ds^2\bigr](u,v) = \frac{1}{(1-\|a\|^2)^2}\bigl[(u,a)(a,v)+(1-\|a\|^2)(u,v)\bigr], \tag{2.2.16}$$

where we are identifying the tangent space of B^n at a with $\mathbf{C}^n$. The distance k_{B^n} associated to $d\kappa^2$ is the *Bergmann distance* on B^n. Using these definitions, Proposition 2.2.17 becomes

Corollary 2.2.18: *Let $f \in \mathrm{Hol}(B^n, B^n)$. Then*

$$\forall z,\, w \in B^n \qquad k_{B^n}\bigl(f(z),f(w)\bigr) \le k_{B^n}(z,w), \tag{2.2.17}$$

and

$$\forall z \in B^n,\ \forall v \in \mathbf{C}^n \qquad d\kappa_{f(z)}\bigl(df_z(v)\bigr) \le d\kappa_z(v). \tag{2.2.18}$$

In particular, every automorphism of B^n is an isometry of the Bergmann metric.

Proof: (2.2.18) is just (2.2.15) in this fancier language, and (2.2.17) is an immediate consequence, **q.e.d.**

The Bergmann metric is the natural generalization of the Poincaré metric on Δ. Indeed, it is easy to check that the restriction of the Bergmann metric to the intersection of B^n with any one-dimensional linear subspace of $\mathbf{C}^n$ is exactly the Poincaré metric (1.1.7). It should be remarked that, by Corollary 2.2.18 and Lemma 2.2.6, this is still true if we consider the restriction of the Bergmann metric to any one-dimensional affine subset of B^n.

Arguing as in the proof of Lemma 1.1.5 it is easy to see that the radii $t \mapsto tx$ for $x \in \partial B^n$ are geodesics for the Bergmann metric. In particular,

$$\forall z \in B^n \qquad k_{B^n}(0, z) = \tfrac{1}{2} \log \frac{1 + \|z\|}{1 - \|z\|}. \tag{2.2.19}$$

To visualize a bit the Bergmann metric, we shall now describe the *open Bergmann balls* $B_k(z_0, R)$, that is the open balls for k_{B^n} of center $z_0 \in B^n$ and radius $R > 0$. Clearly,

$$B_k(z_0, R) = \gamma_{z_0}\big(B_k(0, R)\big).$$

Now, (2.2.19) shows that $B_k(0, R)$ is the euclidean open ball $B(0, \tanh R)$ of center 0 and radius $\tanh R$. Therefore the Bergmann ball $B_k(z_0, R)$ is given by

$$B_k(z_0, R) = \left\{ z \in \mathbf{C}^n \;\middle|\; \frac{\|P_{z_0}(z) - a\|^2}{r^2\rho^2} + \frac{\|Q_{z_0}(z)\|^2}{r^2\rho} < 1 \right\}, \tag{2.2.20}$$

where $r = \tanh R$,

$$a = \frac{1 - r^2}{1 - r^2\|z_0\|^2}\, z_0,$$

and

$$\rho = \frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2}.$$

Hence $B_k(z_0, R)$ is an ellipsoid; its intersection with the subspace $\mathbf{C}z_0$ generated by z_0 is a disk of radius ρr, which is of order $R(1 - \|z_0\|^2)$ when R is small. On the other hand, its intersection with the subspace orthogonal to $\mathbf{C}z_0$ is a ball of the much larger radius $\sqrt{\rho}r$, which is roughly $R(1 - \|z_0\|^2)^{1/2}$ when R is small. Note furthermore that, being every $B_k(z_0, R)$ strictly contained in B^n, the Bergmann metric is complete.

Now we are ready to introduce in this setting one of the most useful tools we used in the one-variable theory. In analogy with the definition in the disk, the *horosphere* $E(x, R)$ of *center* $x \in \partial B^n$ and *radius* $R > 0$ is

$$E(x, R) = \left\{ z \in B^n \;\middle|\; \frac{|1 - (z, x)|^2}{1 - \|z\|^2} < R \right\}. \tag{2.2.21}$$

An easy computation shows that $E(x, R)$ is the ellipsoid

$$E(x, R) = \left\{ z \in \mathbf{C}^n \;\middle|\; \frac{\|P_x(z) - a\|^2}{r^2} + \frac{\|Q_x(z)\|^2}{r} < 1 \right\}, \tag{2.2.22}$$

where $r = R/(1+R) < 1$ and $a = (1-r)x$. In particular,

$$\forall x \in \partial B^n \ \forall R > 0 \qquad \overline{E(x,R)} \cap \partial B^n = \{x\} \tag{2.2.23}$$

and given any two distinct points $x, y \in \partial B^n$ it is always possible to find $R_0 > 0$ so small that

$$E(x, R_0) \cap E(y, R_0) = \emptyset, \tag{2.2.24}$$

and so that $\overline{E(x,R_0)} \cap \overline{E(y,R_0)}$ contains exactly one point.

The horospheres are intended as a natural generalization of the horocycles; so they must enjoy the main properties of the horocycles. First of all, they are limit of Bergmann balls, exactly as in the disk:

Proposition 2.2.19: *Let $B_\nu = B_k(z_\nu, R_\nu)$ be a sequence of Bergmann balls such that $z_\nu \to x \in \partial B^n$ and*

$$\frac{1 - \|z_\nu\|}{1 - r_\nu} \to R \neq 0, \infty, \tag{2.2.25}$$

where $r_\nu = \tanh R_\nu$. Then
(i) *if $z \in B_\nu$ for infinitely many ν then $z \in \overline{E(x,R)}$;*
(ii) *if $z \in E(x,R)$ then $z \in B_\nu$ for all sufficiently large ν.*

Proof: We observe that $z \in B_\nu$ iff $\gamma_{z_\nu}(z) \in B_k(0, R_\nu)$ iff, by (2.2.3),

$$\frac{|1 - (z, z_\nu)|^2}{1 - \|z\|^2} < \frac{1 - \|z_\nu\|^2}{1 - r_\nu^2} = \frac{1 + \|z_\nu\|}{1 + r_\nu} \frac{1 - \|z_\nu\|}{1 - r_\nu}.$$

Then (i) and (ii) follow taking the limit as $\nu \to +\infty$ and using (2.2.25), **q.e.d.**

Proposition 2.2.20: *Let $x \in \partial B^n$ and $R > 0$. Then*

$$E(x,R) = \left\{ z \in B^n \;\middle|\; \lim_{w \to x} \left[k_{B^n}(z,w) - k_{B^n}(0,w) \right] < \tfrac{1}{2} \log R \right\}. \tag{2.2.26}$$

Proof: We have, using (2.2.19)

$$\begin{aligned} k_{B^n}(z,w) - k_{B^n}(0,w) &= k_{B^n}\big(0, \gamma_z(w)\big) - k_{B^n}(0,w) \\ &= \tfrac{1}{2} \log\left(\frac{1 + \|\gamma_z(w)\|}{1 + \|w\|} \cdot \frac{1 - \|w\|}{1 - \|\gamma_z(w)\|} \right), \end{aligned}$$

and the assertion follows from (2.2.3), **q.e.d.**

Clearly, horospheres are sent into horospheres by automorphisms of B^n. Indeed, take $\gamma \in \mathrm{Aut}(B^n)$, $x \in \partial B^n$ and $R > 0$. Set $z_0 = \gamma^{-1}(0)$ and $\alpha = (1 - \|z_0\|^2)/|1 - (z_0, x)|^2$. Then using (2.2.4) it is easy to see that $\gamma\big(E(x,R)\big) = E\big(\gamma(x), \alpha R\big)$.

The two main properties of horocycles were Julia's lemma and Wolff's lemma. Our horospheres should deserve their name only if they enjoy a multidimensional version of these results. This is indeed the case, for we have a *Julia's lemma:*

Theorem 2.2.21: *Let $f\colon B^n \to B^n$ be a holomorphic map and take $x \in \partial B^n$ such that*

$$\liminf_{z\to x} \frac{1-\|f(z)\|}{1-\|z\|} = \alpha < +\infty.$$

Then there exists a unique $y \in \partial B^n$ such that for every $z \in B^n$

$$\frac{\bigl|1-\bigl(f(z),y\bigr)\bigr|^2}{1-\|f(z)\|^2} \le \alpha\,\frac{|1-(z,x)|^2}{1-\|z\|^2}, \tag{2.2.27}$$

that is

$$\forall R>0 \qquad f\bigl(E(x,R)\bigr) \subset E(y,\alpha R). \tag{2.2.28}$$

Proof: Choose a sequence $\{z_\nu\} \subset B^n$ converging to x such that

$$\lim_{\nu\to\infty} \frac{1-\|f(z_\nu)\|}{1-\|z_\nu\|} = \alpha.$$

Up to a subsequence, we can assume that $f(z_\nu) \to y \in \partial B^n$ as $\nu \to +\infty$. Then, by Proposition 2.2.17 for every $z \in B^n$ and $\nu \in \mathbf{N}$ we have

$$\frac{\bigl|1-\bigl(f(z),f(z_\nu)\bigr)\bigr|^2}{1-\|f(z)\|^2} \le \frac{|1-(z,z_\nu)|^2}{1-\|z\|^2}\cdot\frac{1+\|f(z_\nu)\|}{1+\|z_\nu\|}\cdot\frac{1-\|f(z_\nu)\|}{1-\|z_\nu\|},$$

and (2.2.27) follows letting $\nu \to +\infty$. Finally, if (2.2.28) holds for two distinct boundary points y_1, $y_2 \in \partial B^n$, then we get a contradiction taking $R>0$ so small that $E(y_1,\alpha R)\cap E(y_2,\alpha R) = \emptyset$, as in (2.2.24), **q.e.d.**

And even a *Wolff's lemma:*

Theorem 2.2.22: *Let $f\colon B^n \to B^n$ be a holomorphic map without fixed points. Then there is a unique $x \in \partial B^n$ such that for every $z \in B^n$*

$$\frac{\bigl|1-\bigl(f(z),x\bigr)\bigr|^2}{1-\|f(z)\|^2} \le \frac{|1-(z,x)|^2}{1-\|z\|^2}, \tag{2.2.29}$$

that is

$$\forall R>0 \qquad f\bigl(E(x,R)\bigr) \subset E(x,R). \tag{2.2.30}$$

Proof: Choose a sequence $\{r_\nu\} \subset (0,1)$ increasing to 1, and set $f_\nu = r_\nu f$. Clearly, $f_\nu(B^n)$ is relatively compact in B^n, and hence, by Corollary 2.1.32, every f_ν has a fixed point $w_\nu \in B^n$. Up to a subsequence, we can assume $w_\nu \to x \in \overline{B^n}$ as $\nu \to +\infty$. Exactly as in the proof of the one-dimensional Wolff lemma we see that $x \in B^n$ would imply $f(x)=x$, impossible; hence $x \in \partial B^n$.

