An introduction to holomorphic dynamics in one complex variable

Informal notes

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0. Prerequisites.

In this section we list a number of classical theorems of complex analysis we shall need later on.

Definition 0.1: If X and Y are complex manifolds, we shall denote by $\mathrm{Hol}(X,Y)$ the set of holomorphic maps from X into Y. In particular, $\mathrm{Hol}(X,X)$ will denote the set of holomorphic self-maps of a complex manifold X.

Theorem 0.1: (Identity principle) *Let X, Y be two (connected) Riemann surfaces, and f, $g \in \mathrm{Hol}(X,Y)$. If the set $\{z \in X \mid f(z) = g(z)\}$ admits an accumulation point then $f \equiv g$.*

Corollary 0.2: *Let X, Y be two Riemann surfaces, $f \in \mathrm{Hol}(X,Y)$ not constant, and $w \in Y$. Then the set $f^{-1}(w)$ is discrete.*

Theorem 0.3: (Open mapping theorem) *Let X, Y be two Riemann surfaces, and $f \in \mathrm{Hol}(X,Y)$ not constant. Then $f(X)$ is open in Y. In particular, f is an open mapping.*

Theorem 0.4: (Weierstrass) *Let X, Y be two Riemann surfaces, and $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ a sequence of holomorphic functions converging, uniformly on compact subsets, to a function $g\colon X \to Y$. Then g is holomorphic, and the sequence $\{df_\nu\}$ of the differentials converges, uniformly on compact subsets, to the differential dg of g. In particular, if X, Y are open subsets of $\mathbb{C}$ then the sequence $\{f'_\nu\}$ of the derivatives converges, uniformly on compact subsets, to the derivative g' of g.*

Theorem 0.5: (Rouché) *Let f and g be holomorphic functions defined in a neighborhood of a closed disk $\overline{D} \subset \mathbb{C}$ and such that $|f - g| < |g|$ on ∂D. Then f and g have the same number of zeroes (counted with multiplicities) in D.*

Definition 0.2: The *Riemann sphere* $\widehat{\mathbb{C}}$ is the complex projective line $\mathbb{P}^1(\mathbb{C})$. As a set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where we are identifying $z \in \mathbb{C}$ with the point $[z : 1] \in \mathbb{P}^1(\mathbb{C})$, and $\infty \in \widehat{\mathbb{C}}$ with the point $[1 : 0] \in \mathbb{P}^1(\mathbb{C})$. Topologically, $\widehat{\mathbb{C}}$ is homeomorphic (even diffeomorphic) to the unit sphere $S^2 \subset \mathbb{R}^3$.

The *unit disk* $\Delta \subset \mathbb{C}$ is $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$; more generally we shall denote by $\Delta_r \subset \mathbb{C}$ the euclidean disk of center the origin and radius $r > 0$. A (*complex*) *torus* is a Riemann surface of the form $\mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ is a rank-2 lattice in $\mathbb{C}$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 0.6: (Riemann's uniformization theorem) *Let X be a Riemann surface, and $\pi\colon \tilde{X} \to X$ its universal covering map. Then $\tilde{X}$, with its unique complex structure making π a local biholomorphism, is biholomorphic either to $\widehat{\mathbb{C}}$, or to $\mathbb{C}$, or to Δ. More precisely:*

(i) *if $\tilde{X} = \widehat{\mathbb{C}}$ then $X = \widehat{\mathbb{C}}$;*
(ii) *if $\tilde{X} = \mathbb{C}$ then $X = \mathbb{C}$, $\mathbb{C}^*$ or a torus;*
(iii) *in all other cases, $\tilde{X} = \Delta$.*

In particular, if X is an open connected subset of $\widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} \setminus X$ contains at least three points, then $\tilde{X} = \Delta$.

Definition 0.3: A Riemann surface X is *elliptic* if its universal covering space is $\widehat{\mathbb{C}}$ (and hence $X = \widehat{\mathbb{C}}$); *parabolic* if its universal covering space is $\mathbb{C}$; *hyperbolic* if its universal covering space is Δ.

Theorem 0.7: (Liouville) *If X is a elliptic or parabolic Riemann surface and Y is a hyperbolic Riemann surface, then every $f \in \mathrm{Hol}(X,Y)$ is constant. In particular, every bounded entire function is constant.*

Definition 0.4: A sequence $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ of holomorphic maps between two complex manifolds X and Y is *compactly divergent* if for every compact subset $H \subset X$ and every compact subset $K \subset Y$ there is $\nu_0 \in \mathbb{N}$ such that $f_\nu(H) \cap K = \varnothing$ for $\nu \geq \nu_0$. A family $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ is *normal* if every sequence in $\mathcal{F}$ admits a subsequence which is either convergent uniformly on compact sets to a map in $\mathrm{Hol}(X,Y)$ or compactly divergent.

Remark 0.1: If Y is a compact complex manifold (e.g., $X = \widehat{\mathbb{C}}$), then a family $\mathcal{F} \subseteq \mathrm{Hol}(X,Y)$ is normal if and only if every sequence in $\mathcal{F}$ admits a subsequence converging to a map in $\mathrm{Hol}(X,Y)$ (because there are no compactly divergent sequences).

Proposition 0.8: *Let $\mathcal{F} \subset \mathrm{Hol}(X,Y)$, where Y is a compact complex manifold. Assume we have a family $\{U_\alpha\}_{\alpha \in A}$ of open subsets covering X such that $\mathcal{F}|_{U_\alpha} = \{f|_{U_\alpha} \mid f \in \mathcal{F}\} \subset \mathrm{Hol}(U_\alpha, Y)$ is normal for every $\alpha \in A$. Then $\mathcal{F}$ is normal.*

Proof: Since the topology of any complex manifold (and hence of X) has a countable basis, we can assume that $A = \mathbb{N}$. Let $\{f_k\}$ be a sequence in $\mathcal{F}$. Since $\mathcal{F}|_{U_0}$ is normal, the previous remark ensures us that we can find a subsequence $\{f_{0,k}\}$ converging to a holomorphic map in $\mathrm{Hol}(U_0, Y)$. From this sequence, we can extract a subsequence $\{f_{1,k}\}$ converging to a holomorphic map in $\mathrm{Hol}(U_1, Y)$. Proceeding in this way, we get a sequence of subsequences $\{f_{j,k}\}$ of the original sequence such that $\{f_{j,k}\}$ is a subsequence of $\{f_{j-1,k}\}$ converging in U_j. But then $\{f_{k,k}\}$ is a subsequence of the original sequence converging to a map in $\mathrm{Hol}(X,Y)$, as desired. □

Theorem 0.9: (Montel) *If X and Y are hyperbolic Riemann surfaces, then $\mathrm{Hol}(X,Y)$ is a normal family. Furthermore, if Y is an open subset of $\widehat{\mathbb{C}}$ whose complement contains at least three points, then $\mathrm{Hol}(X,Y)$ is normal in $\mathrm{Hol}(X,\widehat{\mathbb{C}})$.*

Theorem 0.10: (Vitali) *Let X and Y be two hyperbolic Riemann surfaces, and $A \subset X$ a subset with at least one accumulation point. Let $\{f_\nu\} \subset \mathrm{Hol}(X,Y)$ be a sequence of holomorphic maps such that $\{f_\nu(a)\}$ converges in Y for each $a \in A$. Then the sequence $\{f_\nu\}$ converges, uniformly on compact subsets, to a holomorphic map $g \in \mathrm{Hol}(X,Y)$.*

Theorem 0.11: (Hurwitz) *Let $\Omega \subseteq \mathbb{C}$ open, and $\{f_\nu\} \subset \mathrm{Hol}(\Omega,\mathbb{C})$ a sequence of injective holomorphic functions converging uniformly on compact subsets to $f \in \mathrm{Hol}(\Omega,\mathbb{C})$. Then f is either injective or constant.*

A topological result we shall sometimes use is

Theorem 0.12: (Baire) *Let X be a locally compact Hausdorff topological space. Then each countable intersection of open dense sets is still dense, and each countable union of closed sets with empty interior has empty interior.*

1. Holomorphic self-maps of the Riemann sphere

The complex manifold we shall be mostly interested in is the one-dimensional Riemann sphere; so we begin our study by describing the structure of the holomorphic self-maps of $\widehat{\mathbb{C}}$.

Proposition 1.1: *Every non-constant $f \in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ is a rational function, that is of the form*

$$f(z) = \frac{P(z)}{Q(z)}\,,$$

where P and Q are polynomials without common factors, uniquely determined up to a multiplicative constant.

Proof: The set $f^{-1}(\infty)$ of poles of f, being a discrete subset of the compact set $\widehat{\mathbb{C}}$, is finite. For the same reason, the set $f^{-1}(0)$ of zeroes of f is finite. Let $z_1, \ldots, z_h \in \mathbb{C}$ the poles (in $\mathbb{C}$) of f, and $w_1, \ldots, w_k \in \mathbb{C}$ the zeroes (in $\mathbb{C}$) of f, both listed as many times as their multiplicity; clearly $\{z_1, \ldots, z_h\} \cap \{w_1, \ldots, w_k\} = \varnothing$. Then

$$g(z) = \frac{(z - z_1) \cdots (z - z_h)}{(z - w_1) \cdots (z - w_k)} f(z)$$

has neither poles nor zeroes in $\mathbb{C}$. If $g(\infty) \in \mathbb{C}$ then $g(\widehat{\mathbb{C}})$ is a bounded (being compact) subset of $\mathbb{C}$; therefore Liouville's theorem says that $g \equiv c$ is constant, and so f can be written in the required form with $P(z) = c(z - w_1) \cdots (z - w_k)$ and $Q(z) = (z - z_1) \cdots (z - z_h)$.

If instead $g(\infty) = \infty$, then $1/g$ is bounded, and hence constant; but this implies that g is constant too, that is $g \equiv \infty$, and hence $f \equiv \infty$, again the assumption that f is not constant.

Finally, the uniqueness statement follows from the fact that a polynomial is completely determined by its zeroes up to a multiplicative constant. □

Definition 1.1: The *degree* of a non-constant $f = P/Q \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ is given by

$$\deg f = \max\{\deg P, \deg Q\} ;$$

the degree of a constant map is 0.

