Chapter 2.5
Common fixed points

In this chapter we shall deal with the second main theme of this book, discussing several topics concerning fixed point sets. We begin with the characterization, due to Wong and Rosay, of B^n by means of its automorphism group; using it, we shall be able to prove that in a strongly convex domain $D \subset\subset \mathbf{C}^n$ not biholomorphic to B^n there exists a common fixed point of $\mathrm{Aut}(D)$.

The next argument is the study of the fixed point set of a holomorphic map sending a convex domain into itself. We shall show that in convex domains fixed point sets and holomorphic retracts are one and the same thing; furthermore, using iteration theory, we shall generalize Shields' theorem, proving that a commuting family of holomorphic maps sending a strongly convex domain into itself (and continuous up to the boundary) has a fixed point.

Finally, we shall study one-parameter semigroups of holomorphic maps. The present state of the theory is far away from the completeness and elegance we saw in chapter 1.4; however, we have a fairly complete description of the asymptotic behavior, obtained by applying in an interesting way the theory of fixed point sets in convex domains.

2.5.1 *Compact groups of automorphisms*

This section is devoted to prove that B^n is the only strongly pseudoconvex domain with non-compact automorphism group, and that a compact group of automorphisms of a convex domain always has a fixed point.

To prove the former theorem, we shall apply the characterization of B^n given in Theorem 2.3.43. To be more specific, let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain, and assume that $\mathrm{Aut}(D)$ is non-compact; then we can find $z_0 \in D$ and a sequence $\{\gamma_\nu\} \subset \mathrm{Aut}(D)$ such that $\gamma_\nu(z_0) \to \partial D$. Using localization theorems, we shall prove that $\{\gamma_\nu^* \tilde{\kappa}_D(z_0)\}$ and $\{\gamma_\nu^* \tilde{\gamma}_D(z_0)\}$ tend to the same limit as $\nu \to +\infty$, where $\tilde{\kappa}_D$ and $\tilde{\gamma}_D$ are the Kobayashi and Carathéodory volume forms; hence $\tilde{\kappa}_D(z_0) = \tilde{\gamma}_D(z_0)$, and D is biholomorphic to B^n by Theorem 2.3.43.

To made precise the previous argument, we need localization theorems. Let D be a domain in $\mathbf{C}^n$; then the *Carathéodory volume element* $C_D\colon D \to \mathbf{R}^+$ and the *Kobayashi volume element* $K_D\colon D \to \mathbf{R}^+$ are defined by

$$\forall z \in D \qquad \tilde{\gamma}_D(z) = C_D(z)\Theta \qquad \text{and} \qquad \tilde{\kappa}_D(z) = K_D(z)\Theta,$$

where Θ is the standard volume form given by (2.3.11). We have already proved a localization theorem for the Kobayashi volume element, Theorem 2.3.61; our first aim is a similar result for the Carathéodory volume element. We need

Lemma 2.5.1: *Let D be a bounded domain of $\mathbf{C}^n$, and let $\{D_\nu\}$ be a sequence of domains relatively compact in D such that $D = \bigcup_\nu D_\nu$ and $D_\nu \subset\subset D_{\nu+1}$ for all $\nu \in \mathbf{N}$. Then*

$$\forall z \in D \qquad \lim_{\nu\to\infty} C_{D_\nu}(z) = C_D(z).$$

Proof: Fix $z_0 \in D$. The sequence $\{C_{D_\nu}(z_0)\}$ is not increasing; therefore the limit exists and

$$\lim_{\nu\to\infty} C_{D_\nu}(z_0) \geq C_D(z_0).$$

Conversely, since B^n is taut, for all $\nu \in \mathbf{N}$ there exists $f_\nu \in \mathrm{Hol}(D_\nu, B^n)$ such that $f_\nu(z_0) = 0$ and $|\det d(f_\nu)_{z_0}|^2 = C_{D_\nu}(z_0)$; up to a subsequence, we can assume that $f_\nu \to f \in \mathrm{Hol}(D, B^n)$. Clearly, $f(z_0) = 0$; then

$$C_D(z_0) \geq |\det df_{z_0}|^2 = \lim_{\nu\to\infty} |\det d(f_\nu)_{z_0}|^2 = \lim_{\nu\to\infty} C_{D_\nu}(z_0),$$

and we are done, **q.e.d.**

Then

Proposition 2.5.2: *Let D be a bounded domain of $\mathbf{C}^n$, $x \in \partial D$ a local peak point for D and U a neighbourhood of x in $\mathbf{C}^n$ such that $U \cap D$ is connected. Assume there are $z_0 \in D$ and a sequence $\{\gamma_\nu\} \subset \mathrm{Aut}(D)$ such that $z_\nu = \gamma_\nu(z_0) \to x$. Then*

$$\lim_{\nu\to\infty} \frac{C_D(z_\nu)}{C_{D\cap U}(z_\nu)} = 1.$$

Proof: Clearly,

$$\limsup_{\nu\to\infty} \frac{C_D(z_\nu)}{C_{D\cap U}(z_\nu)} \leq 1.$$

Fix a sequence $\{D_\nu\}$ of subdomains of D such that $z_0 \in D_0$, $D_\nu \subset\subset D_{\nu+1}$ and $D = \bigcup_\nu D_\nu$. Choose $\varepsilon > 0$; by Lemma 2.5.1, there is $\nu_0 \in \mathbf{N}$ such that

$$C_D(z_0) \leq C_{D_\nu}(z_0) \leq (1+\varepsilon) C_D(z_0)$$

for all $\nu \geq \nu_0$. Now, Corollary 2.3.60 implies that $\gamma_\nu(D_{\nu_0}) \subset D \cap U$ for all ν sufficiently large. Therefore we eventually have

$$\frac{C_D(z_\nu)}{C_{D\cap U}(z_\nu)} \geq \frac{C_D\big(\gamma_\nu(z_0)\big)}{C_{\gamma_\nu(D_{\nu_0})}\big(\gamma_\nu(z_0)\big)} = \frac{C_D(z_0)}{C_{D_{\nu_0}}(z_0)} \geq \frac{1}{1+\varepsilon},$$

and the assertion follows, **q.e.d.**

The idea now is to select a particular local peak point such that the hypotheses of Proposition 2.5.2 are fulfilled, to apply Theorem 2.3.61 and Proposition 2.5.2 obtaining the equalities of C_D and K_D somewhere, and to invoke Theorem 2.3.43 to infer that D is biholomophic to B^n. In other words, our aim is:

Theorem 2.5.3: *Let D be a bounded domain of $\mathbf{C}^n$ and $x_0 \in \partial D$ a strongly pseudoconvex point. Assume there are $z_0 \in D$ and a sequence $\{\gamma_\nu\} \subset \mathrm{Aut}(D)$ such that $\gamma_\nu(z_0) \to x_0$. Then D is biholomorphic to B^n.*

Proof: Let U be a neighbourhood of x_0 such that there exists a strictly plurisubharmonic defining function ρ for $D \cap U$ belonging to $C^2(U)$. For every $x \in \partial D \cap U$ let $p_{\rho,x}$ be the Levi polynomial and $L_{\rho,x}$ the Levi form of ρ at x. Then, by (2.3.23),

$$D \cap U = \bigl\{ z \in U \bigm| 2\,\mathrm{Re}\, p_{\rho,x}(z) + L_{\rho,x}(z-x, z-x) + o(\|z-x\|^2) < 0 \bigr\}.$$

Looking at the proof of Proposition 2.1.13 we see that, shrinking U if necessary, for every $x \in \partial D \cap U$ we can find a biholomorphic map $\Phi_x\colon U \to \Phi_x(U) \subset \mathbf{C}^n$ such that

$$D'_x = \Phi_x(D \cap U) = \bigl\{ w \in \Phi_x(U) \bigm| -2\,\mathrm{Re}\, w_1 + \|w\|^2 + o(\|w\|^2) < 0 \bigr\}.$$

Furthermore, for every ν large enough we can find $x_\nu \in \partial D \cap U$ such that, if for every $\nu \in \mathbf{N}$ we set $z_\nu = \gamma_\nu(z_0)$,

$$\Phi_{x_\nu}(z_\nu) = (a_\nu, 0, \ldots, 0)$$

for a suitable $a_\nu > 0$, with $a_\nu \to 0$ as $\nu \to +\infty$.