Now (2.2.14) says that for all $z \in B^n$ and $\nu \in \mathbf{N}$

$$\frac{\bigl|1-\bigl(f_\nu(z),w_\nu\bigr)\bigr|^2}{1-\|f_\nu(z)\|^2} \le \frac{|1-(z,w_\nu)|^2}{1-\|z\|^2},$$

and (2.2.29) follows taking the limit as $\nu \to +\infty$.

Assume, by contradiction, there is another point $x' \in \partial B^n$ such that (2.2.30) holds. Choose R and R' so that the ellipsoids $E(x,R)$ and $E(x',R')$ are tangent to each other at the point $z \in B^n$. Then (2.2.30) would imply $f(z)=z$, impossible, **q.e.d.**

As we saw in chapter 1.2, Julia's lemma will be used to study angular derivatives, while Wolff's lemma will become the cornerstone of iteration theory. But before probing these questions, we should fit to our needs yet another tool.

2.2.3 *Korányi regions and the Lindelöf theorem*

When in chapter 1.2 we studied the boundary behavior of holomorphic functions in Δ, the natural notion of limit was the non-tangential limit, as naturally expected. In 1967, Korányi and Stein (see Korányi and Stein [1968] and Korányi [1969]) made the amazing discovery that in several variables the non-tangential limit can be replaced by a broader notion: the approach regions may be tangent to the boundary along the complex tangential directions. In this section we shall introduce Korányi's approach regions; however, we shall not spend too much time describing the limit considered by Korányi (a longer and quite better exposition is Rudin [1980]), because the main goal of this section, the several variables version of Lindelöf's Theorem 1.3.23, involves another kind of limit, slightly weaker than Korányi's.

But let's be concrete. In section 1.2.1 we studied non-tangential limits using the Stolz regions $K^\Delta(\tau, M)$ (we placed the exponent Δ to distinguish them from the Korányi regions in B^n to be introduced momentarily) defined in (1.2.16). Their natural generalization in B^n are the *Korányi regions* $K(x, M)$ of *vertex* $x \in \partial B^n$ and *amplitude* $M > 0$ given by

$$K(x, M) = \left\{ z \in B^n \,\middle|\, \frac{|1 - (z, x)|}{1 - \|z\|} < M \right\}. \tag{2.2.31}$$

$K(x, M)$ is empty if $M \leq 1$; on the other hand, for any $x \in \partial B^n$ the regions $K(x, M)$ fill B^n as M approaches $+\infty$.

We now introduce the limit considered by Korányi. Let $f \colon B^n \to \mathbf{C}$ be a function. We shall say that f has *K-limit* (or *admissible limit*) λ at $x \in \partial B^n$ (possibly $\lambda = \infty$) if $f(z) \to \lambda$ as $z \to x$ within $K(x, M)$ for any $M > 1$. If $n = 1$ we saw that this is exactly the non-tangential limit; to understand its meaning in the present context, let us examine more closely the shape of Korányi regions.

First of all, note that every $U \in \mathbf{U}(n)$ permutes the Korányi regions: in fact,

$$U\big(K(x, M)\big) = K(Ux, M).$$

So to sketch $K(x, M)$ we can assume without loss of generality $x = (1, 0, \ldots, 0) = e_1$. Then the intersection of $K(e_1, M)$ with the complex subspace generated by e_1 is

$$K^\Delta(1, M) = \left\{ z_1 \in \Delta \,\middle|\, \frac{|1 - z_1|}{1 - |z_1|} < M \right\},$$

that is the usual Stolz region in Δ. On the other hand, the intersection with the copy of $\mathbf{R}^{2n-1}$ obtained by setting $\operatorname{Im} z_1 = 0$ contains the ball

$$\left(\operatorname{Re} z_1 - \frac{1}{M} \right)^2 + \|z'\|^2 < \left(1 - \frac{1}{M} \right)^2,$$

where $z' = (z_2, \ldots, z_n)$, as usual, which is tangent to ∂B^n in e_1.

In chapter 2.7 we shall need a more precise description of the tangential shape of the Korányi regions:

Lemma 2.2.23: *Let $x \in \partial B^n$; for every $M > 1$ and $\zeta \in K^\Delta(1, M)$ set*

$$\delta_M(\zeta) = \inf\{\|v\| \mid (v, x) = 0 \text{ and } \zeta x + v \in \partial K(x, M)\};$$

$\delta_M(\zeta)$ is the radius of the largest $(n-1)$-dimensional ball centered at ζx contained in the affine $(n-1)$-dimensional subset of B^n passing through ζx and parallel to $T_x^{\mathbf{C}}(\partial B^n)$. Then for any $1 < M < M_1$ and all $\zeta \in K^\Delta(1, M)$ we have

$$\left(1 - \frac{M}{M_1}\right)(1 - |\zeta|)^{1/2} \leq \delta_{M_1}(\zeta) \leq \sqrt{2}(1 - |\zeta|)^{1/2}.$$

Proof: Let $v \in \mathbf{C}^n$ be any vector orthogonal to x; then $\zeta x + v \in \partial K(x, M_1)$ iff

$$|1 - \zeta| = M_1\bigl(1 - (|\zeta|^2 + \|v\|^2)^{1/2}\bigr).$$

Therefore

$$\delta_{M_1}(\zeta)^2 = \left(1 - \frac{|1 - \zeta|}{M_1}\right)^2 - |\zeta|^2.$$

In particular,

$$\delta_{M_1}(\zeta)^2 \leq 1 - |\zeta|^2 \leq 2(1 - |\zeta|).$$

On the other hand, $\zeta \in K^\Delta(1, M)$ means $|1 - \zeta| < M(1 - |\zeta|)$; hence

$$\begin{aligned}
\delta_{M_1}(\zeta)^2 &\geq \left(1 - \frac{M}{M_1}(1 - |\zeta|)\right)^2 - |\zeta|^2 \\
&= \left(1 - \frac{M}{M_1}\right)(1 - |\zeta|)\left[1 - \frac{M}{M_1} + \left(1 + \frac{M}{M_1}\right)|\zeta|\right] \\
&\geq \left(1 - \frac{M}{M_1}\right)^2 (1 - |\zeta|),
\end{aligned}$$

q.e.d.

So the Korányi regions are tangent to ∂B^n along the complex tangential directions. In particular, a function having K-limit always has non-tangential limit; in K-limits, the approach is restricted to be non-tangential only in the radial direction. However, as mentioned before, we shall need a slightly different notion of limit, still generalizing non-tangential limit, and so we refer to Rudin [1980] for a more complete discussion of K-limits in B^n.

Our next goal is a several variables version of Lindelöf's Theorem 1.3.23. The classical statement in Δ says that, for a bounded holomorphic function, the limit along a curve ending at a boundary point determines the behavior along any curve ending at the same boundary point, and in particular determines the non-tangential limit. In B^n this is not anymore true. Take, for instance, $f \in \mathrm{Hol}(B^2, \Delta)$ given by

$$f(z, w) = \frac{w^2}{1 - z^2}. \tag{2.2.32}$$

Then for every $\lambda \in \Delta$ and $t \in (0,1)$ we have

$$f(t, \lambda\sqrt{1-t^2}) = \lambda^2,$$

and so the limit of f along the curve $\sigma_\lambda(t) = (t, \lambda\sqrt{1-t^2})$ ending at $(1,0)$ depends on the parameter λ. Note, furthermore, that f has no K-limit at $(1,0)$, because for every $t \in (0,1)$ and $\lambda \in \Delta$ we have $\sigma_\lambda(t) \in K\big((1,0), 2/(1-|\lambda|^2)\big)$.

The idea is that to extend Lindelöf's theorem to B^n we need some kind of restriction on the curves we want to consider. To describe the precise statement, we introduce some definitions.

For $x \in \partial B^n$, a *x-curve* is a curve $\sigma\colon [0,1) \to B^n$ such that $\sigma(t) \to x$ as $t \to 1$. To every x-curve we associate its orthogonal projection $\sigma_x = (\sigma, x)x$ into $\mathbf{C}x$. Then $(\sigma - \sigma_x) \perp \sigma_x$, so that

$$\|\sigma\|^2 = \|\sigma - \sigma_x\|^2 + \|\sigma_x\|^2,$$

and hence

$$\frac{\|\sigma - \sigma_x\|^2}{1 - \|\sigma_x\|^2} < 1,$$

because $\|\sigma\|^2 < 1$.