Definition 1.2: Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function defined in a neighborhood Ω of a point $z_0 \in \mathbb{C}$ with $f(z_0) = 0$. Then the *multiplicity* $\delta_f(z_0)$ of f at z_0 (as zero of f) is the least $k \geq 1$ such that $f^{(k)}(z_0) \neq 0$. In other words, f has multiplicity k at z_0 if and only if its Taylor series expansion at z_0 is

$$f(z) = a_k(z - z_0)^k + o\big((z - z_0)^k\big)$$

with $a_k \neq 0$.

More generally, we shall say that z_0 is a solution of the equation $f(z) = w$ of *multiplicity* $k \geq 1$ if $f - w$ has multiplicity k at z_0.

Remark 1.1: By definition, $z_0 \in \mathbb{C}$ is a solution of $f(z) = w \in \mathbb{C}$ of multiplicity greater than one if and only if $f(z_0) = w$ and z_0 is a zero of f', that is a critical point of f.

Remark 1.2: The definition of multiplicity can be extended to holomorphic functions defined in a neighborhood of a point in any Riemann surface X. Indeed, let $\varphi\colon \Omega \to \mathbb{C}$ be a chart centered in a point $p \in X$, and $f \in \mathrm{Hol}(\Omega, \mathbb{C})$ with $f(p) = w$, and assume that $f \circ \varphi^{-1} - w$ has multiplicity k at the origin; we claim then that $f \circ \tilde{\varphi}^{-1} - w$ has multiplicity k at the origin for any other chart $\tilde{\varphi}\colon \tilde{\Omega} \to \mathbb{C}$ centered at p. We argue by induction on k, proving the equivalent statement that $f \circ \varphi^{-1} - w$ has multiplicity at least k at the origin if and only if $f \circ \tilde{\varphi}^{-1} - w$ does. We have

$$(f \circ \tilde{\varphi}^{-1})'(z) = (f \circ \varphi^{-1})'\big(\varphi \circ \tilde{\varphi}^{-1}(z)\big) \cdot (\varphi \circ \tilde{\varphi}^{-1})'(z) ; \tag{1.1}$$

since $\varphi \circ \tilde{\varphi}^{-1}(0) = 0$ and $(\varphi \circ \tilde{\varphi}^{-1})'(0) \neq 0$, it follows that $f \circ \tilde{\varphi}^{-1} - w$ has multiplicity at least 2 at the origin if and only $f \circ \varphi^{-1} - w$ does. Assume the claim holds for $k - 1$, and assume that $f \circ \varphi^{-1} - w$ has multiplicity at least k at the origin. This implies that the first $k - 1$ derivatives of $f \circ \varphi^{-1}$ vanish at the origin. Differentiating (1.1) $k - 2$ times and evaluating at the origin we then get

$$(f \circ \tilde{\varphi}^{-1})^{k-1}(0) = (f \circ \varphi^{-1})^{k-1}(0) \cdot [(\varphi \circ \tilde{\varphi}^{-1})'(0)]^{k-1} ,$$

and the assertion follows.

An analogous argument shows that if $f \in \mathrm{Hol}(X, Y)$ then the multiplicity of $p_0 \in X$ as solution of the equation $f(p) = f(p_0)$ is well-defined as the multiplicity of the origin as solution of the equation $\psi \circ f \circ \varphi^{-1} = 0$, where φ is a chart of X centered at p_0 and ψ is a chart of Y centered at $f(p_0)$.

Definition 1.3: Given $f \in \mathrm{Hol}(X, \mathbb{C})$, the *multiplicity* $\delta_f(p_0)$ of f at $p_0 \in X$ is the multiplicity of the origin as solution of the equation $f \circ \varphi^{-1} = f(p_0)$, where φ is any chart centered at p_0. More generally, if $f \in \mathrm{Hol}(X, Y)$ and $p_0 \in X$ then the *multiplicity* $\delta_f(p_0)$ of f at $p_0 \in X$ is the multiplicity of the origin as solution of the equation $\psi \circ \varphi^{-1} = 0$, where φ is a chart of X centered at p_0 and ψ is a chart of Y centered at $f(p_0)$.

For instance, take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ and $p_0 = \infty$. If $f(\infty) = w \in \mathbb{C}$, then the multiplicity of f at ∞ is the multiplicity of the origin as solution of the equation $f(1/z) = w$. In particular, if we write

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0} , \tag{1.2}$$

with a_m, $b_n \neq 0$, then

$$f(1/z) - w = z^{n-m} \frac{a_m + \cdots + a_0 z^m}{b_n + \cdots + b_0 z^n} - w\;;$$

notice that $f(\infty) = w \in \mathbb{C}$ implies $n \geq m$. So

$$\frac{d}{dz}\bigl(f(1/z) - w\bigr)\bigg|_{z=0} = \begin{cases} \frac{a_{m-1}b_n - b_{n-1}a_m}{b_n^2} & \text{if } n = m, \\ \frac{a_m}{b_n} & \text{if } n = m+1, \\ 0 & \text{if } n \geq m+2. \end{cases}$$

In particular, if $n = m+1$ then ∞ is a zero of multiplicity 1; and in a similar way one can check (do it!) that if $n \geq m+1$ then ∞ is a zero of multiplicity $n - m$.

Exercise 1.1: Given $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ written as in (1.2), assuming that $f(\infty) = \infty$ prove that $m > n$ and that the multiplicity of ∞ as pole of f, that is as solution of the equation $f(p) = \infty$, is exactly $m - n$.

Proposition 1.2: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be not constant. Then for every $w \in \widehat{\mathbb{C}}$ the equation $f(z) = w$ has exactly* $\deg f$ *solutions, counted with multiplicities. In other words,*

$$\forall q \in \widehat{\mathbb{C}} \qquad \deg f = \sum_{p \in f^{-1}(q)} \delta_f(p)\;.$$

Proof: Write

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}\;,$$

where P and Q are polynomials without common factors of degree respectively m and n. In $\mathbb{C}$, the map f has exactly m zeroes (the zeroes of P) and n poles (the zeroes of Q).

If $m = n = \deg f$ then $f(\infty) = a_m/b_n \neq 0, \infty$, and thus f has exactly $\deg f$ zeroes and poles in $\widehat{\mathbb{C}}$.

If $m \neq n$, we can write

$$f(z) = z^{m-n} \frac{a_m + \cdots + a_0 z^{-m}}{b_n + \cdots + b_0 z^{-n}}\;;$$

therefore if $m > n$ it follows that ∞ is a pole of multiplicity $m - n$, whereas if $m < n$ then ∞ is a zero of multiplicity $n - m$. In both cases it turns out that the number of zeroes and poles of f, counted with multiplicities, is equal to $\max\{m, n\} = \deg f$.

Now take $w \in \mathbb{C}^*$. Then the number of solutions of $f(z) = w$ is, by definition, the number of zeroes of

$$f(z) - w = \frac{P(z) - wQ(z)}{Q(z)}\;.$$

Since P and Q have no common factors, $P - wQ$ and Q do not too; therefore $\deg(f - w) = \deg f$, and we are done. □

Remark 1.3: In other words, the (algebraic) degree of a non-constant $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ coincides with the topological degree, that is the number of inverse images of a generic point. In fact, Remark 1.1 and Proposition 1.2 imply that if $w \in \widehat{\mathbb{C}}$ is not a critical value (i.e., the image of a critical point) then $f^{-1}(w)$ contains exactly $\deg f$ points. The critical points are the zeroes of f', and thus there are only finitely many critical values, at most $\deg f'$. We remark that the Riemann-Hurwitz formula (see Theorem 3.7) implies that $\deg f' = 2\deg f - 2$.

Definition 1.4: Let X be a complex manifold. If $f \in \mathrm{Hol}(X, X)$ and $k \geq 1$, we shall denote by $f^k = f \circ \cdots \circ f$ the k-th *iterate* of f, that is, the composition of f with itself k times. We shall also put $f^0 = \mathrm{id}_X$.

Corollary 1.3: *If f, $g \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ then* $\deg(g \circ f) = (\deg g)(\deg f)$. *In particular,* $\deg(f^k) = (\deg f)^k$.

Proof: If f or g are constant, the assertion is obvious; so assume $\deg f$, $\deg g > 0$. Now, for all but at most a finite number of $w \in \mathbb{C}$ the solutions of $f(z) = w$ have all multiplicity one, by Remark 1.3; therefore for all but at most a finite number of $w \in \mathbb{C}$ the set $f^{-1}(w)$ contains exactly $\deg f$ points. Analogously, for all but at most a finite number of $w \in \mathbb{C}$ the set $g^{-1}(w)$ contains exactly $\deg g$ points; hence for all but at most a finite number of $w \in \mathbb{C}$ the set $(g \circ f)^{-1}(w)$ contains exactly $(\deg f)(\deg g)$ points, and thus $\deg(g \circ f) = (\deg g)(\deg f)$. □

Definition 1.5: An *automorphism* of a complex manifold X is a biholomorphism of X with itself, that is a holomorphic invertible map $\gamma \in \mathrm{Hol}(X, X)$ with holomorphic inverse. The group of automorphisms of X will be denoted by $\mathrm{Aut}(X)$.

Definition 1.6: A *fixed point* of $f \in \mathrm{Hol}(X, X)$ is a $z_0 \in X$ such that $f(z_0) = z_0$. A *periodic point* is a fixed point of some iterate f^k of f, that is a $z_0 \in X$ such that $f^k(z_0) = z_0$ for some $k \geq 1$; the least such k is the *period* of z_0. A point $z_0 \in X$ is *preperiodic* if $f^h(z_0)$ is periodic for some $h \geq 0$; *strictly preperiodic* if it is preperiodic but not periodic. Finally, we shall denote by $\mathrm{Fix}(f)$ the set of fixed points of f, and by $\mathrm{Per}(f)$ the set of periodic points of f.

Proposition 1.4: *Every $\gamma \in \mathrm{Aut}(\widehat{\mathbb{C}})$ can be written in the form*

$$\gamma(z) = \frac{az + b}{cz + d}\,, \tag{1.3}$$

where a, b, c, $d \in \mathbb{C}$ are such that $ad - bc = 1$, and are uniquely determined up to multiplication by -1. In particular:

(i) *for every pair of triples of distinct points $\{z_0, z_1, z_2\}$ and $\{w_0, w_1, w_2\}$ there exists a unique $\gamma \in \mathrm{Aut}(\widehat{\mathbb{C}})$ such that $\gamma(z_j) = w_j$ for $j = 0, 1, 2$;*

(ii) *every $\gamma \in \mathrm{Aut}(\widehat{\mathbb{C}})$ different from the identity has either two distinct fixed points or one double fixed point;*

(iii) *if γ_1, $\gamma_2 \in \mathrm{Aut}(\widehat{\mathbb{C}})$ are such that γ_1^2, $\gamma_2^2 \neq \mathrm{id}_{\widehat{\mathbb{C}}}$ then $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ if and only if $\mathrm{Fix}(\gamma_1) = \mathrm{Fix}(\gamma_2)$.*

Proof: The formula (1.3) follows immediately from Propositions 1.1 and 1.2, because γ is injective.