Now Theorem 2.3.61 and Proposition 2.5.2 imply

$$1 \le \frac{K_D(z_0)}{C_D(z_0)} = \frac{K_D\bigl(\gamma_\nu(z_0)\bigr)}{C_D\bigl(\gamma_\nu(z_0)\bigr)} = \limsup_{\nu\to\infty} \frac{K_{D\cap U}(z_\nu)}{C_{D\cap U}(z_\nu)}; \tag{2.5.1}$$

we claim that the right-hand term in (2.5.1) is 1. Choose $\varepsilon > 0$, and set

$$B^\pm_\varepsilon = \bigl\{ w \in \mathbf{C}^n \bigm| -2\,\mathrm{Re}\, w_1 + (1 \pm \varepsilon)\|w\|^2 < 0 \bigr\};$$

the $B^\pm_\varepsilon$ are euclidean balls of radius $(1 \pm \varepsilon)^{-1/2}$ such that $0 \in \partial B^\pm_\varepsilon$. In particular, by (2.3.12) we have

$$K_{B^\pm_\varepsilon}(w) = C_{B^\pm_\varepsilon}(w) = \frac{1}{(1\pm\varepsilon)^{n+1}\bigl|(1\pm\varepsilon)\|w\|^2 - 2\,\mathrm{Re}\, w_1\bigr|^{n+1}}.$$

If ε is small enough we have

$$\forall x \in \partial D \cap U \qquad B^-_\varepsilon \cap \Phi_x(U) \subset D'_x \subset B^+_\varepsilon \cap \Phi_x(U),$$

and $\Phi_{x_\nu}(z_\nu) \in B^-_\varepsilon \cap \Phi_{x_\nu}(U)$ for ν large enough. Hence

$$\begin{aligned} \limsup_{\nu\to\infty} \frac{K_{D\cap U}(z_\nu)}{C_{D\cap U}(z_\nu)} &\le \limsup_{\nu\to\infty} \frac{K_{B^-_\varepsilon \cap \Phi_{x_\nu}(U)}\bigl(\Phi_{x_\nu}(z_\nu)\bigr)}{C_{B^+_\varepsilon}\bigl(\Phi_{x_\nu}(z_\nu)\bigr)} \\ &= \limsup_{\nu\to\infty} \frac{K_{B^-_\varepsilon}\bigl(\Phi_{x_\nu}(z_\nu)\bigr)}{C_{B^+_\varepsilon}\bigl(\Phi_{x_\nu}(z_\nu)\bigr)} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{n+1}, \end{aligned} \tag{2.5.2}$$

where we have used Theorem 2.3.61 once again. Now $\varepsilon > 0$ is arbitrary; hence (2.5.1) and (2.5.2) imply

$$K_D(z_0) = C_D(z_0)$$

and, by Theorem 2.3.43, D is biholomorphic to B^n, **q.e.d.**

The corollaries of this result are probably more expressive than the theorem itself:

Corollary 2.5.4: *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain such that* $\mathrm{Aut}(D)$ *is non-compact. Then D is biholomorphic to B^n.*

Proof: If $\mathrm{Aut}(D)$ is non-compact, by Theorem 2.1.26 and Corollary 2.1.14 there is a sequence $\{\gamma_\nu\} \subset \mathrm{Aut}(D)$ converging to a point $x_0 \in \partial D$, and the assertion follows from Theorem 2.5.3, **q.e.d.**

Corollary 2.5.5: *Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain such that* $\mathrm{Aut}(D)$ *is transitive. Then D is biholomorphic to B^n.*

Proof: Assume, for simplicity, that $0 \notin D$. Let $x_0 \in \partial D$ be a point of ∂D farthest from 0, and let $R = \|x_0\|$. Then near x_0 the boundary of D is defined by a C^2 function ρ such that $\rho(z) - (\|z\|^2 - R^2) \geq 0$, with equality possibly only on ∂D. Therefore x_0 is a local minimum for $\rho(z) - (\|z\|^2 - R^2)$, and so the real Hessian of ρ at x_0 is strictly positive definite. In particular, x_0 is a strongly convex point of ∂D.

Now choose $z_0 \in D$; since $\mathrm{Aut}(D)$ is transitive, we can find a sequence $\{\gamma_\nu\} \subset \mathrm{Aut}(D)$ such that $\gamma_\nu(z_0) \to x_0$; by Theorem 2.5.3, D is biholomorphic to B^n, **q.e.d.**

So B^n is really a very special domain: it is the unique homogeneous C^2 domain, and even the unique strongly pseudoconvex domain with non-compact automorphism group. Now, let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex domain not biholomorphic to the ball; we know, by Proposition 2.1.24, that the isotropy group of a point of D is compact; so it would be nice if the whole automorphism group were contained in the isotropy group of a point, that is if $\mathrm{Aut}(D)$ had a fixed point. Unfortunately, without extra topological hypotheses this is a vain hope: a trivial variation of the map (2.4.6) yields an automorphism without fixed points. So, as by now usual, we restrict our attention to strongly convex domains, where at least we know that every automorphism has a fixed point in the closure of the domain (by Brouwer's theorem and Theorem 2.3.58). Actually, the fixed point is in the domain itself:

Corollary 2.5.6: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain not biholomorphic to B^n, and $\gamma \in$* $\mathrm{Aut}(D)$*. Then γ has a fixed point in D.*

Proof: Since D is not biholomorphic to B^n, $\mathrm{Aut}(D)$ is compact, by Corollary 2.5.4. In particular, the sequence of iterates $\{\gamma^k\}$ cannot be compactly divergent, and the assertion follows from Theorem 2.4.20, **q.e.d.**

So we can hope in the existence of a fixed point for the automorphism group of a strongly convex domain. Actually, a much stronger result holds, generalizing Theorem 2.2.36:

Theorem 2.5.7: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and Γ a subgroup of* $\mathrm{Aut}(D)$*. Then Γ is relatively compact in* $\mathrm{Aut}(D)$ *iff it has a fixed point in D.*

Proof: If Γ has a fixed point, then it is relatively compact in $\mathrm{Aut}(D)$ by Proposition 2.1.24. To prove the converse, we can directly assume Γ compact, of course. Take $z_0 \in D$, and set

$$\tau_1 = \inf\Big\{r > 0 \;\Big|\; \bigcap_{\gamma\in\Gamma} \overline{B_k\big(\gamma(z_0), r\big)} \neq \emptyset\Big\} > 0.$$

Then $K_1 = \bigcap_{\gamma\in\Gamma} \overline{B_k\big(\gamma(z_0), \tau_1\big)}$ is a compact, convex, non-empty and Γ-invariant subset of D (by Proposition 2.3.46). Furthermore, the interior of K_1 is empty, for τ_1 is minimal. Hence the real dimension of the affine hull H_1 of K_1 (i.e., of the smallest affine subspace of $\mathbf{C}^n$ containing K_1) is strictly less than $2n$.