A x-curve is *special* if

$$\lim_{t\to 1} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2} = 0, \tag{2.2.33}$$

and *restricted* if it is special and moreover there is $A < \infty$ such that for all $t \in [0,1)$

$$\frac{\|\sigma_x(t) - x\|}{1 - \|\sigma_x(t)\|} \le A. \tag{2.2.34}$$

In other words, a special curve σ is restricted iff its projection σ_x is nontangential. Note that a restricted x-curve can be tangent to ∂B^n at x: take for instance $n = 2$, $x = (1,0)$ and

$$\sigma(t) = \big(t(2-t), (1-t)^{3/2}\big). \tag{2.2.35}$$

Let $f\colon B^n \to \mathbf{C}$ be any function, and $x \in \partial B^n$. We shall say that f has *restricted K-limit* (or *hypoadmissible limit*) λ at x if $f\big(\sigma(t)\big) \to \lambda$ as $t \to 1$ for any restricted x-curve σ. As promised, this is an intermediate notion between non-tangential limit and K-limit, as shown in

Lemma 2.2.24: *Let $\sigma\colon [0,1) \to B^n$ be a x-curve, where $x \in \partial B^n$. Then*
(i) *if σ is non-tangential, then it is restricted;*
(ii) *assume σ is special. If σ satisfies* (2.2.34) *then it lies eventually in $K(x, M)$ for all $M > A$. Conversely, if σ lies in $K(x, M)$ then it satisfies* (2.2.34) *with $A = M$.*

Proof: (i) We recall that a x-curve σ is non-tangential iff there is $C \ge 1$ such that

$$\forall t \in [0,1) \qquad \frac{\|\sigma(t) - x\|}{1 - \mathrm{Re}\big(\sigma(t), x\big)} \le C < +\infty. \tag{2.2.36}$$

Now note that for every curve σ we have

$$\forall t \in [0,1) \qquad \|\sigma(t) - \sigma_x(t)\| \le \|\sigma(t) - x\|$$

and

$$\forall t \in [0,1) \qquad \|\sigma_x(t) - x\| \le \|\sigma(t) - x\|. \tag{2.2.37}$$

Therefore if σ satisfies (2.2.36) we find

$$\forall t \in [0,1) \qquad \bigl|\mathrm{Im}\bigl(\sigma(t), x\bigr)\bigr| \le (C-1)^{1/2}\bigl[1 - \mathrm{Re}\bigl(\sigma(t), x\bigr)\bigr].$$

It follows that since $\mathrm{Re}\bigl(\sigma(t), x\bigr) \to 1$ as $t \to 1$, for every $\delta < 1$ we eventually have

$$\begin{aligned} 1 - \|\sigma_x\| \ge \frac{1 - \|\sigma_x\|^2}{2} &\ge \frac{1}{2}\bigl[1 - \mathrm{Re}(\sigma, x)\bigr]\,\bigl[2 - C\bigl(1 - \mathrm{Re}(\sigma, x)\bigr)\bigr] \\ &\ge \delta\bigl(1 - \mathrm{Re}(\sigma, x)\bigr). \end{aligned}$$

In conclusion, we eventually have

$$\frac{\|\sigma - \sigma_x\|^2}{1 - \|\sigma_x\|^2} = \frac{\|\sigma - \sigma_x\|^2}{\|\sigma - x\|^2} \cdot \frac{\|\sigma - x\|}{1 - \|\sigma_x\|} \cdot \frac{\|\sigma - x\|}{1 + \|\sigma_x\|} \le \frac{C}{\delta}\|\sigma - x\| \to 0,$$

and so every non-tangential curve is special and, by (2.2.37), restricted.

(ii) Assume σ satisfies (2.2.34). Then

$$|1 - (\sigma, x)| = \|\sigma_x - x\| \le A(1 - \|\sigma_x\|),$$

and thus

$$\frac{|1 - (\sigma, x)|}{1 - \|\sigma\|} \le A\left(1 - \frac{\|\sigma - \sigma_x\|^2}{1 - \|\sigma_x\|^2}\right)^{-1} \cdot \frac{1 + \|\sigma\|}{1 + \|\sigma_x\|}. \tag{2.2.38}$$

Since σ is special, the right-hand side of (2.2.38) tends to A as $t \to 1$; therefore σ lies eventually in $K(x, M)$ for all $M > A$.

Conversely, assume σ lies in $K(x, M)$. Then

$$\|\sigma_x - x\| = |1 - (\sigma, x)| \le M(1 - \|\sigma\|) \le M(1 - \|\sigma_x\|),$$

and the assertion follows, **q.e.d.**

So a function having K-limit has restricted K-limit, and a function having restricted K-limit has non-tangential limit. The function $f \in \mathrm{Hol}(B^2, \Delta)$ given in (2.2.32) is an example of a bounded holomorphic function having restricted K-limit but no K-limit. On the other hand, the function $g\colon B^2 \to \mathbf{C}$ given by

$$g(z, w) = \frac{\bigl(|1 - z|^2 + |w|^2\bigr)^{2/3}}{1 - \mathrm{Re}\, z}$$

has non-tangential limit 0 at $(1, 0)$, whereas the curve σ given in (2.2.35) is a restricted $(1, 0)$-curve such that $g\bigl(\sigma(t)\bigr) \to 1$.

The choice of a non-holomorphic (though real analytic) function for the latter counterexample is not casual. In fact, the Lindelöf theorem we shall present in a moment states that for a bounded holomorphic function $f\colon B^n \to \mathbf{C}$ the existence of the limit along a special x-curve implies that f has restricted K-limit at x; in particular, for bounded (and not only bounded, as we shall see) holomorphic functions the non-tangential limit implies the restricted K-limit. This is the content of *Čirka's theorem:*

Theorem 2.2.25: *Let $f \in \mathrm{Hol}(B^n, \mathbf{C})$ be a bounded holomorphic function such that there are $x \in \partial B^n$ and a special x-curve $\sigma^o\colon [0,1) \to B^n$ such that*

$$\lim_{t\to 1} f\bigl(\sigma^o(t)\bigr) = \lambda \in \mathbf{C}.$$

Then f has restricted K-limit λ at x.

Proof: Clearly we can assume $f(B^n) \subset \Delta$. Let σ be any special x-curve; we claim that

$$\lim_{t\to 1}\bigl[f\bigl(\sigma(t)\bigr) - f\bigl(\sigma_x(t)\bigr)\bigr] = 0. \tag{2.2.39}$$

Take $\zeta \in \mathbf{C}$; then $(1-\zeta)\sigma_x(t) + \zeta\sigma(t) \in B^n$ iff

$$|\zeta|^2 < \frac{1 - \|\sigma_x(t)\|^2}{\|\sigma(t) - \sigma_x(t)\|^2}.$$

Fix $R > 1$, set $\Delta_R = \{\zeta \in \mathbf{C} \mid |\zeta| < R\}$, and define $\varphi_{t,R}\colon \Delta_R \to \Delta$ by

$$\varphi_{t,R}(\zeta) = f\bigl((1-\zeta)\sigma_x(t) + \zeta\sigma(t)\bigr);$$

since σ is special, for every $R > 1$ the function $\varphi_{t,R}$ is defined as soon as t is close enough to 1. Then Schwarz's lemma applied to $\varphi_{t,R} - \varphi_{t,R}(0)$ yields

$$\limsup_{t\to 1}\bigl|f\bigl(\sigma(t)\bigr) - f\bigl(\sigma_x(t)\bigr)\bigr| \le 2/R,$$

and (2.2.39) follows, for R is arbitrary.

In particular, (2.2.39) implies that $f\bigl(\sigma_x^o(t)\bigr) \to \lambda$ as $t \to 1$. But then the classical Lindelöf Theorem 1.3.23 implies that $f|_{\mathbf{C}x\cap B^n}$ has non-tangential limit λ at x; so, if σ is any restricted x-curve, $f\bigl(\sigma_x(t)\bigr) \to \lambda$ as $t \to 1$. Hence another application of (2.2.39) implies that $f\bigl(\sigma(t)\bigr) \to \lambda$ as $t \to 1$, and we are done, **q.e.d.**

So for bounded holomorphic functions, non-tangential limit, restricted K-limit and limit along a special curve are one and the same thing. This in an instance of the so-called *Lindelöf principle:* for bounded holomorphic functions, the existence of the limit along one curve forces the existence of the limit along quite larger sets. We shall discuss other examples of Lindelöf's principles in chapter 2.7; for the moment we limit ourselves to the following result, a Lindelöf principle for not necessarily bounded holomorphic functions:

Proposition 2.2.26: *Let $f \in \mathrm{Hol}(B^n, \mathbf{C})$ and $x \in \partial B^n$ be such that f is bounded in every region $K(x, M)$. Suppose furthermore that there is a restricted x-curve σ^o such that*

$$\lim_{t\to 1} f\bigl(\sigma^o(t)\bigr) = \lambda \in \mathbf{C}.$$

Then f has restricted K-limit λ at x.