To prove (i), it is enough (why?) to consider the case $z_0 = 0$, $z_1 = 1$ and $z_2 = \infty$. In this case the required conditions become $b/d = w_0$, $(a + b)/(c + d) = w_1$ and $(a/c) = w_2$, and it is easy to see that these equations, together with $ad - bc = 1$, admit a solution unique up to a sign.

(ii) follows because the fixed point equation is $az + b = (cz + d)z$. In particular, if γ fixes three points then it is the identity (and this proves, in another way, the uniqueness assertion in (i)).

(iii) Assume that $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$. Then γ_1 sends $\mathrm{Fix}(\gamma_2)$ onto itself, and γ_2 does likewise with $\mathrm{Fix}(\gamma_1)$. Hence if $\mathrm{Fix}(\gamma_1) \neq \mathrm{Fix}(\gamma_2)$ the cardinality of $\mathrm{Fix}(\gamma_2)$ must be two, and γ_1 must swap the two fixed points of γ_2; but then γ_1^2 has at least three fixed points, and hence $\gamma_1^2 = \mathrm{id}_{\widehat{\mathbb{C}}}$, against our assumption. Therefore $\mathrm{Fix}(\gamma_1) = \mathrm{Fix}(\gamma_2)$, as claimed.

Finally, assume that $\mathrm{Fix}(\gamma_1) = \mathrm{Fix}(\gamma_2) = F$. If the cardinality of F is two, using (i), up to a conjugation we can assume that $F = \{0, \infty\}$; therefore $\gamma_j(z) = \lambda_j z$ for suitable λ_1, $\lambda_2 \in \mathbb{C}^*$, and we are done. Analogously, if the cardinality of F is one, up to a conjugation we can assume $F = \{\infty\}$, and hence $\gamma_j(z) = z + b_j$ for suitable b_1, $b_2 \in \mathbb{C}^*$, and γ_1 and γ_2 commute in this case too. □

Example 1.1: Let $\gamma_1(z) = 1/z$ and $\gamma_2(z) = -z$; then γ_1, $\gamma_2 \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$ but $\mathrm{Fix}(\gamma_1) \neq \mathrm{Fix}(\gamma_2)$. It is however evident that $\gamma_1^2 = \gamma_2^2 = \mathrm{id}_{\widehat{\mathbb{C}}}$.

2. Julia and Fatou sets

Let X be a complex (connected) manifold, and $f \in \mathrm{Hol}(X, X)$. Our aim is to study the dynamics of the discrete dynamical system $\{f^k\}$; in other words, we would like to understand the asymptotic behavior of the orbits $\{f^k(z)\}$ as z varies in X (and sometimes how this behavior depends on the map f). In particular, we would like to understand which orbits show a regular behavior, and which orbits have a chaotic behavior.

In (one-dimensional) holomorphic dynamics, the notions of “regular” and “chaotic” behavior are described in terms of normal families.

Definition 2.1: Given $f \in \mathrm{Hol}(X, X)$, we shall say that $z \in X$ belongs to the *Fatou set* $\mathcal{F}(f)$ of f if there is a neighbourhood U of z in X such that the sequence $\{f^k|_U\} \subset \mathrm{Hol}(U, X)$ is a normal family. The complement $\mathcal{J}(f) = X \setminus \mathcal{F}(f)$ is the *Julia set* of f.

Roughly speaking, orbits in the Fatou set have regular behavior (in particular, nearby orbits have similar behavior), whereas orbits in the Julia set have chaotic behavior.

The usual aims of holomorphic dynamics are the following:

(a) describe the geometry of Fatou and Julia sets;
(b) describe the dynamics in the Fatou set;
(c) describe the dynamics in the Julia set.

In this short course we shall concentrate on the case $X = \widehat{\mathbb{C}}$; here problem (b) is completely solved, and we know a lot about problems (a) and (c).

Remark 2.1: When X is a hyperbolic Riemann surface (e.g., the unit disk $\Delta \subset \mathbb{C}$) then $\mathcal{F}(f) = X$, essentially by Montel's theorem; thus problems (a) and (c) are (trivially) solved, and a lot is known (almost everything, in fact, except in very wild Riemann surfaces) about problem (b). When X is a torus, $\mathrm{Hol}(X, X)$ consists of linear maps, and thus problems (a)–(c) become mostly trivial. The only one-dimensional cases left are $\mathbb{C}$ and $\mathbb{C}^*$, where several things are known but there are still several open problems. Finally, in several complex variables not very much is known, and the most effective approaches and techniques are quite different from the ones we shall describe here.

To have a first glimpse of what can happen, let us discuss two examples.

EXAMPLE 2.1: Take $X = \widehat{\mathbb{C}}$ and $f(z) = z^2$. In this case $\mathrm{Fix}(f) = \{0, 1, \infty\}$, and it is easy to write the iterates of f:

$$\forall z \in \mathbb{C} \qquad f^k(z) = z^{2^k} .$$

In particular, it is clear that $|z| < 1$ implies $f^k(z) \to 0$ as $k \to \infty$, and $|z| > 1$ implies $f^k(z) \to \infty$ as $k \to +\infty$ (we shall say that 0 and ∞ are *attracting fixed points*). This easily implies (why?) that $\Delta \cup (\widehat{\mathbb{C}} \setminus \overline{\Delta}) \subseteq \mathcal{F}(f)$; but actually we have equality here. Indeed, if $z \in S^1$ then any neighbourhood of z contains points whose orbit converges to 0 as well as points whose orbit converges to ∞; therefore (by the identity principle) no subsequence of iterates can be converging in a neighbourhood of z. Therefore $\mathcal{F}(f) = \widehat{\mathbb{C}} \setminus S^1$, and $\mathcal{J}(f) = S^1$.

This solves problems (a) and (b) in this case. The behavior of the orbit of $z_0 = e^{2\pi i \alpha} \in S^1$ depends on $\alpha \in [0, 1)$:

– if $\alpha = 0$ then $z_0 = 1$ is a fixed point;
– if $\alpha = p/2^r$ with p, $r \in \mathbb{N}^*$ then z_0 is strictly preperiodic and its orbit ends at 1, because $f^r(z_0) = 1$; notice that points of this kind are dense in S^1;
– if $\alpha = p/q$ with $q > 1$ odd then z_0 is periodic: indeed, we have $f^k(z_0) = z_0$ if and only if $2^k \equiv 1 \pmod{q}$, and elementary number theory says that this congruence has a solution (given by $\varphi(q)$, the number of positive integers less than q and relatively prime with q); and again, points of this kind are dense in S^1;
– if $\alpha = p/2^r q$ with $q > 1$ odd, then z_0 is strictly preperiodic, because $f^r(z_0) = e^{2\pi i p/q}$;
– if $\alpha \notin \mathbb{Q}$, then the orbit of z_0 is infinite.

To completely understand the latter case, and to put everything into perspective, let $\chi\colon S^1 \to 2^{\mathbb{N}}$ the map given by the dyadic expansion of $\alpha \in [0, 1)$, that is $\chi(e^{2\pi i \alpha}) = (a_j)_{j \in \mathbb{N}}$ if and only if

$$\alpha = \sum_{j=0}^{\infty} \frac{a_j}{2^{j+1}} ,$$

with each $a_j \in \{0, 1\}$, and no infinite sequence of 1's is allowed. Then the action of f on S^1 becomes the (moltiplication by 2 on α and hence the) *left shift* on sequences:

$$a_0 a_1 a_2 \ldots \mapsto a_1 a_2 a_3 \ldots .$$

The left shift is a self-map of the space of sequences $2^{\mathbb{N}}$, the very first example of *symbolic dynamical system*. Using this representation it is not difficult to understand the dynamics of f on $S^1 = \mathcal{J}(f)$; for instance, there are dense orbits (take α having a dyadic expansion containing all possible finite sequences of 0's and 1's).

EXAMPLE 2.2: Take $X = \widehat{\mathbb{C}}$ and $g(z) = z^2 - 2$. In this case $\mathrm{Fix}(g) = \{-1, 2, \infty\}$. If $|z| \ge 2 + \varepsilon$ we have

$$|g(z)| = |z^2 - 2| \ge |z|^2 - 2 \ge (2 + \varepsilon)|z| - 2 \ge (1 + \varepsilon)|z| + \varepsilon > (1 + \varepsilon)|z| > 2 + \varepsilon ;$$

therefore $|g^k(z)| > (1+\varepsilon)^k|z|$ for all $k \in \mathbb{N}$. In particular $|z| > 2$ implies $g^k(z) \to \infty$. Notice furthermore that $g([-2,2]) = [-2,2] = g^{-1}([-2,2])$, so that $g\big(\widehat{\mathbb{C}} \setminus [-2,2]\big) = \widehat{\mathbb{C}} \setminus [-2,2]$.

Consider now $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$\varphi(\zeta) = \zeta + \frac{1}{\zeta} \; .$$

Clearly $\varphi(1/\zeta) = \varphi(\zeta)$; moreover it is easy to check (do it) that $\varphi^{-1}([-2,2]) = S^1$. Being 2-to-1, φ is injective when restricted to Δ; therefore it is a biholomorphism between Δ and $\widehat{\mathbb{C}} \setminus [-2,2]$. Now,

$$g\big(\varphi(\zeta)\big) = \varphi(\zeta)^2 - 2 = \varphi(\zeta^2) \; ; \tag{2.1}$$

hence $\varphi^{-1} \circ g \circ \varphi(\zeta) = \zeta^2 = f(\zeta)$ on Δ. This implies

$$\forall k \in \mathbb{N} \qquad \varphi^{-1} \circ g^k \circ \varphi = f^k \; ; \tag{2.2}$$

so φ transforms the action of g on $\widehat{\mathbb{C}} \setminus [-2,2]$ into the action of f on Δ. But we know that $f^k(\zeta) \to 0$ for all $\zeta \in \Delta$; hence $f^k(z) \to \varphi(0) = \infty$ for all $z \in \widehat{\mathbb{C}} \setminus [-2,2]$. On the other hand, arguing as in the previous example it is easy to see that $[-2,2] \subseteq \mathcal{J}(g)$; therefore $\mathcal{F}(g) = \widehat{\mathbb{C}} \setminus [-2,2]$ and $\mathcal{J}(g) = [-2,2]$.