If K_1 is not reduced to a point, we repeat the argument. Take $z_1 \in K_1$, and set

$$\tau_2 = \inf\Big\{r > 0 \;\Big|\; K_1 \cap \Big[\bigcap_{\gamma\in\Gamma} \overline{B_k\big(\gamma(z_1), r\big)}\Big] \neq \emptyset\Big\} > 0$$

and

$$K_2 = K_1 \cap \Big[\bigcap_{\gamma\in\Gamma} \overline{B_k\big(\gamma(z_1), \tau_2\big)}\Big].$$

Again, K_2 is compact, convex, non-empty and Γ-invariant. Furthermore, the interior part of K_2 relative to H_1 is empty, again by the minimality of τ_2 (note that $\gamma(z_1) \in K_1$ for every $\gamma \in \Gamma$, and that $\overline{B_k\big(\gamma(z_1), \tau_2\big)} \cap K_1$ is a convex set whose affine hull is still H_1). Hence the real dimension of the affine hull H_2 of K_2 is strictly less than the dimension of H_1.

Iterating this construction, we obtain a decreasing sequence of compact convex non-empty Γ-invariant subsets of D with strictly decreasing affine hulls. This sequence must stop, and the last subset should consist of one point, which is clearly Γ-invariant, **q.e.d.**

Corollary 2.5.8: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain not biholomorphic to B^n. Then $\mathrm{Aut}(D)$ has a fixed point.*

Proof: Theorem 2.5.7 and Corollary 2.5.4, **q.e.d.**

2.5.2 *Fixed point sets*

Now we would like to say something about the shape of a fixed point set. If X is a complex manifold and $f \in \mathrm{Hol}(X, X)$, then the fixed point set $\mathrm{Fix}(f)$ of f in X is clearly a complex analytic subspace of X. Is it smooth? Is it connected? In other words, is $\mathrm{Fix}(f)$ a closed submanifold of X? In this section we shall prove that, if X is taut, $\mathrm{Fix}(f)$ is a not necessarily connected closed submanifold of X. Furthermore, if $D \subset\subset \mathbf{C}^n$ is a convex domain and $f \in \mathrm{Hol}(D, D)$, then $\mathrm{Fix}(f)$ is always a holomorphic retract of D, thus connected, reproducing the situation we saw in B^n. As an application, we shall generalize Shields' theorem to strongly convex domains.

We begin our investigations with the following observation:

Proposition 2.5.9: *Let X be a taut manifold and $z_0 \in X$. Then there exists a local chart φ about z_0 such that*

$$\forall \gamma \in \mathrm{Aut}_{z_0}(X) \qquad \varphi \circ \gamma = d\gamma_{z_0} \circ \varphi, \tag{2.5.3}$$

where we are identifying $T_{z_0}X$ and $\mathbf{C}^n$ by means of the local coordinates induced by φ.

Proof: By Proposition 2.1.24, $\Gamma = \mathrm{Aut}_{z_0}(X)$ is a compact group; let μ be the Haar measure of Γ (i.e., a right-invariant measure of total mass 1; a proof of the existence of the Haar measure is in Rudin [1973]). Up to replace X by a small enough Kobayashi ball of center z_0, we can assume X is a domain in $\mathbf{C}^n$. Then define $\varphi\colon X \to \mathbf{C}^n$ by

$$\forall z \in X \qquad \varphi(z) = \int_\Gamma (d\gamma_{z_0})^{-1}\big(\gamma(z)\big)\, d\mu(\gamma).$$

Since $d\varphi_{z_0} = \mathrm{id}$, φ is a local chart about z_0. Furthermore, for every $\tilde\gamma \in \mathrm{Aut}_{z_0}(X)$ we have

$$\begin{aligned}\varphi \circ \tilde\gamma(z) = \int_\Gamma (d\gamma_{z_0})^{-1}\big(\gamma \circ \tilde\gamma(z)\big)\, d\mu(\gamma) &= d\tilde\gamma_{z_0} \int_\Gamma \big(d(\gamma \circ \tilde\gamma)_{z_0}\big)^{-1}\big(\gamma \circ \tilde\gamma(z)\big)\, d\mu(\gamma)\\ &= d\tilde\gamma_{z_0}\big(\varphi(z)\big),\end{aligned}$$

because of the right-invariance of μ, and we are done, **q.e.d.**

Then we can show that fixed point sets are always (not necessarily connected) submanifolds:

Corollary 2.5.10: *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$. Then $\mathrm{Fix}(f)$ is either empty or a (not necessarily connected) closed submanifold of X.*

Proof: If $\mathrm{Fix}(f) \ne \emptyset$, we can replace X by the limit manifold of f, and then (Corollary 2.1.31) assume $f \in \mathrm{Aut}(X)$. Take $z_0 \in \mathrm{Fix}(f)$; then, by Proposition 2.5.9, we can linearize f in a neighbourhood of z_0, and the assertion follows, **q.e.d.**

In particular, we can easily identify the tangent space to a fixed point set:

Corollary 2.5.11: *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$ such that $\mathrm{Fix}(f) \ne \emptyset$. Let $z_0 \in \mathrm{Fix}(f) = F$; then $T_{z_0}F = \ker(df_{z_0} - \mathrm{id})$.*

Proof: This immediately follows from (2.5.3) and Corollary 2.5.10, **q.e.d.**

In general, the fixed point set is disconnected. For instance, let

$$D = \big\{(z,w) \in \mathbf{C}^2 \bigm| |z|^2 + |w|^2 + |w|^{-2} < 3\big\},$$

and let $f \in \mathrm{Hol}(D,D)$ be given by $f(z,w) = (z, w^{-1})$. Then

$$\mathrm{Fix}(f) = \Delta \times \{1\} \cup \Delta \times \{-1\}.$$

On the other hand, every fixed point set in a convex domain is connected. Actually, we shall prove even more:

Theorem 2.5.12: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and let $f \in \mathrm{Hol}(D, D)$ be such that $\mathrm{Fix}(f) \neq \emptyset$. Then $\mathrm{Fix}(f)$ is a holomorphic retract of D. In particular, $\mathrm{Fix}(f)$ is connected.*

Proof: For every $z \in D$ and $t \in (0, 1)$ define $f_{t,z} \in \mathrm{Hol}(D, D)$ by

$$f_{t,z}(w) = (1 - t)z + tf(w).$$

Clearly, $f_{t,z}(D) \subset\subset D$ for every $z \in D$ and $t \in (0, 1)$; therefore $f_{t,z}$ has a unique fixed point $h_t(z) \in D$, and $(f_{t,z})^k \to h_t(z)$ as $k \to +\infty$ (Corollary 2.1.32). Note that $h_t \in \mathrm{Hol}(D, D)$ for every $t \in (0, 1)$, for h_t is the limit of the sequence of holomorphic maps $\{(f_{t,\cdot})^k(z_0)\}$, where z_0 is a fixed point of f.