Proof: Arguing as in the proof of Theorem 2.2.25 we see that (by Lemma 2.2.24.(ii) and Theorem 1.1.28) it suffices to prove that for every restricted x-curve σ we have

$$\lim_{t\to 1}\bigl[f\bigl(\sigma(t)\bigr) - f\bigl(\sigma_x(t)\bigr)\bigr] = 0. \tag{2.2.40}$$

Since σ is restricted, there is $M > 1$ such that $\sigma(t) \in K(x, M)$ eventually (Lemma 2.2.24). Choose $M_1 > 2M$, and take $\lambda \in \mathbf{C}$. Then $(1 - \lambda)\sigma_x(t) + \lambda\sigma(t) \in K(x, M_1)$ whenever

$$|\lambda|^2 < \frac{1 - \|\sigma_x\|^2}{\|\sigma - \sigma_x\|^2} - \frac{2}{M_1} \frac{|1 - (\sigma, x)|}{\|\sigma - \sigma_x\|^2}.$$

If $\sigma(t) \in K(x, M)$, then $\left|1 - \big(\sigma(t), x\big)\right| < M(1 - \|\sigma(t)\|)$; so $(1-\lambda)\sigma_x(t) + \lambda\sigma(t) \in K(x, M_1)$ whenever

$$|\lambda|^2 < \frac{M_1 - 2M}{M_1} \frac{1 - \|\sigma_x(t)\|^2}{\|\sigma(t) - \sigma_x(t)\|^2}.$$

Since f is bounded in $K(x, M_1)$ (and σ is special), we can now apply the same Schwarz's lemma trick used in the proof of Theorem 2.2.25 to get (2.2.40), and we are done, **q.e.d.**

2.2.4 *Angular derivatives*

The aim of this section is to prove a generalization of the Julia-Wolff-Carathéodory Theorem 1.2.7 to B^n using Theorem 2.2.21. The main difference compared with the one-dimensional case is that if $f \in \mathrm{Hol}(B^n, B^n)$ is such that $f(z) \to y \in B^n$ as $z \to x \in \partial B^n$, then the behavior of the radial component (f, y) of f is quite different from the behavior of the tangential component $f - (f, y)y$ of f, and even more so for the various components of the differential of f.

The theorem we are driving at will be expressed in terms of restricted K-limits. The idea is to use Proposition 2.2.26, but to find suitable restricted curves we shall need two lemmas, specifying even more the shape of Korányi regions:

Lemma 2.2.27: *Choose $x \in \partial B^n$ and fix $M > 1$ and $R > 0$. Then:*
(i) *if $r = (M^2 - R)/(M^2 + R)$ then every $z \in K(x, M)$ such that $\|z\| \geq r$ is contained in $E(x, R)$;*
(ii) *if $z \in E(x, R)$ then $\|z\| \geq (1 - R)/(1 + R)$.*

Proof: (i) If $z \in K(x, M)$ and $\|z\| \geq r$ we have

$$\frac{|1 - (z, x)|^2}{1 - \|z\|^2} = \frac{|1 - (z, x)|^2}{(1 - \|z\|)^2} \cdot \frac{1 - \|z\|}{1 + \|z\|} < M^2 \cdot \frac{1 - r}{1 + r} = R.$$

(ii) If $z \in E(x, R)$ we have

$$R > \frac{|1 - (z, x)|^2}{1 - \|z\|^2} \geq \frac{(1 - \|z\|)^2}{1 - \|z\|^2} = \frac{1 - \|z\|}{1 + \|z\|},$$

and the assertion follows, **q.e.d.**

Lemma 2.2.28: *Choose $M_1 > M > 1$, and set*

$$\delta = \frac{1}{5}\left(\frac{1}{M} - \frac{1}{M_1}\right) \cdot \left(1 - \frac{M}{M_1}\right) \leq \frac{1}{5}.$$

Let $x \in \partial B^n$, and take $z \in K(x, M)$. Then
(i) *If $\lambda \in \Delta$ is such that $|\lambda| \leq \delta |1 - (z, x)|$ then $z + \lambda x \in K(x, M_1)$;*
(ii) *If $w \in B^n$ is such that $(w, x) = 0$ and $\|w\| \leq \delta |1 - (z, x)|^{1/2}$, then $z + w \in K(x, M_1)$.*

Proof: First of all note that $z \in K(x, M)$ implies $|1 - (z, x)| < M$. It is then easy to check that δ is chosen in such a way that for every $z \in K(x, M)$

$$\left[1 - \frac{1}{M}|1 - (z, x)|\right]^2 + 5\delta|1 - (z, x)| \leq \left[1 - \frac{1}{M_1}|1 - (z, x)|\right]^2.$$

For every $z \in B^n$ set $z' = z - (z, x)x = Q_x(z)$. Then $z \in K(x, M)$ iff

$$\|z'\|^2 < \left[1 - \frac{1}{M}|1 - (z, x)|\right]^2 - |(z, x)|^2.$$

Now fix $z \in K(x, M)$. If $\lambda \in \Delta$ is such that $|\lambda| \leq \delta|1 - (z, x)|$ we have

$$\begin{aligned} \|z'\|^2 < \left[1 - \frac{1}{M}|1 - (z, x)|\right]^2 - |(z, x)|^2 &\leq \left[1 - \frac{1}{M_1}|1 - (z, x)|\right]^2 - 5|\lambda| - |(z, x)|^2 \\ &\leq \left[1 - \frac{1}{M_1}|1 - (z + \lambda x, x)|\right]^2 - |(z + \lambda x, x)|^2, \end{aligned}$$

and (i) is proved. If $w \in B^n$ is such that $(w, x) = 0$ and $\|w\| \leq \delta|1 - (z, x)|^{1/2}$, we have

$$\begin{aligned} \|z' + w\|^2 &\leq \|z'\|^2 + 2\|z'\|\,\|w\| + \|w\|^2 \\ &\leq \left[1 - \frac{1}{M}|1 - (z, x)|\right]^2 - |(z, x)|^2 + (3\delta + \delta^2)|1 - (z, x)| \\ &< \left[1 - \frac{1}{M_1}|1 - (z + w, x)|\right]^2 - |(z + w, x)|^2, \end{aligned}$$

where we have used the fact that $2\|z'\|^2 < 3|1 - (z, x)|^{1/2}$ for all $z \in B^n$, and (ii) is proved, **q.e.d.**

And now:

Theorem 2.2.29: *Let $f \in \mathrm{Hol}(B^n, B^n)$ and $x \in \partial B^n$ be such that*

$$\liminf_{z \to x} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < \infty.$$

Then f has K-limit $y \in \partial B^n$ at x, and the following functions are bounded in every Korányi region:

(i) $\big(1 - (f(z), y)\big)/\big(1 - (z, x)\big)$,

(ii) $Q_y\big(f(z)\big)/\big(1 - (z, x)\big)^{1/2}$,

(iii) $\big(df_z x, y\big)$,

(iv) $\big(1 - (z, x)\big)^{1/2} Q_y(df_z x)$,

(v) $\big(df_z x^\perp, y\big)/\big(1 - (z, x)\big)^{1/2}$,

(vi) $Q_y(df_z x^\perp)$,

where $x^\perp$ is any non-zero vector orthogonal to x, and $Q_y(z) = z - (z, y)y$ is the orthogonal projection on the othogonal complement of $\mathbf{C}y$. Moreover, the functions (i) *and* (iii) *have restricted K-limit α at x, and the functions* (ii), (iv) *and* (v) *have restricted K-limit* 0 *at x.*

Proof: We shall divide the proof in five steps.

Step (a): The behavior of f. Let $y \in \partial B^n$ be given by Theorem 2.2.21; we claim that f has K-limit y at x. Let $\{w_\nu\}$ be a sequence contained in some $K(x, M)$ converging to x; we have to prove that $f(w_\nu) \to y$. Fix $r < 1$, and let $R \in (0, 1)$ be such that $(1 - \alpha R)/(1 + \alpha R) > r$. Since $w_\nu \to x$, by Lemma 2.2.27.(i) we have $w_\nu \in E(x, R)$ for any ν large enough; therefore, by Theorem 2.2.21, $f(w_\nu) \in E(y, \alpha R)$ and, by Lemma 2.2.27.(ii), $\|f(w_\nu)\| > r$ for any ν large enough. In other words, we have proved that $\|f(w_\nu)\| \to 1$ as $\nu \to +\infty$. This implies, by Theorem 2.2.21 and Lemma 2.2.27.(i), that every limit point of $\{f(w_\nu)\}$ is contained in

$$\overline{E(y, \alpha R)} \cap \partial B^n = \{y\},$$

by (2.2.23), and so f has K-limit y at x, as claimed.

Step (b): Radial behavior. Our next aim is to prove that

$$\lim_{\xi \to 1} \frac{1 - \big(f(\xi x), y\big)}{1 - \xi} = \alpha, \tag{2.2.41}$$

and

$$\lim_{\xi \to 1} \frac{f(\xi x) - \big(f(\xi x), y\big) y}{(1 - \xi)^{1/2}} = 0, \tag{2.2.42}$$

where ξ varies in $(0, 1)$.

Put $1 - \xi = 2R/(1 + R)$. Then $\xi x \in \partial E(x, R)$ and, by Theorem 2.2.21, we have $f(\xi x) \in \overline{E(y, \alpha R)}$, that is

$$\alpha R \geq \frac{\big|1 - \big(f(\xi x), y\big)\big|^2}{1 - \|f(\xi x)\|^2} \geq \frac{1 - \|f(\xi x)\|}{1 + \|f(\xi x)\|}.$$

In particular,

$$1-\|f(\xi x)\| \le \frac{2\alpha R}{1+\alpha R}.$$

Hence

$$\frac{1-\|f(\xi x)\|^2}{1-\xi^2} \le \alpha\,\frac{1+R}{1+\alpha R}\,\frac{1+\|f(\xi x)\|}{1+\xi} \le \alpha\,\frac{1+R}{1+\alpha R}\,\frac{2}{1+\xi}. \tag{2.2.43}$$

Since $R\to 0$ as $\xi\to 1$,

$$\limsup_{\xi\to 1}\frac{1-\|f(\xi x)\|^2}{1-\xi^2} \le \alpha,$$

and so, by definition of α and by Step (a),

$$\lim_{\xi\to 1}\frac{1-\|f(\xi x)\|^2}{1-\xi^2} = \alpha = \lim_{\xi\to 1}\frac{1-\|f(\xi x)\|}{1-\xi}. \tag{2.2.44}$$

Now, by Theorem 2.2.21 and (2.2.43),

$$\frac{\left|1-\big(f(\xi x),y\big)\right|^2}{(1-\xi)^2} \le \alpha\,\frac{1-\|f(\xi x)\|^2}{1-\xi^2} \le \alpha^2\,\frac{1+R}{1+\alpha R}\,\frac{2}{1+\xi}. \tag{2.2.45}$$

Since $1-\|f(\xi x)\| \le 1-\left|\big(f(\xi x),y\big)\right| \le \left|1-\big(f(\xi x),y\big)\right|$, (2.2.44) and (2.2.45) imply

$$\lim_{\xi\to 1}\frac{1-\left|\big(f(\xi x),y\big)\right|}{1-\xi} = \lim_{\xi\to 1}\frac{\left|1-\big(f(\xi x),y\big)\right|}{1-\xi} = \alpha. \tag{2.2.46}$$

In particular, the ratio of the two numerators in (2.2.46) converges to 1 as $\xi\to 1$. Now

$$\left|\frac{1-a}{1-|a|}\right|^2 = 1+\frac{2}{|a|+\operatorname{Re} a}\cdot\left|\frac{\operatorname{Im} a}{1-|a|}\right|^2$$

for every $a\in\Delta$ with $\operatorname{Re} a>0$; therefore we have

$$\lim_{\xi\to 1}\frac{1-\big(f(\xi x),y\big)}{1-\left|\big(f(\xi x),y\big)\right|} = 1,$$

and (2.2.41) follows from (2.2.46).