This shows the usefulness of *conjugations*, that is of replacing a map g by a (dynamically equivalent, because of (2.2)) map of the form $\varphi^{-1} \circ g \circ \varphi$ for a suitable biholomorphism φ. In this way we solved problems (a) and (b); using another conjugation (and techniques from real dynamics) it is also possible to prove that $g|_{\mathcal{J}(g)}$ is *chaotic* in a very precise sense (because it is conjugated to the action of $4x(1-x)$ on the interval $[0,1]$); we shall prove this later on by complex methods for all self-maps of the Riemann sphere. By the way, notice that (2.1) holds for $\zeta \in S^1$ too, but $\varphi\colon S^1 \to [-2,2]$ is not invertible.

Exercise 2.1: Study as much as possible the dynamics on $\widehat{\mathbb{C}}$ of $\varphi(\zeta) = \zeta + \zeta^{-1}$, and of $\psi(\zeta) = \zeta/(1+\zeta^2)$. Why are they related?

Let us now start to study the general theory.

Definition 2.2: Let X be a complex manifold, $f \in \mathrm{Hol}(X,X)$, and $z_0 \in X$. The *(forward) orbit* of z_0 is the set

$$O^+(z) = \{f^k(z) \mid k \in \mathbb{N}\} \; ;$$

its *inverse orbit* is

$$O^-(z_0) = \bigcup_{k \in \mathbb{N}} f^{-k}(z_0) = \{z \in X \mid \exists k \in \mathbb{N} : f^k(z) = z_0\} \; .$$

Finally, its *grand orbit* is

$$GO(z_0) = \bigcup_{k \in \mathbb{N}} O^-\big(f^k(z_0)\big) = \{z \in X \mid \exists h,k \in \mathbb{N} : f^h(z) = f^k(z_0)\} \; .$$

Definition 2.3: Let X be a complex manifold, and $f \in \mathrm{Hol}(X,X)$. A subset $A \subseteq X$ is *f-invariant* if $f(A) \subseteq A$; it is *completely f-invariant* if $f^{-1}(A) = A$.

Remark 2.2: It is easy to check (do it!) that $A \subseteq X$ is completely f-invariant if and only if $f(A) \subseteq A$ and $f^{-1}(A) \subseteq A$ if and only if $GO(z) \subseteq A$ for all $z \in A$.

The next proposition contains a few basic properties of Julia and Fatou sets of rational maps.

Proposition 2.1: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be of degree at least 1. Then:*

(i) *$\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is compact;*
(ii) *the sequence $\{f^k|_{\mathcal{F}(f)}\} \subset \mathrm{Hol}\big(\mathcal{F}(f), \widehat{\mathbb{C}}\big)$ is normal;*
(iii) *$\mathcal{F}(f^p) = \mathcal{F}(f)$ and $\mathcal{J}(f^p) = \mathcal{J}(f)$ for every $p \geq 1$;*
(iv) *$\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely f-invariant.*

Proof: (i) Obvious.

(ii) It follows from Proposition 0.8.

(iii) Since $\{f^{kp}\}_{k\in\mathbb{N}} \subset \{f^k\}$, we clearly have $\mathcal{F}(f) \subseteq \mathcal{F}(f^p)$ and $\mathcal{J}(f) \supseteq \mathcal{J}(f^p)$. For the converse, put $g = f^p$, and fix $0 \le n \le p-1$. Then the family $\mathcal{F}_n = \{f^n \circ g^k\}_{k\in\mathbb{N}}$ is normal on $\mathcal{F}(g)$; therefore it is so the *finite* union $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{p-1} = \{f^k\}_{k\in\mathbb{N}}$, and so $\mathcal{F}(f^p) \subseteq \mathcal{F}(f)$, as desired.

(iv) It suffices (why?) to show that $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$.

Assume that $z \in \mathcal{F}(f)$; then there is a neighborhood U of z such that $\{f^k|_U\}$ is normal. Since f is not constant, the open mapping theorem implies that $f(U)$ is a neighborhood of $f(z)$, and $\{f^k|_{f(U)}\}$ is clearly normal; so $f(z) \in \mathcal{F}(f)$.

Conversely, if $f(z) \in \mathcal{F}(f)$, there is a neighborhood U of $f(z)$ such that $\{f^k|_U\}$ is normal; therefore $V = f^{-1}(U)$ is a neighborhood of z such that $\{f^k|_V\}$ is normal, and thus $z \in \mathcal{F}(f)$. □

Definition 2.4: Let $z_0 \in \widehat{\mathbb{C}}$ be a periodic point of period p of $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$. The *multiplier* of z_0 (or of its periodic orbit) is $\lambda = (f^p)'(z_0)$. We shall say that z_0 (or its orbit) is *attracting* if $|\lambda| < 1$; *repelling* if $|\lambda| > 1$; *superattracting* if $\lambda = 0$; *hyperbolic* if $|\lambda| \ne 0, 1$; *parabolic* (or *rationally indifferent*) if λ is a root of unity; and *elliptic* (or *irrationally indifferent*) if $|\lambda| = 1$ but λ is not a root of unity. If z_0 is attracting, its *basin of attraction* is the set $\Omega = \{z \in \widehat{\mathbb{C}} \mid \exists 0 \le j \le p-1 : f^{kp}(z) \to f^j(z_0) \in O^+(z_0)\}$.

Proposition 2.2: *Let $z_0 \in \widehat{\mathbb{C}}$ be a periodic point of period $p \ge 1$ and multiplier λ for $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$. Then:*

(i) *if z_0 is attracting, then its basin of attraction Ω is an open subset of $\mathcal{F}(f)$, and $\partial\Omega \subseteq \mathcal{J}(f)$;*

(ii) *if z_0 is repelling or parabolic then $GO(z_0) \subset \mathcal{J}(f)$.*

Proof: (i) Let us start by showing that there exists a neighborhood U of $O^+(z_0)$ such that $f(U) \subset U \subset \Omega$. First of all, $|\lambda| < 1$ implies that there are a neighborhood V_0 of z_0 and $0 < c < 1$ such that

$$|f^p(z) - z_0| < c|z - z_0|$$

for all $z \in V_0 \setminus \{z_0\}$. It follows that $f^{kp}|_{V_0} \to z_0$, and we can also assume that $f^p(V_0) \subset V_0$ (it suffices to choose as V_0 a disk centered at z_0 of a small enough radius). Set $U = \bigcup_{j=0}^{p-1} f^j(V_0)$; then clearly $f(U) \subset U$. Furthermore, if $z \in U$ then $z = f^j(w)$ for some $w \in V_0$ and $0 \le j \le p-1$; therefore $f^{kp}(z) = f^j\bigl(f^{kp}(w)\bigr) \to f^j(z_0) \in O^+(z_0)$, and thus $U \subseteq \Omega$.

Now take $z \in \Omega$ and choose $0 \le j \le p-1$ so that $f^{kp}(z) \to z_j = f^j(z_0)$. There exists k_0 such that $f^{k_0 p}(z) \in f^j(V_0)$, again by the open mapping theorem. Let W be a neighborhood of z such that $f^{k_0 p}(W) \subseteq f^j(V_0) \subset U$; it follows that $f^{kp}(W) \subset f^j(V_0) \subset \Omega$ for all $k \ge k_0$, and so $W \subset \Omega$. In particular, Ω is open.

To prove the rest of the assertion we can assume $p = 1$, thanks to Proposition 2.1.(iii). First of all, by definition $f^k|_\Omega \to z_0$ (at least) pointwise; then Vitali's Theorem 0.10 implies that the convergence is uniform on compact subsets, and hence $\Omega \subseteq \mathcal{F}(f)$. Finally, if $z \in \partial\Omega$ then $f^k(z) \not\to z_0$ (otherwise z would be in Ω) but every neighborhood of z intersects Ω; it follows (why?) that no subsequence of iterates can converge in a neighborhood of z, and thus $\partial\Omega \subseteq \mathcal{J}(f)$.

(ii) Again, we can assume $p = 1$. Suppose first that z_0 is repelling, that is $|\lambda| > 1$. If a subsequence $\{f^{k_j}\}$ of iterates converges to a holomorphic map g in a neighborhood of z_0, then Weierstrass' Theorem 0.4 implies $|\lambda|^{k_j} = |(f^{k_j})'(z_0)| \to |g'(z_0)|$, and so $|\lambda| \le 1$, contradiction. Thus $z_0 \in \mathcal{J}(f)$; Proposition 2.1.(iv) then implies $GO(z_0) \subset \mathcal{J}(f)$.

Suppose now that z_0 is parabolic, that is λ is a root of unity. Up to replacing f by a suitable iterate, we can assume $\lambda = 1$ and (without loss of generality) $z_0 = 0$. Therefore we can write

$$f(z) = z + az^n + \cdots$$

for suitable $n \ge 2$ and $a \ne 0$. Then

$$f^k(z) = z + kaz^n + \cdots$$

If a subsequence $\{f^{k_j}\}$ of iterates converges to a holomorphic function g in a neighborhood of the origin, again by Weierstrass' Theorem 0.4 the n-th derivative of f^{k_j} must converge to the n-th derivative of g. This implies that the sequence $\{k_j a\}$ must be convergent in $\mathbb{C}$; and this is possible only if $a = 0$, contradiction. Thus we again have $z_0 \in \mathcal{J}(f)$, and Proposition 2.1.(iv) again implies $GO(z_0) \subset \mathcal{J}(f)$. □

Exercise 2.2: Describe Julia and Fatou sets of $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f=1$.

We shall now consider only self-maps of $\widehat{\mathbb{C}}$ of degree at least 2. Our next aim is to prove that a Julia set is never empty. To do so we shall need the

Lemma 2.3: *Let* $\{f_\nu\}\subset \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ *be a sequence converging to* $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$. *Then we have* $\deg f_\nu=\deg f$ *for* ν *large enough.*

Proof: Let $d=\deg f$. If $d=0$ then f is constant; on the other hand, if $\deg f_\nu\ge 1$ then f_ν is surjective. Since a sequence of surjective maps cannot converge uniformly on $\widehat{\mathbb{C}}$ (which is compact) to a constant map, the assertion follows if $d=0$.