Now we have

$$\forall z \in D \qquad \sup_{t\in(0,1)} k_D\big(h_t(z), z\big) \le 2k_D(z, z_0) < +\infty,$$

because $f_{t,z_0}(z_0) = z_0$ for every $t \in (0, 1)$. In particular, no sequence of h_t's can be compactly divergent; then there exists a sequence $\{t_\nu\} \subset (0, 1)$ converging to 1 such that $h_{t_\nu} \to \rho \in \mathrm{Hol}(D, D)$.

We claim that ρ is a holomorphic retraction of D onto $\mathrm{Fix}(f)$. Indeed, by definition

$$\forall z \in D\ \forall t \in (0, 1) \qquad f_{t,z}\big(h_t(z)\big) = h_t(z);$$

therefore $f \circ \rho = \rho$, and $\rho(D) \subset \mathrm{Fix}(f)$. But if $z_0 \in \mathrm{Fix}(f)$ then $h_t(z_0) = z_0$ for all $t \in (0, 1)$; thus $\rho(D) = \mathrm{Fix}(f)$ and $\rho|_{\mathrm{Fix}(f)} = \mathrm{id}_{\mathrm{Fix}(f)}$, **q.e.d.**

It should be noticed that if $f \in \mathrm{Hol}(D, D) \cap C^0(\overline{D})$, in general the fixed point set of f in $\overline{D}$ is *not* connected (and thus it is not the closure of the fixed point set of f in D): take for instance $D = B^2$ and $f(z, w) = (z^3, w)$.

So in convex domains holomorphic retracts and fixed point sets are one and the same thing. Together with iteration theory, this yields a neat generalization of Shields' theorem (and of Theorem 2.2.34) to strongly convex domains. We begin with

Lemma 2.5.13: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, $\rho: D \to M$ a holomorphic retraction of D, and $f \in \mathrm{Hol}(D, D)$ such that $f(M) \subset M$. If $\mathrm{Fix}(f) \neq \emptyset$, then $\mathrm{Fix}(f) \cap M$ is a non-empty holomorphic retract of D.*

Proof: Set $g = \rho \circ f$. For every $z \in M$ and $k \in \mathbf{N}$ we have $g^k(z) = f^k(z)$; hence $\{g^k\}$ cannot be compactly divergent and, by Theorem 2.4.20, $\mathrm{Fix}(g) \neq \emptyset$. Since $\mathrm{Fix}(g) = \mathrm{Fix}(f) \cap M$, the assertion follows from Theorem 2.5.12, **q.e.d.**

In particular we have

Proposition 2.5.14: *Let $D \subset\subset \mathbf{C}^n$ be a convex domain, and $\mathcal{F} \subset \mathrm{Hol}(D, D)$ a commuting family of holomorphic maps such that $\mathrm{Fix}(f) \neq \emptyset$ for every $f \in \mathcal{F}$. Then $\mathcal{F}$ has a fixed point in D.*

Proof: First of all note that if f, $g \in \mathcal{F}$ then $g\bigl(\mathrm{Fix}(f)\bigr) \subset \mathrm{Fix}(f)$, for f and g commute. Then using Lemma 2.5.13, Theorem 2.5.12, Lemma 2.1.28 and an induction argument it is easy to show that $\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)$ is a non-empty closed complex submanifold of D for every $r \in \mathbf{N}$ and $f_1, \ldots, f_r \in \mathcal{F}$. Set

$$d = \min\bigl\{\dim\bigl[\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)\bigr] \bigm| r \in \mathbf{N},\ f_1, \ldots, f_r \in \mathcal{F}\bigr\} \geq 0,$$

and choose $f_1, \ldots, f_{r_0} \in \mathcal{F}$ so that the dimension of $F = \mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_{r_0})$ is exactly d. This implies that $F \cap \mathrm{Fix}(f) = F$ for every $f \in \mathcal{F}$, and so every point in F is a fixed point of $\mathcal{F}$, **q.e.d.**

Then

Theorem 2.5.15: *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, and $\mathcal{F}$ a family of continuous self-maps of $\overline{D}$ which are holomorphic in D and commute with each other under composition. Then $\mathcal{F}$ has a fixed point in $\overline{D}$.*

Proof: If there is $f \in \mathcal{F}$ such that $f(D) \cap \partial D \neq \emptyset$, then $f \equiv x_0 \in \partial D$ (for D has simple boundary), and x_0 is clearly a fixed point of $\mathcal{F}$. So we can suppose, without loss of generality, that $\mathcal{F} \subset \mathrm{Hol}(D, D)$.

Assume there is $f \in \mathcal{F}$ without fixed points in D. Then, by Theorem 2.4.23, the sequence of iterates of f converges to a point $x_0 \in \partial D$. It follows that for any $g \in \mathcal{F}$ we have

$$g(x_0) = \lim_{k\to\infty} g \circ f^k = \lim_{k\to\infty} f^k \circ g = x_0,$$

and x_0 is a fixed point of $\mathcal{F}$.

Assume finally that $\mathrm{Fix}(f) \neq \emptyset$ for every $f \in \mathcal{F}$; then the assertion follows from Proposition 2.5.14, **q.e.d.**

We can even generalize Shields' theorem to product domains:

Theorem 2.5.16: *Let $D_1 \subset\subset \mathbf{C}^{n_1}, \ldots, D_r \subset\subset \mathbf{C}^{n_r}$ be strongly convex domains, and $\mathcal{F}$ a family of continuous self-maps of $\overline{D_1} \times \cdots \times \overline{D_r}$ which are holomorphic in $D_1 \times \cdots \times D_r$ and commute with each other under composition. Then $\mathcal{F}$ has a fixed point in $\overline{D_1} \times \cdots \times \overline{D_r}$.*

Proof: If every map $f \in \mathcal{F}$ sends $D_1 \times \cdots \times D_r$ into itself and has a fixed point there, then we can apply Proposition 2.5.14. If this is not the case, we proceed by induction on r. For $r = 1$, apply Theorem 2.5.15; for $r > 1$ we have two cases.

Case (a): there is $f_0 \in \mathcal{F}$ such that $f_0(D_1 \times \cdots \times D_r) \not\subset D_1 \times \cdots \times D_r$. Then, using weak peak functions (cf. Corollary 2.1.11), this implies $f_0(D_1 \times \cdots \times D_r) \subset \partial(D_1 \times \cdots \times D_r)$. Without loss of generality, we may assume

$$f_0(D_1 \times \cdots \times D_r) \subset \{a_1\} \times \cdots \times \{a_p\} \times D_{p+1} \times \cdots \times D_r, \tag{2.5.4}$$

for suitable $1 \le p \le r$ and $a_1 \in \partial D_1, \ldots, a_p \in \partial D_p$, because every D_j is strongly convex. If $p = r$, f_0 is constant, and thus it is a fixed point of $\mathcal{F}$; so assume $p < r$.