Since $\big(f(\xi x),y\big)\to 1$ as $\xi\to 1$, (2.2.41) is the same as

$$\lim_{\xi\to 1}\frac{1-\left|\big(f(\xi x),y\big)\right|^2}{1-\xi^2} = \alpha. \tag{2.2.47}$$

Now (2.2.42) follows from (2.2.44) and (2.2.47) because

$$\|f\|^2 = |(f,y)|^2 + \|f-(f,y)y\|^2. \tag{2.2.48}$$

Step (c): The functions (i) *and* (ii). Fix $M > 1$, and take $z \in K(x, M)$. Put $R = M|1 - (z, x)|$. Then

$$|1 - (z, x)|^2 = \frac{R}{M}|1 - (z, x)| < R(1 - \|z\|) < R(1 - \|z\|^2),$$

that is $z \in E(x, R)$. Then $f(z) \in E(y, \alpha R)$, by Theorem 2.2.21; in particular, $\big(f(z), y\big)$ belongs to the horocycle in Δ of center 1 and radius αR. Now, the euclidean diameter of this horocycle is $2\alpha R/(1 + \alpha R)$, and thus

$$\big|1 - \big(f(z), y\big)\big| \leq \frac{2\alpha R}{1 + \alpha R} < 2\alpha R = 2\alpha M|1 - (z, x)|.$$

Therefore $\big(1 - (f(z), y)\big)/\big(1 - (z, x)\big)$ is bounded in every Korányi region, and Proposition 2.2.26 and (2.2.41) imply that its restricted K-limit at x is α.

Finally, if we write $\|f\|^2$ as in (2.2.48), $f(z) \in E(x, \alpha R)$ implies, by (2.2.22)

$$\big\|Q_y\big(f(z)\big)\big\|^2 < \alpha R = \alpha M|1 - (z, x)|.$$

Therefore $Q_y\big(f(z)\big)/\big(1 - (z, x)\big)^{1/2}$ is bounded in every Korányi region, and again Proposition 2.2.26 and (2.2.42) imply that its restricted K-limit at x is 0.

Step (d): The functions (iii) *and* (iv). This time we should deal with differentiation in the direction x. Let $1 < M < M_1$, choose δ as in Lemma 2.2.28, take $z \in K(x, M)$ and put

$$r = r(z) = \delta|1 - (z, x)|. \tag{2.2.49}$$

Then $z + \lambda x \in K(x, M_1)$ for all $\lambda \in \mathbf{C}$ with $|\lambda| \leq r$. By the Cauchy formula,

$$(df_z x, y) = \frac{1}{2\pi i} \int\limits_{|\lambda|=r} \frac{\big(f(z + \lambda x), y\big)}{\lambda^2}\, d\lambda. \tag{2.2.50}$$

Replacing $\big(f(z + \lambda x), y\big)$ by $\big(f(z + \lambda x), y\big) - 1$ (and the right-hand side of (2.2.50) is unchanged), multiplying and dividing the integrand by $(z + \lambda x, x) - 1$ and setting $\lambda = re^{i\theta}$, we get

$$(df_z x, y) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - \big(f(z + re^{i\theta}x), y\big)}{1 - (z + re^{i\theta}x, x)} \cdot \left\{1 - \frac{1 - (z, x)}{re^{i\theta}}\right\} d\theta. \tag{2.2.51}$$

The first factor in the integrand is bounded, by Step (c); the second one is at most $1 + \delta^{-1}$, by (2.2.49). Therefore $(df_z x, y)$ is bounded in every Korányi region.

When $z = \xi x$ in (2.2.51), the second factor in the integrand is $1 - \delta^{-1}e^{-i\theta}$, and the first factor converges boundedly to α as $\xi \to 1$ (for $\xi + r(\xi x)e^{i\theta}$ tends to 1 non-tangentially for every $\theta \in \mathbf{R}$). Therefore $(df_{\xi x} x, y) \to \alpha$ as $\xi \to 1$, by the dominated convergence theorem, and again Proposition 2.2.26 shows that $(df_z x, y)$ has restricted K-limit α at x.

Another application of the Cauchy formula yields

$$Q_y(df_z x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Q_y\bigl(f(z+re^{i\theta}x)\bigr)}{\bigl(1-(z+re^{i\theta}x,x)\bigr)^{1/2}} \cdot \frac{\bigl(1-(z+re^{i\theta}x,x)\bigr)^{1/2}}{re^{i\theta}}\,d\theta.$$

Then exactly as before we find that $\bigl(1-(z,x)\bigr)^{1/2}Q_y(df_z x)$ is bounded in every Korányi region, and that its restricted K-limit at x is 0.

Step (e): The functions (v) *and* (vi). This time we should deal with differentiation in the direction $x^\perp$. Clearly, we can assume $\|x^\perp\| = 1$. Let $1 < M < M_1$, choose δ as in Lemma 2.2.28, take $z \in K(x,M)$ and put

$$\rho = \rho(z) = \delta|1-(z,x)|^{1/2}. \tag{2.2.52}$$

Then $z + \lambda x^\perp \in K(x, M_1)$ for all $\lambda \in \mathbf{C}$ with $|\lambda| \le \rho$. An application of the Cauchy formula as in Step (d) yields

$$\frac{(df_z x^\perp, y)}{\bigl(1-(z,x)\bigr)^{1/2}} = -\frac{\bigl(1-(z,x)\bigr)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\bigl(f(z+\rho e^{i\theta}x^\perp),y\bigr)}{1-(z+\rho e^{i\theta}x^\perp,x)} e^{-i\theta}\,d\theta \tag{2.2.53}$$

and

$$Q_y(df_z x^\perp) = \frac{\bigl(1-(z,x)\bigr)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Q_y\bigl(f(z+\rho e^{i\theta}x^\perp)\bigr)}{\bigl(1-(z+\rho e^{i\theta}x^\perp,x)\bigr)^{1/2}} e^{-i\theta}\,d\theta. \tag{2.2.54}$$

The integrands are bounded, by Step (c) applied in $K(x, M_1)$. Hence (2.2.52) implies that the left-hand sides of (2.2.53) and (2.2.54) are bounded in every Korányi region.

To finish, we have to prove that the left-hand side of (2.2.53) has radial limit 0 at x (again by Proposition 2.2.26). Note that the curve $\sigma(\xi) = \xi x + \rho(\xi x)e^{i\theta}x^\perp$ is *not* special, and thus a new argument is needed.

Let $\varphi\colon B^n \to \Delta$ be given by $\varphi(z) = \bigl(f(z), y\bigr)$. Restricting the attention to the subspace generated by x and $x^\perp$ we can assume $n = 2$, $x = (1,0)$ and $x^\perp = (0,1)$. In particular,

$$\frac{(df_{\xi x}x^\perp, y)}{(1-\xi)^{1/2}} = \frac{\partial\varphi}{\partial z_2}(\xi, 0)\cdot(1-\xi)^{-1/2}.$$

We can expand φ in the form

$$\varphi(z_1,z_2) = \psi(z_1) + 2z_2(1-z_1)^{1/2}g(z_1) + \sum_{j=2}^{\infty} g_j(z_1)z_2^j,$$

where

$$g(z_1) = \frac{1}{2}\frac{\partial\varphi}{\partial z_2}(z_1,0)\cdot(1-z_1)^{-1/2}.$$

Then we have to show that $g(\xi) \to 0$ as $\xi \to 1$. Note that, by (2.2.41), $\psi(\xi) \to 1$ as $\xi \to 1$ in such a way that $\big(1 - \psi(\xi)\big)/(1 - \xi) \to \alpha$. Set

$$h(z, w) = \psi(z) + w(1 - z)^{1/2} g(z);$$

since h is the arithmetic mean of the first two partial sums of the power series expansion of φ, and $|\varphi| < 1$ on B^2, we have $|h| < 1$ on B^2.