So assume $d\ge 1$; up to replace f and f_ν by $1/f$ and $1/f_\nu$ if necessary, we can suppose that $f(\infty)\ne 0$. So all zeroes of f are in $\mathbb{C}$; let us call them $z_1,\dots,z_q$, with $q\le d$. For each $j=1,\dots,q$, let D_j be a small disk of center z_j not containing poles of f; furthermore, we can assume $\overline{D_h}\cap\overline{D_k}=\varnothing$ if $h\ne k$. Put $K=\widehat{\mathbb{C}}\setminus\bigcup_{j=1}^q D_j$. Since f has no zeroes in K, for ν large enough every f_ν has no zeroes in K too. Analogously, for ν large enough every f_ν has no poles in $\bigcup_{j=1}^q \overline{D_j}$. We also have

$$\max_{z\in\partial D_j}|f_\nu(z)-f(z)|<\min_{z\in\partial D_j}|f(z)|$$

for all $j=1,\dots,q$ if ν is large enough; therefore, by Rouché's Theorem 0.5, the number of zeroes (counted with multiplicities) of f_ν in each D_j is equal to the number of zeroes (counted with multiplicities) of f in the same D_j. Since there are no zeroes in K, Proposition 1.2 yields $\deg f_\nu=\deg f$ for ν large enough, as claimed. □

Corollary 2.4: *Let* $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ *with* $\deg f\ge 2$. *Then* $\mathcal{J}(f)\ne\varnothing$.

Proof: If $\mathcal{J}(f)=\varnothing$, then $\{f^k\}$ is normal in $\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$; hence there is a subsequence $\{f^{k_j}\}$ converging to $g\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$. By Lemma 2.3, this implies that $\deg f^{k_j}=\deg g$ for j large enough. But Corollary 1.3 yields $\deg f^{k_j}=(\deg f)^{k_j}$, and hence we must have $\deg f=1$. □

Remark 2.3: We shall see that, on the other hand, $\mathcal{F}(f)=\varnothing$ might happen.

Definition 2.5: Given $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$, a point $z_0\in\widehat{\mathbb{C}}$ is *exceptional* if $GO(z_0)$ is finite. We shall denote by $\mathcal{E}(f)$ the set of exceptional points of f.

Proposition 2.5: *Let* $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ *with* $\deg f\ge 2$. *Then:*

(i) *the cardinality of* $\mathcal{E}(f)$ *is at most two;*
(ii) *every exceptional point of* f *is a critical point;*
(iii) $\mathcal{E}(f)\subset\mathcal{F}(f)$.

Proof: Take $z_0\in\mathcal{E}(f)$. Since $f\big(GO(z_0)\big)=GO(z_0)$ and $GO(z_0)$ is finite, it follows that f is a bijection of $GO(z_0)$ with itself, and this can happen (why?) if and only if $GO(z_0)$ consists of a unique periodic orbit. In particular the cardinality of $f^{-1}(z)$ is 1 for each $z\in GO(z_0)$; since $\deg f\ge 2$, it follows (Remark 1.1) that every $z\in GO(z_0)$ is a critical point of f. In particular, $GO(z_0)$ is a superattracting periodic orbit, and hence $GO(z_0)\subset\mathcal{F}(f)$ by Proposition 2.2.(i). So we have proven (ii) and (iii).

Assume, by contradiction, that $\mathcal{E}(f)$ contains three distinct points z_1, z_2 and z_3. Then the open connected set $D=\widehat{\mathbb{C}}\setminus\big(GO(z_1)\cup GO(z_2)\cup GO(z_3)\big)$ is completely invariant and hyperbolic; Montel's Theorem 0.9 then implies that $\{f^k\}$ is normal in $\mathrm{Hol}(D,\widehat{\mathbb{C}})$, and thus $D\subset\mathcal{F}(f)$. But we saw that $\mathcal{E}(f)\subset\mathcal{F}(f)$; therefore this would imply $\mathcal{F}(f)=\widehat{\mathbb{C}}$, against Corollary 2.4. □

Corollary 2.6: *Let* $f\in \mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ *with* $\deg f\ge 2$. *Then* $\mathcal{J}(f)$ *is infinite.*

Proof: If $\mathcal{J}(f)$ were finite, being completely invariant, it would be contained in $\mathcal{E}(f)\subset\mathcal{F}(f)$, contradiction. □

Exercise 2.3: Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$.

(i) Prove that if f is a polynomial then $\infty \in \mathcal{E}(f)$.
(ii) Prove that if $f(z) = z^d$ with $d \in \mathbb{Z}^*$ then $\mathcal{E}(f) = \{0, \infty\}$.
(iii) Prove that if $\mathcal{E}(f)$ contains exactly one point, then f is conjugated to a polynomial.
(iv) Prove that if $\mathcal{E}(f)$ contains exactly two points, then f is conjugated to $g(z) = z^d$ for a suitable $d \in \mathbb{Z}^*$.

In a sense, the Julia set is the minimal closed completely invariant subset of $\widehat{\mathbb{C}}$:

Theorem 2.7: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$, and take $E \subseteq \widehat{\mathbb{C}}$ closed and completely f-invariant. Then either E has at most two elements and $E \subseteq \mathcal{E}(f) \subset \mathcal{F}(f)$, or E is infinite and $E \supseteq \mathcal{J}(f)$.*

Proof: Proposition 2.5 says that if E is finite then it has at most two elements and it is contained in the Fatou set. If E is infinite, then $D = \widehat{\mathbb{C}} \setminus E$ is a hyperbolic completely f-invariant open set; Montel's Theorem 0.9 thus implies that $\{f^k|_D\}$ is normal in $\mathrm{Hol}(D, \widehat{\mathbb{C}})$, that is $D \subseteq \mathcal{F}(f)$, and so $E \supseteq \mathcal{J}(f)$. □

Corollary 2.8: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$, and $\Omega \subseteq \mathcal{F}(f)$ a completely f-invariant open subset. Then $\partial\Omega \supseteq \mathcal{J}(f)$. In particular, if Ω is a basin of attraction then $\partial\Omega = \mathcal{J}(f)$.*

Proof: Since f is continuous and open, $\partial\Omega$ is closed and completely f-invariant; furthermore, it is infinite, because if it were finite it would be contained in $\mathcal{F}(f)$ by Proposition 2.5, and hence $\mathcal{J}(f) = \varnothing$ against Corollary 2.6. Therefore the assertion follows from Theorem 2.7.(ii) and Proposition 2.2.(i). □

Example 2.3: In particular, if $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ has $n \geq 2$ attracting fixed points, then we get n open sets (the basins of attraction) sharing exactly the same boundary. For instance, this happens for

$$f(z) = z - \frac{z^n - 1}{nz^{n-1}} \;,$$

the Newton map associated to the polynomial $z^n - 1$, because the n n-roots of unity are superattracting fixed points for f.

Remark 2.4: This corollary provides one of the standard way of drawing the Julia set of a polynomial $p \in \mathbb{C}[z]$: indeed Corollary 2.8 implies that $\mathcal{J}(p)$ is the boundary of the set of points with bounded orbits (the complementary of the basin of attraction of ∞).

In practice, one proceeds as follows. First of all one determines $R > 0$ such that $|z| \geq R$ implies $|p(z)| > R$; so the complementary of the closed disk $\overline{\Delta_R}$ of radius R and center the origin is contained in the basin of attraction Ω of the superattracting point ∞. Then one assigns to $z \in \overline{\Delta_R}$ the color black if $|p^k(z)| \leq R$ for all $k = 0, \ldots, N_0$ (where N_0 is a suitable large number), and a different color otherwise, depending on the first j such that $|p^j(z)| > R$. Then $\mathcal{J}(p)$ is (approximated by) the boundary of the black region (it is exactly equal to the boundary of the black region if $N_0 = +\infty$).

Exercise 2.4: Find $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$, and $\Omega \subseteq \mathcal{F}(f)$ a completely f-invariant open subset, such that $\partial\Omega \neq \mathcal{J}(f)$. Is it always true that $\partial\Omega \subseteq \mathcal{J}(f) \cup \mathcal{E}(f)$?

Definition 2.6: A *perfect* topological space is a topological space without isolated points.

Remark 2.5: A perfect locally compact Hausdorff space X is necessarily uncountable. Indeed, having no isolated point, no point is open. If X were countable, it would then be countable union of closed sets (its points) with empty interior, and this contradicts Baire's Theorem 0.12.

Corollary 2.9: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$. Then either $\mathcal{J}(f) = \widehat{\mathbb{C}}$ or $\mathcal{J}(f)$ is a perfect set with empty interior. In particular, $\mathcal{J}(f)$ is always uncountable.*

Proof: If $\mathcal{F}(f) \neq \varnothing$, then $\overline{\mathcal{F}(f)} = \mathcal{F}(f) \cup \partial\mathcal{J}(f)$ is closed, infinite, and completely f-invariant; therefore, by Theorem 2.7, $\mathcal{F}(f) \cup \partial\mathcal{J}(f) \supset \mathcal{J}(f)$, that is $\partial\mathcal{J}(f) \subseteq \mathcal{J}(f)$; being $\mathcal{J}(f)$ closed by definition, we get $\mathcal{J}(f) = \partial\mathcal{J}(f)$, and so $\mathcal{J}(f)$ has empty interior.

Let $\mathcal{J}_0 \subseteq \mathcal{J}(f)$ be the set of accumulation points of $\mathcal{J}(f)$. By definition $\mathcal{J}_0$ is closed; being f continuous and open, it is also (why?) completely f-invariant. Furthermore it is also infinite, because $\mathcal{E}(f) \cap \mathcal{J}(f) = \varnothing$. Therefore, by Theorem 2.7, $\mathcal{J}_0 = \mathcal{J}(f)$, that is $\mathcal{J}(f)$ has no isolated points. So $\mathcal{J}(f)$ is perfect and thus, by Remark 2.5, uncountable. □

3. Topology of the Fatou set

In this section we shall get some basic results on the topology of the connected components of the Fatou set. We recall the following standard facts on the topology of the Riemann sphere:

Theorem 3.1: (i) *An open connected set $D \subseteq \widehat{\mathbb{C}}$ is simply connected if and only if $\widehat{\mathbb{C}} \setminus D$ is connected if and only if ∂D is connected.*
(ii) *Let $D \subseteq \widehat{\mathbb{C}}$ be open. Then $\widehat{\mathbb{C}} \setminus D$ is connected if and only if every connected component of D is simply connected.*

Corollary 3.2: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$. Then $\mathcal{J}(f)$ is connected if and only if every connected component of $\mathcal{F}(f)$ is simply connected.*

Proof: It is just Theorem 3.1.(ii). □

Definition 3.1: A *domain* is an open connected subset of $\widehat{\mathbb{C}}$. We shall say that a domain $D \subset \widehat{\mathbb{C}}$ is *k-connected* (with $k \geq 1$) if $\widehat{\mathbb{C}} \setminus D$ has exactly k connected components; that it is *∞-connected* if $\widehat{\mathbb{C}} \setminus D$ has infinitely many connected components.