Take $f \in \mathcal{F}$, and for $j = 1, \ldots, p$ let $\pi_j \colon D_1 \times \cdots \times D_r \to D_j$ be the canonical projection. By (2.5.4) we have

$$f\big(f_0(D_1 \times \cdots \times D_r)\big) \subset \{a_1\} \times \cdots \times \{a_p\} \times \overline{D_{p+1}} \times \cdots \times \overline{D_r};$$

therefore for every $j = 1, \ldots, p$ the holomorphic map $h_j \colon D_{p+1} \times \cdots \times D_r \to \mathbf{C}^{n_j}$ given by

$$h_j(z_{p+1}, \ldots, z_r) = \pi_j \circ f(a_1, \ldots, a_p, z_{p+1}, \ldots, z_r)$$

is such that $h_j(D_{p+1} \times \cdots \times D_r) \subset \overline{D_j}$ and $a_j \in h_j(D_{p+1} \times \cdots \times D_r)$. Since D_j is strongly convex, this implies $h_j \equiv a_j$; so for every $f \in \mathcal{F}$ we have

$$f(\{a_1\} \times \cdots \times \{a_p\} \times D_{p+1} \times \cdots \times D_r) \subset \{a_1\} \times \cdots \times \{a_p\} \times \overline{D_{p+1}} \times \cdots \times \overline{D_r},$$

and, by the induction hypothesis, $\mathcal{F}$ has a fixed point.

Case (b): $f(D_1 \times \cdots \times D_r) \subset D_1 \times \cdots \times D_r$ for every $f \in \mathcal{F}$, and there is $f_0 \in \mathcal{F}$ with $\mathrm{Fix}(f_0) = \emptyset$. By Theorem 2.4.20, the sequence $\{f_0^k\}$ is compactly divergent; hence there is a subsequence $\{f_0^{k_\nu}\}$ converging to a holomorphic map g such that

$$g(D_1 \times \cdots \times D_r) \subset \partial(D_1 \times \cdots \times D_r).$$

But $g \circ f = f \circ g$ or every $f \in \mathcal{F}$, and the assertion follows from the argument used in Case (a), **q.e.d.**

So we have a Shields' theorem for certain convex domains, like Δ^n. At present, it is not known if such a theorem holds for every convex domain; the best results in this direction are Proposition 2.5.14 and Theorem 2.5.16, of course.

2.5.3 *One-parameter semigroups*

We end this chapter with an account of the theory of one-parameter semigroups of holomorphic maps. We shall be mainly concerned with the description of the asymptotic behavior, both on taut manifolds and on strongly convex domains, but we shall also present a characterization of the infinitesimal generators of one-parameter semigroups on B^n.

We shall need a general fact about linear semigroups, which is well-known in the general setting of linear norm-continuous semigroups in Banach spaces (or in the setting of linear Lie groups). We shall prove it here in the finite-dimensional case:

Proposition 2.5.17: *Let $T \colon \mathbf{R}^+ \to \mathbf{GL}(n, \mathbf{C})$ be a continuous semigroup homomorphism. Then $T_t = \exp(tA)$ for some $n \times n$ complex matrix A.*

Proof: Since T_t tends to I, the identity operator, as $t \to 0$, we have

$$\frac{1}{t} \int_0^t T_s \, ds \longrightarrow I$$

as $t \to 0$. In particular, there is $\delta > 0$ such that $\int_0^t T_s\,ds$ is invertible for $t \in (0,\delta)$. Fix $t_0 \in (0,\delta)$; then

$$\begin{aligned}\frac{1}{h}(T_h - I)\int_0^{t_0} T_s\,ds - (T_{t_0} - I) &= \frac{1}{h}\left[\int_h^{t_0+h} T_s\,ds - \int_0^{t_0} T_s\,ds\right] - (T_{t_0} - I)\\ &= \frac{1}{h}\left[\int_{t_0}^{t_0+h} (T_s - T_{t_0})\,ds - \int_0^{h} (T_s - I)\,ds\right]\\ &= (T_{t_0} - I)\left[\frac{1}{h}\int_0^h T_s\,ds - I\right] \longrightarrow 0\end{aligned}$$

as $h \to 0$; hence

$$\lim_{h\to 0}\frac{1}{h}(T_h - I) = (T_{t_0} - I)\left(\int_0^{t_0} T_s\,ds\right)^{-1}. \tag{2.5.5}$$

Set $A = (T_{t_0} - I)\left(\int_0^{t_0} T_s\,ds\right)^{-1}$; by (2.5.5) A does not depend on t_0. Moreover, (2.5.5) implies that for every $z \in \mathbf{C}^n$ the map $t \mapsto T_t(z)$ is C^1, and satisfies

$$\begin{cases}\dfrac{\partial T_t}{\partial t}(z) = AT_t(z),\\ T_0(z) = z.\end{cases}$$

Since $(t,z) \mapsto \exp(tA)z$ is another solution of the same Cauchy problem, it follows that $T_t = \exp(tA)$, **q.e.d.**

We recall that a *one-parameter semigroup* on a complex manifold X is a continuous semigroup homomorphism $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$. If the image of Φ is contained in $\mathrm{Aut}(X)$, then Φ extends to a continuous group homomorphism of $\mathbf{R}$ into $\mathrm{Aut}(X)$, and we shall say that Φ is a *one-parameter group*. The following two propositions are straightforward generalizations of their one-variable counterparts:

Proposition 2.5.18: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a complex manifold X. Then Φ_t is one-to-one for all $t \geq 0$.*

Proof: Copy the proof of Proposition 1.4.6, **q.e.d.**

Proposition 2.5.19: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a taut manifold X. Assume $\Phi_{t_0} \in \mathrm{Aut}(X)$ for some $t_0 > 0$; then Φ is a one-parameter group.*

Proof: Copy the proof of Proposition 1.4.7, clearly replacing Corollary 1.1.47 by Proposition 2.1.24, **q.e.d.**

We get immediately rid of the compact case:

Proposition 2.5.20: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a compact hyperbolic manifold X. Then $\Phi_t = \mathrm{id}_X$ for all $t \ge 0$.*

Proof: By Proposition 2.5.18, every Φ_t is injective, and hence open; since X is compact, every Φ_t is an automorphism of X. But $\mathrm{Aut}(X)$ is finite (Corollary 2.4.7), and the assertion follows, **q.e.d.**

Now we shall study the asymptotic behavior of one-parameter semigroups on taut manifolds and, in particular, of one-parameter semigroups with a fixed point.

Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a complex manifold X. A point $z_0 \in X$ is a *fixed point* of Φ if $\Phi_t(z_0) = z_0$ for all $t \ge 0$. If z_0 is a fixed point of Φ, then $t \mapsto T_t = d(\Phi_t)_{z_0}$ is a linear semigroup on $T_{z_0}X$; by Proposition 2.5.17, there exists a linear operator A_Φ acting on $T_{z_0}X$ such that $T_t = \exp(tA_\Phi)$. A_Φ is the *spectral generator* of Φ at z_0, and the eigenvalues of A_Φ are the *spectral values* of Φ at z_0. Note that, by the Cartan-Carathéodory Theorem 2.1.21, if X is taut then the spectral values of Φ at z_0 are contained in the closed left half-plane of $\mathbf{C}$. Then we can prove

Theorem 2.5.21: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a taut manifold X. Then Φ_t converges as $t \to +\infty$ to a map $\rho \in \mathrm{Hol}(X,X)$ iff Φ has a fixed point $z_0 \in X$ and its spectral values at z_0 are contained in*

$$\{\zeta \in \mathbf{C} \mid \mathrm{Re}\,\zeta < 0\} \cup \{0\} = iH^+ \cup \{0\}.$$

Proof: Assume $\Phi_t \to \rho \in \mathrm{Hol}(X,X)$ as $t \to +\infty$. Then for all $t_0 > 0$ the map ρ is necessarily the limit retraction of Φ_{t_0}, and $\Phi_{kt_0} = (\Phi_{t_0})^k \to \rho$. In particular, every Φ_{t_0} fixes the points of the image of ρ, and thus every point of $M = \rho(X)$ is a fixed point of Φ. Now choose $z_0 \in M$; since $\Phi_k \to \rho$ as $k \to +\infty$, we know (by Theorem 2.4.1) that $\mathrm{sp}\big(d(\Phi_1)_{z_0}\big) \subset \Delta \cup \{1\}$. Since

$$\mathrm{sp}\big(d(\Phi_1)_{z_0}\big) = \exp\big(\mathrm{sp}(A_\Phi)\big),$$

where A_Φ is the spectral generator of Φ at z_0, it follows that $\mathrm{sp}(A_\Phi) \subset iH^+ \cup \{0\}$, and one direction is proved.