Now choose $\varepsilon > 0$ and set $c = \alpha^2/\varepsilon^2$. We wish to estimate

$$\limsup_{\xi \to 1} \big| g\big(\xi + ic(1 - \xi)\big)\big|.$$

Set $\zeta_\xi = \xi + ic(1 - \xi)$; it is easy to check that $\zeta_\xi \in \Delta$ whenever $1 - \xi < 2/(1 + c^2)$. Furthermore,

$$1 - |\zeta_\xi|^2 > 1 - \xi$$

if $1 - \xi < 1/(1 + c^2)$. Hence if ξ is sufficiently close to 1 we can find $\eta_\xi \in \mathbf{C}$ such that

$$1 - |\zeta_\xi|^2 > |\eta_\xi|^2 > 1 - \xi \tag{2.2.55}$$

and

$$\eta_\xi (1 - \zeta_\xi)^{1/2} g(\zeta_\xi) \in \mathbf{R}. \tag{2.2.56}$$

Now (2.2.55) implies in particular that $(\zeta_\xi, \eta_\xi) \in B^2$ as soon as $1 - \xi < 1/(1 + c^2)$. By definition,

$$|1 - \zeta_\xi| = (1 - \xi)\sqrt{1 + c^2} \geq c(1 - \xi);$$

hence (2.2.55) yields, by (2.2.56),

$$\operatorname{Re}[\eta_\xi (1 - \zeta_\xi)^{1/2} g(\zeta_\xi)] \geq c^{1/2}(1 - \xi)|g(\zeta_\xi)|. \tag{2.2.57}$$

Now $\zeta_\xi \in K^\Delta(1, 2\sqrt{1 + c^2})$ if $1 - \xi < 1/(1 + c^2)$; hence

$$\frac{1 - \psi(\zeta_\xi)}{1 - \zeta_\xi} = \alpha + o(1)$$

as $\xi \to 1$, that is

$$\psi(\zeta_\xi) = 1 - \big(\alpha + o(1)\big)(1 - ic)(1 - \xi). \tag{2.2.58}$$

Putting together (2.2.56), (2.2.57) and (2.2.58) we get

$$1 \geq \operatorname{Re} h(\zeta_\xi, \eta_\xi) \geq 1 - \big(\alpha + o(1)\big)(1 - \xi) + c^{1/2}(1 - \xi)|g(\zeta_\xi)|,$$

that is

$$|g(\zeta_\xi)| \leq \frac{\alpha + o(1)}{c^{1/2}}.$$

Therefore

$$\limsup_{\xi \to 1} \big| g\big(\xi + ic(1 - \xi)\big)\big| \leq \alpha c^{-1/2} = \varepsilon.$$

Clearly the same estimate holds for $\zeta'_\xi = \xi - ic(1 - \xi)$. Since $|g(\zeta)|$ is bounded in any Stolz region, it follows that

$$\limsup_{\xi \to 1} |g(\xi)| \leq \varepsilon.$$

But $\varepsilon > 0$ is arbitrary; hence $g(\xi) \to 0$ as $\xi \to 1$, and the theorem is proved, **q.e.d.**

We end this section with two examples showing that we cannot improve the statement of Theorem 2.2.29; in other words, $1/2$ is the best exponent, we cannot replace restricted K-limits by K-limits, and the function (vi) may have no restricted K-limit at x.

Let $\psi \in \mathrm{Hol}(\Delta, \Delta)$ be given by

$$\psi(\zeta) = \exp\left(-\frac{\pi}{2} - i \log(1-\zeta)\right).$$

Note that

$$\psi'(\zeta) = \frac{i}{1-\zeta}\,\psi(\zeta).$$

As $\zeta \to 1$, $\psi(\zeta)$ spirals around the origin without limit.

Define $f \in \mathrm{Hol}(B^2, B^2)$ by

$$f(z,w) = \bigl(z + \tfrac{1}{2} w^2 \psi(z), 0\bigr).$$

Since $f(z,0) = (z,0)$, the hypotheses of Theorem 2.2.29 are satisfied with $x = y = (1,0)$ and $\alpha = 1$. Now,

$$\begin{aligned} \frac{1 - \bigl(f(z,w), y\bigr)}{1-z} &= 1 - \frac{w^2}{2(1-z)}\,\psi(z), \\ \bigl(df_{(z,w)}x, y\bigr) &= 1 + \frac{iw^2}{2(1-z)}\,\psi(z), \\ \frac{\bigl(df_{(z,w)}x^\perp, y\bigr)}{(1-z)^{1/2}} &= \frac{w}{(1-z)^{1/2}}\,\psi(z), \end{aligned}$$

where $x^\perp = (0,1)$. Therefore the functions (i), (iii) and (v) have restricted K-limit at x but they need not have K-limit at x (consider for instance the curve $\sigma(t) = \bigl(t, \lambda\sqrt{1-t^2}\bigr)$, where $\lambda \in \Delta^*$). Furthermore, the boundedness assertion made about (v) becomes false replacing $1/2$ by a larger exponent.

For the second example, define $f \in \mathrm{Hol}(B^2, B^2)$ by

$$f(z,w) = \bigl(z, w\psi(z)\bigr).$$

Again, the hypotheses of Theorem 2.2.29 are verified for $x = y = (1,0)$ and $\alpha = 1$. Now, taking $x^\perp = (0,1)$,

$$\begin{aligned} \frac{Q_y\bigl(f(z,w)\bigr)}{(1-z)^{1/2}} &= \frac{w}{(1-z)^{1/2}}\,\psi(z)\,x^\perp, \\ (1-z)^{1/2} Q_y(df_{(z,w)}x) &= \frac{iw}{(1-z)^{1/2}}\,\psi(z)\,x^\perp, \\ Q_y(df_{(z,w)}\,x^\perp) &= \psi(z)x^\perp. \end{aligned}$$

Therefore the functions (ii) and (iv) need not have K-limit at x, and (vi) need not even have radial limit at x. Furthermore, the boundedness assertion made about (iv) becomes false replacing $1/2$ by a smaller exponent, and the one made about (ii) becomes false replacing $1/2$ by a larger exponent.

2.2.5 *Iteration theory and common fixed points in the ball*

In this section we shall study iteration theory in B^n, using Wolff's lemma and following the guidelines provided by Theorem 2.1.29. We shall not be very detailed, because the general theory for strongly convex domains we shall discuss in chapter 2.4 is very similar to the one in the ball. The section will end with the generalization of Shields' theorem to the ball and the construction of a fixed point for any compact group of automorphisms of B^n.

So take $f \in \mathrm{Hol}(B^n, B^n)$. The first step in the study of iterates is to decide whether $\{f^k\}$ is compactly divergent. If f has a fixed point, clearly the sequence $\{f^k\}$ cannot be compactly divergent. The interesting fact is that the converse is true too:

Proposition 2.2.30: *Let $f \in \mathrm{Hol}(B^n, B^n)$. Then the sequence $\{f^k\}$ is not compactly divergent iff f has a fixed point in B^n.*

Proof: One direction is obvious. Conversely, assume that $\{f^k\}$ is not compactly divergent; then, by Theorem 2.1.29, there is a subsequence $\{f^{k_\nu}\}$ converging toward a holomorphic retraction $\rho\colon B^n \to M$. Since $f \circ \rho = \rho \circ f$, f sends M into itself. If M is a point, the proof is finished; otherwise, M is an affine subset of B^n (by Corollary 2.2.16), and thus biholomorphic to B^m for some $0 < m \le n$; since it suffices to show that f has a fixed point in M, we can directly assume $M = B^n$. In particular, by Lemma 2.1.20 f is a pseudoperiodic automorphism of B^n.

Assume, by contradiction, that f has no fixed points in B^n. Then using the Cayley transform we can transfer everything to H^n and assume f is either of the form (2.2.8) or of the form (2.2.9). But in both cases it is easy to check that $f^k(i, 0, \dots, 0) \to \infty$ as $k \to +\infty$, and so f cannot be pseudoperiodic, **q.e.d.**

So the main distinction is between maps with fixed points and maps without, exactly as in the one-dimensional case. Again, Wolff's lemma is the right tool to deal with the fixed point free case:

Theorem 2.2.31: *Let $f \in \mathrm{Hol}(B^n, B^n)$ be without fixed points. Then the sequence of iterates of f converges to a point of ∂B^n.*

Proof: By Proposition 2.2.30, the sequence $\{f^k\}$ is compactly divergent. Let $x \in \partial B^n$ be the point given by Theorem 2.2.22; if we show that x is the unique limit point of $\{f^k\}$ we are done.

Let $\{f^{k_\nu}\}$ be a subsequence of $\{f^k\}$ converging to a holomorphic map $h\colon B^n \to \mathbf{C}^n$. Since $\{f^k\}$ is compactly divergent, $h(B^n) \subset \partial B^n$. But then by Theorem 2.2.22

$$h\big(E(x,R)\big) \subset \overline{E(x,R)} \cap \partial B^n = \{x\}$$

for any $R > 0$, by (2.2.23). Therefore $h \equiv x$, and the assertion is proved, **q.e.d.**

Now let $f \in \mathrm{Hol}(B^n, B^n)$ have a fixed point $z_0 \in B^n$; the idea is that in this case the behavior of $\{f^k\}$ is completely described by the spectrum of df_{z_0}. Indeed

Theorem 2.2.32: *Let $f \in \mathrm{Hol}(B^n, B^n)$, and assume f has a fixed point $z_0 \in B^n$. Then the sequence of iterates $\{f^k\}$ converges iff $\mathrm{sp}(df_{z_0}) \subset \Delta \cup \{1\}$.*

Proof: Let $\rho\colon B^n \to M$ be the limit retraction of f. Clearly, $z_0 \in M$, and $\mathrm{sp}(d\rho_{z_0}) \subset \{0,1\}$. If the sequence $\{f^k\}$ converges, necessarily to ρ, and if $\lambda \in \mathrm{sp}(df_{z_0})$, then the sequence $\{\lambda^k\}$ of powers of λ should converge to an eigenvalue of $d\rho_{z_0}$, and hence $\lambda \in \Delta \cup \{1\}$.