Example 3.1: By Theorem 3.1.(i), a domain is 1-connected if and only if it is simply connected. An annulus is 2-connected.

Definition 3.2: A *Fatou component* is a connected component of the Fatou set.

Proposition 3.3: *Take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$, and let F_0 be a completely f-invariant Fatou component. Then:*
(i) *$\partial F_0 = \mathcal{J}(f)$;*
(ii) *either F_0 is simply connected or it is ∞-connected;*
(iii) *all other Fatou components (if any) are simply connected.*

Proof: (i) We have $\partial F_0 \subseteq \mathcal{J}(f)$, because F_0 is a connected component of $\mathcal{F}(f)$; the assertion then follows from Corollary 2.8.

(ii) Let assume that F_0 is k-connected with $k \geq 1$, and let $E_1, \ldots, E_k$ be the connected components of $\widehat{\mathbb{C}} \setminus F_0$; we would like to prove that $k = 1$.

Since F_0 is completely f-invariant, so is $\widehat{\mathbb{C}} \setminus F_0$. Since f is continuous and surjective, it must permute the connected components of $\widehat{\mathbb{C}} \setminus F_0$; so there is $p \geq 1$ such that $f^p(E_j) = E_j$ for all $j = 1, \ldots, k$. But the E_j are disjoint, and $\widehat{\mathbb{C}} \setminus F_0$ is completely f^p-invariant; so each E_j is completely f^p-invariant. But $\mathcal{J}(f) \subseteq \widehat{\mathbb{C}} \setminus F_0$ is infinite; so at least one of the E_j, for instance E_1, is infinite. So E_1 is an infinite closed completely f^p-invariant set; Theorem 2.7 and Proposition 2.1.(iii) then imply $\mathcal{J}(f) = \mathcal{J}(f^p) \subseteq E_1$. But each E_j must intersect $\mathcal{J}(f) = \partial F_0$; therefore the only possibility is $k = 1$, and F_0 is simply connected.

(iii) Let $D = \widehat{\mathbb{C}} \setminus \big(F_0 \cup \mathcal{J}(f)\big)$. Then D is open, and its complement $F_0 \cup \mathcal{J}(f) = \overline{F_0}$ is connected; the assertion then follows from Theorem 3.1.(ii). □

Corollary 3.4: *Take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$. If $\mathcal{F}(f)$ is connected, then either it is simply connected (and thus $\mathcal{J}(f)$ is connected) or it is ∞-connected (and thus $\mathcal{J}(f)$ has infinitely many connected components).*

Proof: It follows from Proposition 3.3 with $F_0 = \mathcal{F}(f)$. □

Corollary 3.5: *Let $p \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be a polynomial of degree at least 2, and F_0 the Fatou component containing ∞. Then F_0 is completely f-invariant, $\partial F_0 = \mathcal{J}(f)$, and F_0 is either simply connected or ∞-connected.*

Proof: Since ∞ is a superattracting fixed point, F_0 is the connected component containing ∞ of the basin of attraction of ∞; in particular, $p(F_0) \subseteq F_0$, that is $F_0 \subseteq p^{-1}(F_0)$. Take now $z_1 \in p^{-1}(F_0)$; then z_1 belongs to a Fatou component $F_1 \subset \mathcal{F}(p)$ such that $p(F_1) \subseteq F_0$. If $p(F_1) \neq F_0$, then $p(\partial F_1) \cap F_0 \neq \varnothing$, but this is impossible because $\partial F_1 \subseteq \mathcal{J}(p)$. Therefore $p(F_1) = F_0$; in particular, there must exists $z_2 \in F_1$ such that $p(z_2) = \infty$. But this implies $z_2 = \infty$, and hence $F_1 = F_0$ and $z_1 \in F_0$. So $p^{-1}(F_0) \subseteq F_0$, and F_0 is completely p-invariant. The rest of the assertion then follows from Proposition 3.3. □

Definition 3.3: The *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$ is defined by

$$\mathcal{M} = \{c \in \mathbb{C} \mid \mathcal{J}(z^2 + c) \text{ is connected}\} .$$

By the previous corollary, $c \notin \mathcal{M}$ if and only if the Fatou component of $z^2 + c$ containing ∞ is ∞-connected.

It is possible to prove the following theorem, giving in particolar a criterium to establish whether $c \in \mathbb{C}$ belongs to the Mandelbrot set:

Theorem 3.6: *Let $p \in \mathbb{C}[z]$ be a polynomial of degree at least 2. Then $J(p)$ is connected if and only if all critical points of p have bounded orbit.*

Example 3.2: Both possibilities in Corollary 3.5 can occur. We have seen in Example 2.2 that if $p(z) = z^2 - 2$ then $\mathcal{J}(p) = [-2, 2]$, and hence $F_0 = \mathcal{F}(p)$ is simply connected. On the other hand, if $c \in \mathbb{C}$ does not belong to the Mandelbrot set, then $\mathcal{J}(z^2 + c)$ is disconnected, and hence F_0 is ∞-connected. This happens, for instance, if $|c| > 2$. Indeed, in this case it is not difficult to prove by induction (exercise) that

$$|p_c^k(0)| \geq |c|(|c| - 1)^{2^{k-1}} ,$$

where $p_c(z) = z^2 + c$; thus the orbit of the origin (the only critical point of p_c) tends to infinity, and the claim follows from Theorem 3.6.

To get more informations about the topology of Fatou components, we need the Riemann-Hurwitz formula.

Definition 3.4: A continuous map $f\colon X \to Y$ between two topological spaces is *proper* if the inverse image of any compact subset of Y is compact in X.

Remark 3.1: If $f \in \mathrm{Hol}(X, Y)$ is a holomorphic proper map between Riemann surfaces, then for all $q \in Y$ the set $f^{-1}(q)$ is compact and discrete — and hence finite. It is possible to prove that there exists $m \in \mathbb{N}^*$ such that the cardinality of $f^{-1}(q)$ is equal to m for almost all $q \in Y$.

Definition 3.5: Let $f \in \mathrm{Hol}(X, Y)$ be a holomorphic proper map. The number m just defined is the *degree* $\deg f$ of f.

Remark 3.2: Clearly, every $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ is automatically proper, and the degree just defined coincides with the degree of f introduced in Definition 1.1, thanks to Remark 1.3.

Recalling the definition of multiplicity given in Remark 1.4, we can now state the following

Theorem 3.7: (Riemann-Hurwitz) *If $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ we have*

$$\sum_{z \in \widehat{\mathbb{C}}} \bigl(\delta_f(z) - 1\bigr) = 2 \deg f - 2 . \tag{3.1}$$

More generally, if X and Y are Riemann surfaces with finite Euler-Poincaré characteristic, and $f \in \mathrm{Hol}(X, Y)$ is proper, then

$$\chi(X) + \sum_{z \in X} \bigl(\delta_f(z) - 1\bigr) = \chi(Y) \deg f , \tag{3.2}$$

where χ is the Euler-Poincaré characteristic.

Remark 3.3: We recall that the Euler-Poincaré characteristic of a Riemann surface X can be computed using the formula

$$\chi(X) = f - l + v ,$$

where f (respectively, l and v) is the number of faces (respectively, of sides and vertices) of a triangulation of X. Furthermore, it is known that $\chi(X) = 2$ if and only if $X = \widehat{\mathbb{C}}$, and that the Euler-Poincaré characteristic of a compact Riemann surface X homeomorphic to a sphere with g handles (that is, X has genus g) is $\chi(X) = 2 - 2g$. In particular, if $\mathbf{T}$ is a torus then $\chi(\mathbf{T}) = 0$. Finally, if $\Omega \subset \widehat{\mathbb{C}}$ is a m-connected domain in $\widehat{\mathbb{C}}$ then $\chi(\Omega) = 2 - m$.

An $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ restricted to a Fatou component is necessarily proper:

Lemma 3.8: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$, and F_0 a Fatou component. Then there exists a unique Fatou component such that $f(F_0) = F_1$. Furthermore, $f|_{F_0}: F_0 \to F_1$ is proper with $\deg f|_{F_0} \le \deg f$.*

Proof: First of all, $f(F_0)$ is a connected open set contained in $\mathcal{F}(f)$; therefore it must be contained in a Fatou component F_1.

Now, $\partial f(F_0) = f(\partial F_0)$. Indeed, f continuous easily implies $f(\partial F_0) \subseteq \partial f(F_0)$. Conversely, take $w \in \partial f(F_0)$ and choose $\{z_j\} \subset F_0$ such that $f(z_j) \to w$. Up to a subsequence, we can assume that $z_j \to z \in \overline{F_0}$; clearly, $f(z) = w$. If $z \in F_0$, then $w \in f(F_0)$, which is open, being f an open map, against the assumption $w \in \partial f(F_0)$; therefore $z \in \partial F_0$, and the claim is proved.

Since f is an open map, $f(F_0)$ is an open connected subset of F_1; furthermore, it is closed in F_1, because $\partial f(F_0) = f(\partial F_0) \subseteq \mathcal{J}(f)$ is disjoint from F_1. Being F_1 connected, this yields $f(F_0) = F_1$.

Now take $K \subset F_1$ compact. If $(f|_{F_0})^{-1}(K) \subset F_0$ were not compact, we would find a sequence $\{z_j\} \subset F_0$ converging to $z \in \partial F_0$ with $f(z_j) \in K$ for all j. Up to a subsequence, we can assume $f(z_j) \to w \in K \subset F_1$; but, by continuity, $w = f(z) \in f(\partial F_0) \subseteq \mathcal{J}(f)$, contradiction.