Conversely, assume that Φ has a fixed point $z_0 \in X$ such that the spectral values of Φ at z_0 are contained in $iH^+ \cup \{0\}$. Then for every $t_0 > 0$ the spectrum of the differential $d(\Phi_{t_0})_{z_0}$ is contained in $\Delta \cup \{1\}$; therefore, by Theorem 2.4.1, the sequence $\{\Phi_{kt_0}\}$ converges. In particular, for every given $p \in \mathbf{N}$ the sequence $\{\Phi_{k/2^p}\}$ converges, and the limit does not depend on p — for if $p < q$ then $\{\Phi_{k/2^p}\}$ is a subsequence of $\{\Phi_{k/2^q}\}$. Since $\{k/2^p \mid k,\, p \in \mathbf{N}\}$ is dense in $\mathbf{R}^+$, this implies that the whole semigroup converges, as we wanted to show, **q.e.d.**

As we did for the iterates, to study the compactly divergent case we shall limit ourselves to convex domains. Again, we have a few general facts:

Proposition 2.5.22: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ be a one-parameter semigroup on a domain $D \subset \mathbf{C}^n$. Then there is a holomorphic map $F\colon D \to \mathbf{C}^n$ such that*

$$\frac{\partial \Phi}{\partial t} = F \circ \Phi. \tag{2.5.6}$$

In particular, Φ is analytic in t.

Proof: Copy the proof of Theorem 1.4.11, replacing derivatives by differentials, **q.e.d.**

The map F verifying (2.5.6) is the *infinitesimal generator* of the one-parameter semigroup Φ. There is a relation between infinitesimal and spectral generators, of course:

Proposition 2.5.23: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ be a one-parameter semigroup on a domain $D \subset \mathbf{C}^n$, and let $F \in \mathrm{Hol}(D, \mathbf{C}^n)$ be its infinitesimal generator. Then:*
(i) *$z_0 \in D$ is a fixed point of Φ iff $F(z_0) = 0$;*
(ii) *if Φ has a fixed point z_0, then its spectral generator at z_0 is dF_{z_0}.*

Proof: (i) is proved exactly as in Proposition 1.4.13. Regarding (ii), if A is the spectral generator of Φ at z_0 then

$$dF_{z_0} \circ d(\Phi_t)_{z_0} = d(F \circ \Phi_t)_{z_0} = d\left(\frac{\partial \Phi_t}{\partial t}\right)(z_0) = \frac{\partial}{\partial t}(d\Phi_t)_{z_0} = A \circ d(\Phi_t)_{z_0},$$

and hence $dF_{z_0} = A$, because $d(\Phi_t)_{z_0}$ is invertible, by Proposition 2.5.18, **q.e.d.**

Let $D \subset\subset \mathbf{C}^n$ be a strongly convex domain, and $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ a one-parameter semigroup on D. We shall say that Φ is *fixed point free* if $\mathrm{Fix}(\Phi_t) = \emptyset$ for all $t > 0$. We shall see in a moment that this is equivalent to saying that Φ has no fixed points.

The first step in the investigation of the asymptotic behavior is:

Theorem 2.5.24: *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ be a one-parameter semigroup on a strongly convex domain $D \subset\subset \mathbf{C}^n$. Assume $\mathrm{Fix}(\Phi_{t_0}) = \emptyset$ for some $t_0 > 0$. Then Φ_t tends to a point $x \in \partial D$ as $t \to +\infty$.*

Proof: By Theorem 2.4.23, the sequence $\{\Phi_{kt_0}\} = \{(\Phi_{t_0})^k\}$ converges, uniformly on compact sets, to a point $x \in \partial D$.

Fix $z_0 \in D$, and let $K = \{\Phi_s(z_0) \mid 0 \le s \le t_0\}$. By continuity, K is a compact subset of D; therefore, for all $\varepsilon > 0$ there is $k_\varepsilon \in \mathbf{N}$ such that

$$k \ge k_\varepsilon \Longrightarrow \sup_{z \in K} \|\Phi_{kt_0}(z) - x\| < \varepsilon \Longrightarrow \sup_{0 \le s \le t_0} \|\Phi_{kt_0+s}(z_0) - x\| < \varepsilon,$$

that is $\|\Phi_t(z_0) - x\| < \varepsilon$ for all $t \ge k_\varepsilon t_0$. In other words, $\Phi_t(z_0)$ converges to x for all $z_0 \in D$; by Corollary 2.1.17, $\Phi_t \to x$, **q.e.d.**

In particular we have:

Corollary 2.5.25: *Let* $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ *be a one-parameter semigroup on a strongly convex domain* $D \subset\subset \mathbf{C}^n$. *Then* Φ *is fixed point free iff* $\mathrm{Fix}(\Phi_{t_0}) = \emptyset$ *for some* $t_0 > 0$.

Proof: One direction is trivial. Conversely, if $\mathrm{Fix}(\Phi_{t_0}) = \emptyset$ for some $t_0 > 0$, then by Theorem 2.5.24 the semigroup converges to a point in the boundary of D; hence no Φ_t with $t > 0$ can have a fixed point, **q.e.d.**

The next step is crucial:

Proposition 2.5.26: *Let* $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ *be a one-parameter semigroup on a strongly convex domain* $D \subset\subset \mathbf{C}^n$. *Assume* $\mathrm{Fix}(\Phi_{t_0}) \neq \emptyset$ *for some* $t_0 > 0$. *Then there is a non-empty closed connected submanifold* F *of* D *contained in* $\mathrm{Fix}(\Phi_t)$ *for every* $t \in \mathbf{R}^+$. *In particular,* Φ *has fixed points.*

Proof: Put $f_k = \Phi_{t_0/2^k}$; then $f_0 = \Phi_{t_0}$ and $(f_{k+1})^2 = f_k$. Let $F_k = \mathrm{Fix}(f_k)$; by Theorem 2.5.12 every F_k is a closed connected submanifold of D, and $F_k \supset F_{k+1}$. Moreover, $F_0 \neq \emptyset$; then, by Corollary 2.5.25 every F_k is not empty.

So we have constructed a decreasing sequence of non-empty closed connected submanifolds of D; therefore $\dim F_k$ should eventually become constant. But F_{k+1} is a closed submanifold of F_k, which is connected; hence $\dim F_{k+1} = \dim F_k$ implies $F_{k+1} = F_k$, and the sequence $\{F_k\}$ itself is eventually constant. Let F be its limit.