Conversely, assume $\mathrm{sp}(df_{z_0}) \subset \Delta \cup \{1\}$, and let $\mathbf{C}^n = L_N \oplus L_U$ be the df_{z_0}-invariant splitting of $\mathbf{C}^n \cong T_{z_0}B^n$ constructed in Theorem 2.1.21.(iv). Then $df_{z_0}|_{L_U} = \mathrm{id}$ and $(df_{z_0}|_{L_N})^k \to 0$ as $k \to +\infty$; in particular, every limit point h of $\{f^k\}$ fixes z_0 and is such that $dh_{z_0} = d\rho_{z_0}$. Hence $d(h|_M)_{z_0} = \mathrm{id}$ and, by Theorem 2.1.21.(iii), $h|_M = \mathrm{id}_M$. Therefore Theorem 2.1.29 implies that ρ is the unique limit point of $\{f^k\}$, **q.e.d.**

Summing up, we have proved that the sequence of iterates of a map $f \in \mathrm{Hol}(B^n, B^n)$ converges iff either f has no fixed points or f has a fixed point $z_0 \in B^n$ such that $\mathrm{sp}(df_{z_0}) \subset \Delta \cup \{1\}$. In chapter 2.4 we shall show how to extend this result to strongly convex domains; for the moment, we prefer spending our time describing the set of limit points of the sequence of iterates when it does not converge.

Take $f \in \mathrm{Hol}(B^n, B^n)$ with a fixed point $z_0 \in B^n$; then the idea is that the limit points of $\{f^k\}$ are completely determined by $\mathrm{sp}(df_{z_0}) \cap \partial\Delta$:

Proposition 2.2.33: *Take $f \in \mathrm{Hol}(B^n, B^n)$ with a fixed point $z_0 \in B^n$, and let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of modulus 1 of df_{z_0}, listed according to their multiplicity. Then:*
(i) *the limit manifold of f is the affine subset of B^n passing through z_0 and parallel to the unitary space of df_{z_0};*
(ii) *the limit manifold of f coincides with $\mathrm{Fix}(f)$ iff the sequence $\{f^k\}$ converges, and in this case it is $\big(z_0 + \mathrm{Fix}(df_{z_0})\big) \cap B^n$;*
(iii) *the sequence $\{f^k\}$ converges to z_0 iff $\mathrm{sp}(df_{z_0}) \subset \Delta$;*
(iv) *the limit points of $\{f^k\}$ are in one-to-one correspondence with the limit points of the sequence of powers of $(\lambda_1, \ldots, \lambda_r) \in \mathbf{T}^r = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$, the torus group of (real) dimension r.*

Proof: Up to an automorphism of B^n, we can assume $z_0 = 0$. Let $\rho\colon B^n \to B^n$ be the limit retraction of f. Then the unitary space of df_0 coincides with the eigenspace associated to the eigenvalue 1 of $d\rho_0$ and hence, by Lemma 2.2.13, we have (i) and (iii).

(ii) follows immediately from (i), Lemma 2.2.13 and Theorem 2.2.32.

Finally, by Theorem 2.1.29 and Corollary 2.1.22, any limit point of $\{f^k\}$ is completely determined by its differential at 0, and (iv) follows from the decomposition given in Theorem 2.1.21.(iv), **q.e.d.**

We have seen that, as in the one-dimensional situation, iteration theory is mingled with fixed point sets. We shall now explore the reverse of the coin: how to construct fixed points using iteration theory.

The first result is the generalization of Shields' theorem to B^n:

Theorem 2.2.34: *Let $\mathcal{F}$ be a family of continuous self-maps of $\overline{B^n}$ which are holomorphic in B^n and commute with each other under composition. Then $\mathcal{F}$ has a fixed point in $\overline{B^n}$.*

Proof: If there is $f \in \mathcal{F}$ such that $f(B^n) \cap \partial B^n \neq \emptyset$, then $f \equiv x_0 \in \partial B^n$, and x_0 is clearly a fixed point of $\mathcal{F}$. So we can suppose without loss of generality that $\mathcal{F} \subset \mathrm{Hol}(B^n, B^n)$.

Assume there is $f \in \mathcal{F}$ without fixed points in B^n. By Theorem 2.2.31, the sequence $\{f^k\}$ converges to a point $x \in \partial B^n$, and for any $g \in \mathcal{F}$

$$g(x) = \lim_{k\to\infty} g \circ f^k = \lim_{k\to\infty} f^k \circ g = x,$$

and we have found a fixed point of $\mathcal{F}$.

Now assume that every $f \in \mathcal{F}$ has a fixed point in B^n, and denote by $\mathrm{Fix}(f)$ the fixed point set of f in B^n. In particular, no sequence $\{f^k(z)\}$, where $f \in \mathcal{F}$ and $z \in B^n$, can have a limit point in ∂B^n, by Corollary 2.2.18.

Let E be any affine subset of B^n, and $g \in \mathcal{F}$ be such that $g(E) \subset E$; then g has a fixed point in E. In fact, otherwise $\{(g|_E)^k\}$ should converge to a point of ∂B^n, and this is prohibited by the previous observation.

Take f, $g \in \mathcal{F}$. Then g sends $\mathrm{Fix}(f)$ into itself: indeed, if $z \in \mathrm{Fix}(f)$, then

$$f\big(g(z)\big) = g\big(f(z)\big) = g(z).$$

Since $\mathrm{Fix}(f)$ is affine (by Corollary 2.2.15), g has a fixed point in $\mathrm{Fix}(f)$, that is the intersection $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is not empty.

By induction it follows that, for any finite number of maps $f_1, \ldots, f_r \in \mathcal{F}$ the intersection $\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)$ is not empty. Let

$$d = \min\Big\{\dim\big(\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)\big) \,\Big|\, f_1, \ldots, f_r \in \mathcal{F},\ r \in \mathbf{N}\Big\} \geq 0,$$

and choose $f_1, \ldots, f_r \in \mathcal{F}$ such that $\dim\big(\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)\big) = d$. Set

$$E = \mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r);$$

then E is a non-empty affine subset of B^n; we claim that $E \subset \mathrm{Fix}(g)$ for all $g \in \mathcal{F}$. In fact, otherwise

$$0 \leq \dim\big(\mathrm{Fix}(g) \cap E\big) < \dim E = d,$$

for some $g \in \mathcal{F}$, contradiction, **q.e.d.**

The second application is the construction of a fixed point for a compact group acting on B^n. The proof we present here is somehow involved, resting on a sort of uniform convexity lemma and even on Zorn's lemma. In chapter 2.5 we shall describe a simpler proof in the setting of convex domains, but for the moment let's begin with

Lemma 2.2.35: *Let $\mathcal{F}$ be a family of closed Bergmann balls in B^n such that $K = \bigcap \mathcal{F}$ contains two distinct points. Then the interior part of K is non-empty.*

Proof: Since a closed Bergmann ball is a compact convex subset of B^n, K is a compact convex subset of B^n. Up to an automorphism of B^n, then, we can assume there is a point $z_0 \in K$, $z_0 \neq 0$, such that the whole segment from $-z_0$ to z_0 is contained in K; in particular, $0 \in K$. We shall show that there exists $\varepsilon > 0$ such that $B_k(0, \varepsilon)$ is contained in every closed Bergmann ball $\overline{B_k(a, R)}$ containing z_0 and $-z_0$, and the assertion will clearly follow.

We recall — cf. (2.2.20) — that $B_k(a, R)$ is given by

$$B_k(a,R) = \left\{ z \in \mathbf{C}^n \;\middle|\; \frac{\|P_a(z) - \alpha\|^2}{r^2\rho^2} + \frac{\|Q_a(z)\|^2}{r^2\rho} < 1 \right\},$$

where $r = \tanh R \in (0, 1)$,

$$\alpha = \frac{1 - r^2}{1 - r^2\|a\|^2}\, a \in B^n$$

and

$$\rho = \frac{1 - \|a\|^2}{1 - r^2\|a\|^2} \in (0, 1).$$

Let $\mathcal{F} = \left\{ \overline{B_k(a,R)} \;\middle|\; z_0,\, -z_0 \in \overline{B_k(a,R)} \right\}$; we need to show that

$$\varepsilon = \inf\left\{ \|z\| \;\middle|\; z \in \partial B,\, B \in \mathcal{F} \right\}$$

is strictly positive.

Now, $B_k(a, R)$ contains the euclidean ball

$$B(\alpha, r) = \left\{ z \in \mathbf{C}^n \;\middle|\; \|z - \alpha\|^2 < r^2\rho^2 \right\};$$

hence it is enough to estimate the euclidean distance between 0 and $\partial B(\alpha, r)$ for every $\overline{B_k(a,R)} \in \mathcal{F}$.

The point of $\partial B(\alpha, r)$ nearest to 0 is of the form $t\alpha$, where $t \in \mathbf{R}$ satisfies

$$|t - 1| = \frac{r\rho}{\|\alpha\|}. \tag{2.2.59}$$

Since $0 \in B_k(a, R)$ implies $r > \|a\|$, (2.2.59) yields

$$d\big(0, \partial B(\alpha, r)\big) = \frac{r - \|a\|}{1 - r\|a\|} \geq \frac{r^2 - \|a\|^2}{2(1 - r^2\|a\|^2)}.$$

Therefore it suffices to show that there exists $c > 0$ such that

$$(r^2 - \|a\|^2)/(1 - r^2\|a\|^2) \geq c \tag{2.2.60}$$

for every $\overline{B_k(a,R)} \in \mathcal{F}$.