The last assertion follows trivially from the definition of degree. □

Using this we can prove

Theorem 3.9: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with $\deg f \ge 2$. Then:*

(i) *f admits at most two completely invariant Fatou components;*
(ii) *$\mathcal{F}(f)$ can have 0, 1, 2 or infinitely many connected components.*

Proof: (i) Let assume that there are $k \ge 2$ completely invariant Fatou components $F_1, \dots, F_k$; we should prove that $k = 2$. Proposition 3.3.(iii) shows that they all are simply connected; hence $\chi(F_j) = 1$ for all $j = 1, \dots, k$. Furthermore, $f|_{F_j}: F_j \to F_j$ is proper by the previous Lemma, and $\deg f|_{F_j} = \deg f$ because F_j is completely invariant; therefore (3.2) yields

$$\sum_{z \in F_j} \big(\delta_f(z) - 1\big) = \deg f - 1 \;.$$

But then (3.1) implies

$$k(\deg f - 1) = \sum_{j=1}^{k} \sum_{z \in F_j} \big(\delta_f(z) - 1\big) \le \sum_{z \in \widehat{\mathbb{C}}} \big(\delta_f(z) - 1\big) = 2(\deg f - 1) \;,$$

and hence $k = 2$.

(ii) If $\mathcal{F}(f)$ has a finite number of connected components, say $F_1, \dots, F_k$, then Lemma 3.8 implies that f permutes them; therefore there exists $m \ge 1$ such that they all are f^m-invariant, and hence completely f^m-invariant, because $\mathcal{F}(f^m) = \mathcal{F}(f)$ is. Part (i) then gives $k \le 2$, and we are done. □

All cases of Theorem 3.9 can be realized:

Example 3.3: We have already seen that $\mathcal{F}(z^2 - 2)$ is connected, and that $\mathcal{F}(z^2)$ has two completely invariant connected components; it is easy to check that $\mathcal{F}(z^{-2})$ has instead two not invariant connected components, and no completely invariant connected components.

Example 3.4: Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be the Newton map associated to the polynomial $z^n - 1$, with $n \ge 3$. In Example 2.3 we saw that f has (at least) n basins of attraction; therefore $\mathcal{F}(f)$ has at least $n \ge 3$ connected components, and hence infinitely many.

Example 3.5: (Lattes' example) We would like to find $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ such that $\mathcal{F}(f) = \varnothing$. Choose $\tau \notin \mathbb{R}$, and let $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$; then $\mathbf{T} = \mathbb{C}/\Lambda$ is a torus. Let $g: \mathbf{T} \to \mathbf{T}$ be induced by $z \mapsto -z$ in $\mathbb{C}$. It is easy to check (do it) that $g^{-1} = g$ and that g has exactly four fixed points: 0, 1/2, $\tau/2$ and $(1+\tau)/2$ (mod Λ). Put $S = \mathbf{T}/\sim$, where the equivalence relation $\sim$ is defined by $p \sim q$ if and only if $g(p) = g(q)$. Let $\pi: \mathbf{T} \to S$ be the canonical projection.

We claim that S is a Riemann surface. If $p \in S$ is not in $\pi^{-1}\big(\mathrm{Fix}(g_0)\big)$, then $\pi^{-1}(p)$ contains exactly two points, z_1 and $z_2 = g(z_1)$; furthermore, we can find a chart (U, φ) centered in z_1 with U disjoint from

Fix(g) such that $\big(g(U),\varphi\circ g^{-1}\circ\big)$ is a chart centered in z_2. It follows that we can find a local chart $\big(\pi(U),\psi\big)$ centered at p, where ψ is defined by $\psi\circ\pi|_U=\varphi$.

If instead $p=\pi(z_0)$ with $z_0\in$ Fix(g), a chart in $\mathbf{T}$ centered at z_0 is given by $\varphi(z)=z-z_0$, defined in a suitable neighborhood U of z_0. Then we can define a chart $\big(\pi(U),\psi\big)$ in S centered at p by imposing $\psi\circ\pi(z)=(z-z_0)^2$. It is easy to check that these charts define a Riemann surface atlas on S such that $\pi\colon\mathbf{T}\to S$ is holomorphic and proper (because $\mathbf{T}$ and S are compact).

By construction, $\pi\colon\mathbf{T}\to S$ has degree 2, and all points of $\mathbf{T}$ have multiplicity 1 but for the fixed points of g, having multiplicity 2. Since $\chi(\mathbf{T})=0$, (3.2) yields

$$4=\sum_{z\in\mathbf{T}}\big(\delta_\pi(z)-1\big)=2\chi(S)\;;$$

therefore $\chi(S)=2$, and thus S is biholomorphic to $\widehat{\mathbb{C}}$.

Now, the holomorphic map $f_0\colon\mathbf{T}\to\mathbf{T}$ given by $f_0(z)=2z \pmod\Lambda$ commutes with g; therefore it defines a holomorphic map $f\colon S\to S$, that is $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$. We claim that $\mathcal{J}(f)=\widehat{\mathbb{C}}$. To prove this, by Proposition 2.2.(ii) it suffices to show that the repelling periodic points of f are dense in S; and to get this it suffices to show that the repelling periodic points of f_0 are dense in $\mathbf{T}$.

Take r, $s\in\mathbb{Q}$ with odd denominator larger than 1; we claim that $z=r+s\tau \pmod\Lambda$ is a periodic point for f_0. In fact, if p is the least common multiple of the denominators of r and s, elementary number theory shows that there exists $k\in\mathbb{N}$ such that p divides 2^k-1; but this means that $(2^k-1)r$, $(2^k-1)s\in\mathbb{Z}$, that is $2^k(r+s\tau)\equiv r+s\tau \pmod\Lambda$, that is $f^k(z)=z$. Since f^k is given by the multiplication by $2^k>1$, the periodic point z is necessarily repelling; and since rational numbers with odd denominator larger than 1 are dense in $\mathbb{R}$, we have proved our claim.

4. Dynamics on the Julia set

The dynamics on the Julia set is expanding. A way to express this fact is given by the following important

Theorem 4.1: *Take $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f\ge 2$, and $U\subseteq\widehat{\mathbb{C}}$ an open set intersecting $\mathcal{J}(f)$. Then:*

(i) *we have*
$$\bigcup_{k=0}^{\infty}f^k(U)\supseteq\widehat{\mathbb{C}}\setminus\mathcal{E}(f)\;,$$
with equality if and only if $U\cap\mathcal{E}(f)=\varnothing$;

(ii) *$f^k(U)\supset\mathcal{J}(f)$ as soon as k is large enough.*

Proof: (i) Put $W=\bigcup_{k=0}^\infty f^k(U)$ and $E=\widehat{\mathbb{C}}\setminus W$. By construction, $f(W)\subseteq W$, and so $f^{-1}(E)\subseteq E$. Since W intersects $\mathcal{J}(f)$, the family $\{f^k\}$ cannot be normal on W; therefore the cardinality of E is at most 2. Being f surjective, it follows that $f(E)=E$; so $f^{-1}(E)=E$, and $E\subseteq\mathcal{E}(f)$. Finally, the complete invariance of $\mathcal{E}(f)$ implies that $W\cap\mathcal{E}(f)=\varnothing$ if and only if $U\cap\mathcal{E}(f)=\varnothing$, and (i) is proved.

(ii) Since $\mathcal{J}(f)$ is infinite (Corollary 2.9), we can find three open sets U_1, U_2, $U_3\subset U$ with disjoint closures and intersecting $\mathcal{J}(f)$. We claim that for every $h=1$, 2, 3 there are $k\ge1$ and $1\le j\le3$ such that $f^k(U_h)\supset U_j$. If this were not the case, for each $k\ge1$ and $1\le j\le3$ we would find $z_{j,k}\in U_j\setminus f^k(U_h)$. Let $\gamma_k\in\mathrm{Aut}(\widehat{\mathbb{C}})$ be such that $\gamma_k(z_{1,k})=0$, $\gamma_k(z_{2,k})=1$, and $\gamma_k(z_{3,k})=\infty$; notice that γ_k is unique, by Proposition 1.4.(i). By construction, $\gamma_k\circ f^k(U_h)\subseteq\widehat{\mathbb{C}}\setminus\{0,1,\infty\}$; therefore, by Theorem 0.9, $\{\gamma_k\circ f^k\}$ is normal in U_h. If we show that this implies that $\{f^k\}$ is normal in U_h we would have a contradiction, because $U_h\cap\mathcal{J}(f)\neq\varnothing$ by assumption.

Let $\{f^{k_\nu}\}$ be a sequence of iterates. Up to a subsequence, we can assume that $\gamma_{k_\nu}\circ f^{k_\nu}\to h\in\mathrm{Hol}(U_h,\widehat{\mathbb{C}})$ as $\nu\to+\infty$; furthermore, always up to a subsequence, we can also assume that $z_{j,k_\nu}\to w_j\in\overline{U_j}$ as $\nu\to+\infty$ for $j=1$, 2, 3. Since w_1, w_2 and w_3 are distinct by the assumptions on the U_j's, we also have that $\gamma_{k_\nu}\to\gamma\in\mathrm{Aut}(\widehat{\mathbb{C}})$, where γ is the unique automorphism of $\widehat{\mathbb{C}}$ with $\gamma(w_1)=0$, $\gamma(w_2)=1$ and $\gamma(w_3)=\infty$. It follows that
$$f^{k_\nu}=(\gamma_{k_\nu})^{-1}\circ(\gamma_{k_\nu}\circ f^{k_\nu})\to\gamma^{-1}\circ h\in\mathrm{Hol}(U_h,\widehat{\mathbb{C}})\;,$$
and thus $\{f^k\}$ is normal on U_h, contradiction.

Thus we have a map $\tau\colon\{1,2,3\}\to\{1,2,3\}$ such that for every $h=1,\,2,\,3$ there is $k\ge 1$ such that $f^k(U_h)\supseteq U_{\tau(h)}$. Since τ is a self-map of a finite set, it must have a periodic point; therefore we can find $1\le h\le 3$ and $k_1\ge 1$ such that $f^{k_1}(U_h)\supseteq U_h$.