By construction, $F \subset \mathrm{Fix}(\Phi_{t_0/2^k})$ for all $k \in \mathbf{N}$; hence $F \subset \mathrm{Fix}(\Phi_{pt_0/2^k})$ for all p and $k \in \mathbf{N}$. Since $\{pt_0/2^k \mid p,\, k \in \mathbf{N}\}$ is dense in $\mathbf{R}^+$, we finally get $F \subset \mathrm{Fix}(\Phi_t)$ for all $t \in \mathbf{R}^+$, **q.e.d.**

Corollary 2.5.25 and Proposition 2.5.26 show, as announced, that a one-parameter semigroup in a strongly convex domain either has a fixed point or is fixed point free. In particular, we can collect what we did and prove

Theorem 2.5.27: *Let* $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(D,D)$ *be a one-parameter semigroup on a strongly convex domain* $D \subset\subset \mathbf{C}^n$. *Then* Φ_t *converges as* $t \to +\infty$ *to a map in* $\mathrm{Hol}(D,\mathbf{C}^n)$ *iff either*

(i) Φ *has a fixed point* $z_0 \in D$, *and the spectral values of* Φ *at* z_0 *belong to* $iH^+ \cup \{0\}$, *or*

(ii) Φ *is fixed point free.*

Proof: This follows from Theorem 2.5.21, Proposition 2.5.26 and Theorem 2.5.24, **q.e.d.**

We end this chapter characterizing the infinitesimal generators of one-parameter semigroups on B^n. Let $F \in \mathrm{Hol}(B^n,\mathbf{C}^n)$; since $TB^n = B^n \times \mathbf{C}^n$, F can be thought of as a holomorphic section of TB^n, and thus it makes sense to consider $\kappa_{B^n} \circ F\colon B^n \to \mathbf{R}^+$. Furthermore, κ_{B^n} is a smooth function out of the zero section of TB^n; hence $d(\kappa_{B^n} \circ F)$ is defined out of the zero set of F. Now, using (2.2.16) we see that

$$\frac{\partial \kappa_{B^n}}{\partial z_j}(z;v) = \frac{1}{2(1-\|z\|^2)}\left[2\bar{z}_j \kappa_{B^n}(z;v) + \frac{(v,z)\bar{v}_j - \|v\|^2 \bar{z}_j}{\kappa_{B^n}(z;v)(1-\|z\|^2)}\right],$$

and
$$\frac{\partial\kappa_{B^n}}{\partial v_j}(z;v)=\frac{1}{2\kappa_{B^n}(z;v)(1-\|z\|^2)^2}\big[(z,v)\bar z_j+(1-\|z\|^2)\bar v_j\big];$$
therefore for every $z\in B^n$ we have
$$\begin{aligned}d(\kappa_{B^n}\circ F)\cdot F\\ =\frac{1}{\kappa_{B^n}(z;F)(1-\|z\|^2)^4}\operatorname{Re}\Big[(2\|G\|^2-|(G,z)|^2)(G,z)+(1-\|z\|^2)^2(dF\cdot F,G)\Big],\end{aligned}\tag{2.5.7}$$
where
$$G(z)=(1-\|z\|^2)F(z)+\big(F(z),z\big)z.$$
Note that if $F(z_0)=0$ then
$$\lim_{z\to z_0}d(\kappa_{B^n}\circ F)(z)\cdot F(z)=0.\tag{2.5.8}$$

Then the sought characterization of infinitesimal generators is

Theorem 2.5.28: *A holomorphic map $F\colon B^n\to\mathbf{C}^n$ is the infinitesimal generator of a one-parameter semigroup on B^n iff for every $z\in B^n$ we have*
$$\Big[2\|G(z)\|^2-\big|\big(G(z),z\big)\big|^2\Big]\operatorname{Re}\big(G(z),z\big)+(1-\|z\|^2)^2\operatorname{Re}\big(dF\cdot F(z),G(z)\big)\le 0,\tag{2.5.9}$$
where $G(z)=(1-\|z\|^2)F(z)+\big(F(z),z\big)z$.

Proof: (2.5.9) is equivalent to
$$\forall z\in B^n\qquad\qquad d(\kappa_{B^n}\circ F)(z)\cdot F(z)\le 0,\tag{2.5.10}$$
by (2.5.7) and (2.5.8), and we shall actually show that F is an infinitesimal generator iff (2.5.10) holds.

Choose $z_0\in B^n$. Then for every $t_1>t_2>0$ and every $v\in\mathbf{C}^n$ we have
$$\begin{aligned}\kappa_{B^n}\big(\Phi_{t_1}(z_0);\,d(\Phi_{t_1})(z_0)\cdot v\big)&=\kappa_{B^n}\Big(\Phi_{t_1-t_2}\big(\Phi_{t_2}(z_0)\big);\,d(\Phi_{t_1-t_2})\big(\Phi_{t_2}(z_0)\big)\cdot\big(d\Phi_{t_2}(z_0)\cdot v\big)\Big)\\&\le\kappa_{B^n}\big(\Phi_{t_2}(z_0);\,d(\Phi_{t_2})(z_0)\cdot v\big).\end{aligned}$$

Therefore for every $v\in\mathbf{C}^n$ and $z_0\in B^n$ the function $t\mapsto\kappa_{B^n}\big(\Phi_t(z_0);\,d\Phi_t(z_0)\cdot v\big)$ is not increasing. Thus if $v\ne 0$ we have
$$\begin{aligned}0&\ge\frac{d}{dt}\Big[\kappa_{B^n}\big(\Phi_t(z_0);\,d\Phi_t(z_0)\cdot v\big)\Big]\Big|_{t=0}\\&=2\operatorname{Re}\sum_{j=1}^n\left[\frac{\partial\kappa_{B^n}}{\partial z_j}(z_0;v)F_j(z_0)+\frac{\partial\kappa_{B^n}}{\partial v_j}(z_0;v)\,dF_j(z_0)\cdot v\right].\end{aligned}\tag{2.5.11}$$

In particular, we can take $v = F(z_0)$ obtaining

$$0 \geq 2\,\mathrm{Re}\sum_{j=1}^{n}\left[\frac{\partial\kappa_{B^n}}{\partial z_j}\bigl(z_0;F(z_0)\bigr)F_j(z_0) + \frac{\partial\kappa_{B^n}}{\partial v_j}\bigl(z_0;F(z_0)\bigr)\,dF_j(z_0)\cdot F(z_0)\right]$$
$$= d(\kappa_{B^n}\circ F)(z_0)\cdot F(z_0),$$

and (2.5.10) is proved.

Conversely, assume (2.5.10) holds. Fix $z_0 \in B^n$, and let $\phi_{z_0}\colon[0,\delta_{z_0})\to B^n$ be the unique maximal solution of the Cauchy problem

$$\begin{cases} \dfrac{d\phi}{dt} = F\circ\phi, \\ \phi(0) = z_0. \end{cases}$$

To show that F is an infinitesimal generator, it suffices to prove that $\delta_{z_0} = +\infty$ for all $z_0 \in B^n$.

If $F(z_0) = 0$, then $\phi_{z_0} \equiv z_0$, and so there is nothing to prove. If $F(z_0) \neq 0$, then ϕ_{z_0} is a non-constant real analytic map; therefore it cannot be eventually constant, and thus $F\bigl(\phi_{z_0}(t)\bigr) \neq 0$ for all $t \in [0,\delta_{z_0})$. So to prove that $\delta_{z_0} = +\infty$ it suffices to show that if we assume, by contradiction, $\delta_{z_0} < +\infty$ then there is $M > 0$ such that $k_{B^n}\bigl(z_0,\phi_{z_0}(t)\bigr) \leq M$ for all $t \in [0,\delta_{z_0})$. Indeed, in this case $\phi_{z_0}\bigl([0,\delta_{z_0})\bigr)$ is contained in a compact subset of B^n, and thus δ_{z_0} cannot be maximal.