Take $\overline{B_k(a,R)} \in \mathcal{F}$; we can restrict our attention to the balls such that we have both $z_0 \in \partial B_k(a,R)$ and $-z_0 \in \overline{B_k(a,R)}$. This is equivalent to requiring

$$\operatorname{Re}(z_0, a) \le 0, \tag{2.2.61}$$

and

$$\|P_a(z_0) - \alpha\|^2 + \rho\|Q_a(z_0)\|^2 = r^2\rho^2. \tag{2.2.62}$$

Now, plugging the definitions of P_a, Q_a, ρ and α in (2.2.62), we find

$$r^2 - \|a\|^2 = (1 - \|a\|^2)\|z_0\|^2 + (1 - r^2)\big[|(z_0, a)|^2 - 2\operatorname{Re}(z_0, a)\big].$$

Finding r^2 in terms of $\|a\|^2$ and then computing the left-hand side of (2.2.60) we finally obtain

$$\frac{r^2 - \|a\|^2}{1 - r^2\|a\|^2} = \frac{\|z_0\|^2 + |(z_0, a)|^2 - 2\operatorname{Re}(z_0, a)}{1 + |(z_0, a)|^2 - 2\operatorname{Re}(z_0, a) + \|a\|^2(1 - \|z_0\|^2)} \ge \frac{\|z_0\|^2}{4},$$

since $\operatorname{Re}(z_0, a)$ is nonpositive by (2.2.61), and the assertion follows, **q.e.d.**

Then:

Theorem 2.2.36: *Let $G \subset \operatorname{Hol}(B^n, B^n)$ be a group under composition. Then G has a fixed point z_0 in B^n iff G is relatively compact in $\operatorname{Hol}(B^n, B^n)$. Moreover, in this case there is a holomorphic retraction $\rho\colon B^n \to M$ such that every $g \in G$ is of the form $g = \gamma \circ \rho$, where γ belongs to a given subgroup of the isotropy group of z_0 in M; in particular, G is isomorphic to a subgroup of $\mathbf{U}(m)$ for some $0 \le m \le n$. Conversely, every subgroup of $\mathbf{U}(m)$ with $0 \le m \le n$ can be realized as a group $G \subset \operatorname{Hol}(B^n, B^n)$ with a fixed point.*

Proof: Let $e \in G$ be the identity of G. Since $e^2 = e$, e is a holomorphic retraction of B^n onto an affine subset E of B^n. Now $g \circ e = g = e \circ g$ if $g \in G$; hence $g = \gamma \circ e$ for a suitable $\gamma \in \operatorname{Hol}(E, E)$. Moreover, since G is a group, γ belongs to $\operatorname{Aut}(E)$, and we have established an isomorphism between G and a subgroup of $\operatorname{Aut}(E)$. If E is a point, there is nothing else to prove; otherwise, E is biholomorphic to B^m for some $0 < m \le n$ (by Corollary 2.2.16), and we can assume without loss of generality that $E = B^n$, that is that G is a subgroup of $\operatorname{Aut}(B^n)$.

If G has a fixed point, then it is relatively compact in $\operatorname{Hol}(B^n, B^n)$ by Corollary 2.1.27. Conversely, assume G is relatively compact in $\operatorname{Hol}(B^n, B^n)$; actually, we can directly assume G compact (taking the closure, by Theorem 2.1.26). We want to show that G has a fixed point.

Let K be a compact subset of B^n. Put

$$\widehat{K} = \bigcap \left\{ \overline{B_k(z,R)} \;\middle|\; z \in B^n,\ R > 0,\ K \subset \overline{B_k(z,R)} \right\};$$

clearly $\widehat{K}$ is a compact subset of B^n, and $\widehat{\gamma(K)} = \gamma(\widehat{K})$ for all $\gamma \in \mathrm{Aut}(B^n)$. We shall (temporarily) say that K is *B-convex* if $\widehat{K} = K$. Note that a B-convex set is convex (but, obviously, the converse is false).

Let K be a compact G-invariant subset of B^n (for instance, the G-orbit of a point). Then $\widehat{K}$ is compact, G-invariant and B-convex. The intersection of any descending chain of compact, G-invariant, B-convex subsets of $\widehat{K}$ is still compact, G-invariant and B-convex; by Zorn's lemma, there exists a minimal compact G-invariant B-convex subset H of $\widehat{K}$.

Suppose, by contradiction, that H contains more than one point. Then the interior part $\mathring{H}$ of H is not empty, by Lemma 2.2.35. Take $z \in \mathring{H}$; since $\mathring{H}$ is clearly G-invariant, $K_1 = G(z)$ is a compact G-invariant subset of $\mathring{H}$. In particular, $K_1 \cap \partial H = \emptyset$. Then there exists a closed Bergmann ball B with center at 0 such that $K_1 \subset B$ and $H \not\subset B$. But then $\widehat{K_1}$ is strictly contained in $\widehat{H} = H$, and this contradicts the minimality of H. So H is a G-invariant subset of B^n containing just one point, that is, a fixed point of G.

Finally, If G is a subgroup of $\mathbf{U}(m)$ for some $m \le n$, take a m-dimensional affine subset E of B^n, and a point $z_0 \in E$. Then we can realize G as a subgroup of the isotropy group of z_0 in $\mathrm{Aut}(E)$, and thus as a subgroup of $\mathrm{Hol}(B^n, B^n)$, using a holomorphic retraction of B^n onto E, **q.e.d.**

Notes

The study of B^n as a particularly meaningful example is relatively recent, and evolved during the shift in complex analysis from the algebraic techniques of sheaf theory to the analytic techniques linked to the $\overline{\partial}$-equation; this caused a need for simple examples of strongly pseudoconvex domains as gymnasium where testing conjectures should be easy. The beatiful book Rudin [1980] is the best evidence of this situation, and indeed the informed reader will easily recognize the debt we owe to that book for the first four sections of this chapter.

The automorphisms of B^2 are first described by Poincaré [1907b]; the general case appears in Sommer [1949] and Erwe and Peschl [1953]. Another realization of $\mathrm{Aut}(B^n)$ can be found in Franzoni and Vesentini [1980].

The Siegel upper half-space is named after Siegel [1943], and it is the main example of Siegel domain of second kind; see Piatetsky-Shapiro [1966].

The translations h_a defined in (2.2.6) induce a group structure on the boundary ∂H^n of the Siegel upper half-space. ∂H^n with this structure, which is homeomorphic to $\mathbf{R} \times \mathbf{C}^{n-1}$, is called the *Heisenberg group* of order $n-1$; see for instance Rothschild and Stein [1976].

Proposition 2.2.9 is in Hayden and Suffridge [1971]; our proof is due to D. Ullrich, and is taken from Rudin [1980]. Proposition 2.2.10 is implicit in MacCluer [1983], where it is ascribed to D. Ullrich.

Schwarz's lemma in B^n is an easy generalization of the classical Schwarz lemma in Δ; see Reinhardt [1921]. Lemma 2.2.13, Proposition 2.2.14 and Corollaries 2.2.15 and 2.2.16 are due to Hervé [1963a], while Proposition 2.2.17 for B^2 appeared in Bergmann [1937]; cf. also Bureau [1952], Schieferdecker [1957], Suffridge [1974] and Rudin [1978].

The Bergmann metric of B^n is the concrete instance of three different general objects. The first two are the invariant Carathéodory and Kobayashi metrics, that we shall thoroughly discuss in the next chapter. The third one is the Bergmann metric, introduced by Bergmann [1933, 1935] by means of the space of holomorphic square-integrable functions, which is a Kähler metric invariant under automorphisms that can be defined on a large class of complex manifolds, including for instance bounded domains of $\mathbf{C}^n$. For more details, see Kobayashi [1959], Lichnerowicz [1965] and Fefferman [1974].

The definition (2.2.21) of horospheres in the ball is in Hervé [1963a]; it is essentially differential geometric in character, as we shall see in the notes to chapter 2.4. Proposition 2.2.20 is due to Yang [1978], and it will be very important in chapter 2.4.

Theorem 2.2.21 appears in Minialoff [1935] for B^2; the general statement is due to Hervé [1963a], where it is also discussed what happens in case of equality at one point in (2.2.27).

Theorem 2.2.22 appears with a slightly different proof in MacCluer [1983], and with a much more different proof in Chen [1984]. In MacCluer [1983] there is also a discussion of the case of equality at one point in (2.2.29).

The Korányi regions were introduced by Korányi [1969] and Korányi and Stein [1968] in the study of boundary behavior of harmonic functions in B^n. Later, Stein [1972] generalized Korányi's approach to a large class of domains, exploiting the asimmetry between the radial direction and the complex tangential directions appearing in horospheres and Korányi regions; see also the notes to chapter 2.7.

The definition of restricted K-limit is an elaboration due to Rudin [1980] of a more general notion introduced by Čirka [1973] in bounded C^1 domains of $\mathbf{C}^n$; Theorem 2.2.25 and Proposition 2.2.26 have the same origin.

Hervé [1963a] proved Theorem 2.2.29, but he dealt only with non-tangential limits and considered only the functions (i) and (ii). The complete statement and the final examples are due to Rudin [1980].

Theorem 2.2.31 was first proved by Hervé [1963a], and later rediscovered by MacCluer [1983]. Hervé's proof relies on Julia's lemma instead of Wolff's lemma, with an approach similar to the one used in the second proof of the Wolff-Denjoy theorem described in section 1.3.2. Another paper dealing with the iterates in B^n is Kubota [1983].

Theorem 2.2.32 and Proposition 2.2.33 are in Hervé [1963a]; see also Vesentini [1985].

Suffridge [1974] proved Theorem 2.2.34 but only for two commuting maps, using a different argument.

The proof we gave of Theorem 2.2.36 is modelled on Mitchell's proof (Mitchell [1979]) of Theorem 1.3.28. Another proof, shorter but not self-contained, goes as follows: the Bergmann metric has constant sectional curvature -4, and B^n is simply connected. Then we can end the proof quoting É. Cartan's theorem, which assures that a compact group of isometries of a simply connected Riemannian manifold of negative sectional curvature has a fixed point. A proof of É. Cartan's theorem can be found, for instance, in Kobayashi and Nomizu [1968].

Finally, it should be mentioned that a large amount of the material described in this chapter has been generalized to the unit ball of a complex Banach space; see Renaud [1973], Franzoni and Vesentini [1980], Goebel and Reich [1984], Stachura [1985] and references therein.