Now we have (setting $\phi = \phi_{z_0}$, and dropping t in the computations)

$$\frac{d}{dt}\kappa_{B^n}(\phi;\dot\phi) = 2\,\mathrm{Re}\sum_{j=1}^{n}\left[\frac{\partial\kappa_{B^n}}{\partial z_j}(\phi;\dot\phi)\,\dot\phi_j + \frac{\partial\kappa_{B^n}}{\partial v_j}(\phi;\dot\phi)\ddot\phi_j\right]$$
$$= 2\,\mathrm{Re}\sum_{j=1}^{n}\left[\frac{\partial\kappa_{B^n}}{\partial z_j}(\phi;F\circ\phi)\,F_j\circ\phi + \frac{\partial\kappa_{B^n}}{\partial v_j}(\phi;F\circ\phi)\,dF_j\circ\phi\cdot(F\circ\phi)\right]$$
$$= d(\kappa_{B^n}\circ F)\circ\phi\cdot(F\circ\phi) \leq 0,$$

by (2.5.10). Hence the function $t \mapsto \kappa_{B^n}\bigl(\phi_{z_0}(t);\dot\phi_{z_0}(t)\bigr)$ is not increasing; therefore for all $t \in [0,\delta_{z_0})$ we have

$$k_{B^n}\bigl(z_0,\phi_{z_0}(t)\bigr) \leq \int_0^t \kappa_{B^n}\bigl(\phi_{z_0}(s);\dot\phi_{z_0}(s)\bigr)\,ds \leq \delta_{z_0}\kappa_{B^n}\bigl(z_0;F(z_0)\bigr),$$

and we are done, **q.e.d.**

The scrupulous reader will check that (2.5.9) reduces to (1.4.9) if $n = 1$. Furthermore, we remark (see Abate [1988g]) that the condition

$$d(\kappa_D\circ F)(z)\cdot F(z) \leq 0$$

characterizes the infinitesimal generators of one-parameter semigroups on any domain D such that κ_D is a C^1 function out of the zero section of TD; for instance (by Lempert [1981, 1984]), in strongly convex C^3 domains. However, since we have not studied the regularity of κ_D, we preferred to state our results for B^n only.

We end with two standard corollaries of Theorem 2.5.28:

Corollary 2.5.29: *The set of infinitesimal generators of one-parameter semigroups on B^n is a cone in* $\mathrm{Hol}(B^n, \mathbf{C}^n)$ *with vertex at* 0.

Proof: In the proof of Theorem 2.5.28 we saw that $F \in \mathrm{Hol}(B^n, \mathbf{C}^n)$ is an infinitesimal generator iff (2.5.11) holds for every $z_0 \in B^n$ and $v \in \mathbf{C}^n \setminus \{0\}$. The assertion is then clear, **q.e.d.**

Corollary 2.5.30: *A holomorphic map $F\colon B^n \to \mathbf{C}^n$ is the infinitesimal generator of a one-parameter group on B^n iff for every $z \in B^n$ we have*

$$\Big[2\|G(z)\|^2 - \big|\big(G(z), z\big)\big|^2\Big] \operatorname{Re}\big(G(z), z\big) + (1 - \|z\|^2)^2 \operatorname{Re}\big(dF \cdot F(z), G(z)\big) = 0,$$

where $G(z) = (1 - \|z\|^2)F(z) + \big(F(z), z\big)z$.

Proof: Copy the proof of Corollary 1.4.16, **q.e.d.**

Notes

As anticipated in the notes to chapter 2.3, Theorem 2.5.3 for strongly pseudoconvex domains (as well as Corollary 2.5.4) is due to Wong [1977], while the complete statement (as well as Corollary 2.5.5) was proved shortly later by Rosay [1979]. Greene and Krantz [1985] have shown that any hyperbolic manifold X with a point $z_0 \in X$ so that $\dim_{\mathbf{R}} X/\operatorname{Aut}_{z_0}(X) \le 1$ is biholomorphic to B^n. Since Hsiang and Straume [1986] have proved that a compact group K acting topologically on a contractible manifold X so that $\dim_{\mathbf{R}} X/K \le 4$ has a fixed point, we can infer, for instance, that every topologically contractible strongly pseudoconvex domain D such that $\dim_{\mathbf{R}} D/\operatorname{Aut}(D) \le 1$ is biholomorphic to the ball. In Abate, Geatti and Hsiang [1987] there is a complete classification of the bounded topologically contractible domains D of $\mathbf{C}^n$ where a compact group K acts holomorphically in such a way that $\dim_{\mathbf{R}} D/K = 2$. Bedford and Dadok [1987] have shown that every compact Lie group is the automorphism group of a strongly pseudoconvex domain with analytic boundary (the automorphism group of a hyperbolic manifold is always a Lie group: see Kobayashi [1970]). On the other hand, Greene and Krantz [1982a, b] have proved a very precise statement showing that, roughly speaking, the automorphism group of a generic strongly pseudoconvex smooth domain reduces to the identity.

Theorem 2.5.7 is due to Lempert (unpublished); the proof is very similar to a standard proof of the É. Cartan theorem about fixed points of a compact group acting on a simply connected Riemannian manifold of negative curvature mentioned in the notes to chapter 2.2.

Proposition 2.5.9 is due to H. Cartan [1931]; the proof, as well as Corollaries 2.5.10 and 2.5.11, is taken from Vigué [1986]. It should be noticed that the only property of taut manifolds used to prove Proposition 2.5.9 is the compactness of the isotropy group of a point. Since this holds in any hyperbolic manifold (Kobayashi [1970]), it is easy to check that Proposition 2.5.9 and Corollaries 2.5.10 and 2.5.11 are still valid for hyperbolic manifolds.

Theorem 2.5.12 for generic convex domains is in Vigué [1984a, 1985]; Heath and Suffridge [1981] proved it for Δ^n, and Kuczumow [1986] for balanced convex domains (a domain $D \subset\subset \mathbf{C}^n$ is *balanced* if $\lambda D \subset D$ for all $\lambda \in \overline{\Delta}$). The proof described here is due to Kuczumow and Stachura [1989]; see also Kuczumow [1985], and the notes to chapter 2.2.

Theorem 2.5.15 has been originally proved in Abate [1988d], by means of an involved argument making use of the notion of complex geodesic to be introduced in the next chapter. The proof presented here, as well as Theorem 2.5.16, is in Kuczumow and Stachura [1989]. Proposition 2.5.14 has been obtained independently by Vigué. Theorem 2.5.16 for Δ^2 is due to Eustice [1972]; for Δ^n to Heath and Suffridge [1981]; consult also the notes to chapter 2.2. The infinite dimensional situation is studied in Bruck [1973] and Kuczumow [1984].

Proposition 2.5.17 is an *ad hoc* proof of the fact that for linear groups the exponential map in the sense of Lie groups coincides with the usual exponential of matrices. See Warner [1983] for another proof. The rest of section 2.5.3 is either a straightforward generalization of chapter 1.4 or adapted from Abate [1988b, g]. Finally, Vesentini [1987] has completely described the one-parameter semigroups of automorphisms of B^n.