Let $g=f^{k_1}$. Part (i) implies $\bigcup_{k=0}^\infty g^k(U_h)\supset\mathcal{J}(f)$; but $g^k(U_h)$ is an increasing sequence of open sets, and $\mathcal{J}(f)$ is compact; therefore there exists $k_2\in\mathbb{N}$ such that $f^{k_1k_2}(U_h)=g^{k_2}(U_h)\supseteq\mathcal{J}(f)$. Put $k_0=k_1k_2$; then $f^{k_0}(U)\supset f^{k_0}(U_h)\supset\mathcal{J}(f)$, and $f^k(U)\supset f^{k-k_0}\bigl(\mathcal{J}(f)\bigr)=\mathcal{J}(f)$ for all $k\ge k_0$. □

Corollary 4.2: *Take $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f\ge 2$, and $z_0\notin\mathcal{E}(f)$. Then $\overline{O^-(z_0)}\supseteq\mathcal{J}(f)$. In particular, if $z_0\in\mathcal{J}(f)$ then $O^-(z_0)$ is dense in $\mathcal{J}(f)$.*

Proof: If U is an open set intersecting $\mathcal{J}(f)$, Theorem 4.1.(i) says that $z_0\in f^k(U)$ for some $k\in\mathbb{N}$. Therefore $O^-(z_0)\cap U\ne\varnothing$, and the arbitrariness of U implies $\overline{O^-(z_0)}\supseteq\mathcal{J}(f)$. Finally, if $z_0\in\mathcal{J}(f)$ then $O^-(z_0)\subset\mathcal{J}(f)$, and hence $\overline{O^-(z_0)}=\mathcal{J}(f)$. □

Remark 4.1: This corollary provides a way to draw $\mathcal{J}(f)$; it suffices to find $z_0\in\mathcal{J}(f)$, for instance a repelling periodic point, and then plot the solutions of $f^k(z)=z_0$ for $k\to+\infty$.

This corollary also explains why Julia sets are self-similar, as explained by the following definition and exercise.

Definition 4.1: Let $J_1,\,J_2\subset\widehat{\mathbb{C}}$ be subsets of the Riemann sphere, and $z_j\in J_j$ for $j=1,\,2$. We say that (J_1,z_1) is *locally biholomorphic* to (J_2,z_2) if there exists a biholomorphism $\varphi\colon U_1\to U_2$ of a neighborhood U_1 of z_1 with a neighborhood U_2 of z_2 such that $\varphi(z_1)=z_2$ and $\varphi(U_1\cap J_1)=U_2\cap J_2$.

Exercise 4.1: Take $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f\ge 2$, and $z_0\in\mathcal{J}(f)$. Using Corollary 4.2 prove that the set of points $z\in\mathcal{J}(f)$ such that $\bigl(\mathcal{J}(f),z\bigr)$ is locally biholomorphic to $\bigl(\mathcal{J}(f),z_0\bigr)$ is dense in $\mathcal{J}(f)$ unless every backward orbit terminating at z_0 (i.e., every sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_{n-1}=f(z_n)$ for $n\ge 1$) contains a critical point of f. Prove moreover that the latter quite special condition is satisfied for $z_0=\pm 2$ when $f(z)=z^2-2$.

Exercise 4.2: Take $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f\ge 2$, and $U\subseteq\widehat{\mathbb{C}}$ an open set such that $U\cap\mathcal{J}(f)\ne\varnothing$. Prove that no subsequence of iterates of f can converge uniformly on compact sets of U.

Forward orbits are often dense too.

Definition 4.2: Let X be a locally compact Hausdorff topological space. We shall say that a property $\mathcal{P}$ holds for a *generic* point of X if it holds for all points belonging to a countable intersection of open dense sets (which is still dense by Baire's Theorem 0.12).

Corollary 4.3: *Take $f\in\mathrm{Hol}(\widehat{\mathbb{C}},\widehat{\mathbb{C}})$ with $\deg f\ge 2$. Then $O^+(z)$ is dense in $\mathcal{J}(f)$ for generic $z\in\mathcal{J}(f)$.*

Proof: Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable family of open sets of $\widehat{\mathbb{C}}$ such that $\{B_n\cap\mathcal{J}(f)\}$ is a countable basis for the topology of $\mathcal{J}(f)$. For every $n\in\mathbb{N}$ let

$$U_n=\bigcup_{k=0}^\infty f^{-k}(B_n)\;;$$

Corollary 4.2 implies that $U_n\cap\mathcal{J}(f)$ is open and dense in $\mathcal{J}(f)$, because it contains $O^-(z)$ for each $z\in B_n\cap\mathcal{J}(f)$. Put

$$W=\bigcap_{n=0}^\infty\bigl(U_n\cap\mathcal{J}(f)\bigr)\subseteq\mathcal{J}(f)\;;$$

by construction, W is the intersection of a countable family of open dense sets of $\mathcal{J}(f)$. If $z\in W$, then for every $n\in\mathbb{N}$ there is $k_n\in\mathbb{N}$ such that $z\in f^{-k_n}(B_n)$, that is $f^{k_n}(z)\in B_n$. Thus $O^+(z)$ intersects all open subsets of $\mathcal{J}(f)$, that is is dense in $\mathcal{J}(f)$, and we are done. □

Remark 4.2: If $z_0\in\mathcal{J}(f)$ is periodic, then its orbit clearly cannot be dense in $\mathcal{J}(f)$.

A slightly more complicated consequence of Theorem 4.1 is the following:

Proposition 4.4: *Take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$. Then $\overline{\mathrm{Per}(f)} \supseteq \mathcal{J}(f)$. In particular, f has infinitely many periodic points.*

Proof: Let $U \subseteq \widehat{\mathbb{C}}$ be an open set intersecting $\mathcal{J}(f)$; we must show that U contains at least one periodic point.

Take $w_0 \in U \cap \mathcal{J}(f)$ which is neither a fixed point nor a critical value of f; since there are only a finite number of fixed points and critical values, this can be done. Since $\deg f \geq 2$ and w_0 is not a critical value, $f^{-1}(w_0)$ contains at least two distinct points $w_1 \neq w_2$; since w_0 is not fixed, we also have w_1, $w_2 \neq w_0$. We can find neighborhoods U_j of w_j for $j = 0, 1, 2$ with disjoint closures such that $U_0 \subseteq U$ and $f|_{U_j}: U_j \to U_0$ is a biholomorphism for $j = 1, 2$. Let $g_j: U_0 \to U_j$ be the inverse of $f|_{U_j}$.

Assume, by contradiction, that for all $z \in U_0$, $j = 1, 2$, and $k \geq 1$ we have $f^k(z) \neq z, g_j(z)$. Then the maps $h_k \in \mathrm{Hol}(U_0, \widehat{\mathbb{C}})$ defined by setting

$$h_k(z) = \frac{\big(f^k(z) - g_1(z)\big)\big(z - g_2(z)\big)}{\big(f^k(z) - g_2(z)\big)\big(z - g_1(z)\big)}$$

have image in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$; therefore, by Montel's Theorem 0.9, the family $\{h_k\}$ is normal in $\mathrm{Hol}(U_0, \widehat{\mathbb{C}})$. Since

$$f^k(z) = \frac{h_k(z) g_2(z)\big(z - g_1(z)\big) - g_1(z)\big(z - g_2(z)\big)}{h_k(z)\big(z - g_1(z)\big) - \big(z - g_2(z)\big)} \ ,$$

then every subsequence of $\{f^k\}$ contains a subsequence converging in $\mathrm{Hol}(U_0, \widehat{\mathbb{C}})$, and thus $U_0 \subseteq \mathcal{F}(f)$, contradiction.

So we have proved that there exist $z_0 \in U_0$ and $k \geq 1$ such that $f^k(z_0) = z_0$ or $f^k(z_0) = g_1(z_0)$ or $f^k(z_0) = g_2(z_0)$. This implies $z_0 \in \mathrm{Per}(f) \cap U$: in the first case this is clear, in the latter two cases follows from $f^{k+1}(z_0) = f\big(g_j(z_0)\big) = z_0$, and we are done. □

Definition 4.3: Let $f: X \to X$ be a continuous self-map of a metric space (X, d). We shall say that f is *topologically transitive* if for every open subsets U, $V \subseteq X$ there is $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$. We shall say that f has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for each $z \in X$ and each neighborhood $U \subseteq X$ of z there are $w \in U$ and $k \in \mathbb{N}$ such that $d\big(f^k(z), f^k(w)\big) \geq \delta$. We shall say that f is *chaotic* if it is topologically transitive, has sensitive dependence on the initial conditions, and $\mathrm{Per}(f)$ is dense in X.

Remark 4.3: It is possible to prove that if X is a perfect locally compact metric space with a countable basis then topological transitivity and density of periodic points imply the sensitive dependence on the initial conditions. Furthermore, in this case topological transitivity is equivalent to the existence of a dense orbit.

Corollary 4.5: *Take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$. Then f restricted to $\mathcal{J}(f)$ is topologically transitive and has sensitive dependence on initial conditions.*

Proof: Let $U, V \subseteq \widehat{\mathbb{C}}$ be two open sets intersecting $\mathcal{J}(f)$. By Corollary 4.3, we can find $z_0 \in U \cap \mathcal{J}(f)$ with a dense orbit. In particular, there is $k \in \mathbb{N}$ such that $f^k(z_0) \in V \cap \mathcal{J}(f)$; hence $f^k\big(U \cap \mathcal{J}(f)\big) \cap \big(V \cap \mathcal{J}(f)\big) \neq \varnothing$, and thus $f|_{\mathcal{J}(f)}$ is topologically transitive.

Now take $\delta = \mathrm{diam}\big(\mathcal{J}(f)\big)/2$, where the diameter is computed with respect to any distance d on $\widehat{\mathbb{C}}$ inducing the usual topology (for instance, the spherical distance). Let $z \in \mathcal{J}(f)$, and $U \subseteq \widehat{\mathbb{C}}$ be any open neighborhood of z. Theorem 4.1.(ii) implies that we can find $k \in \mathbb{N}$ such that $f^k(U) \supset \mathcal{J}(f)$. In particular there must exists $w \in U \cap \mathcal{J}(f)$ such that $d\big(f^k(z), f^k(w)\big) \geq \delta$, and so $f|_{\mathcal{J}(f)}$ has sensitive dependence on initial conditions. □

Remark 4.4: We shall show later that the repelling periodic points are dense in $\mathcal{J}(f)$, and hence $f|_{\mathcal{J}(f)}$ is chaotic.

We end this section with a result on the connectivity of the Julia set:

Proposition 4.6: *Take* $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ *with* $\deg f \geq 2$. *Then either* $\mathcal{J}(f)$ *is connected or it has uncountably many connected components.*

Proof: If $\mathcal{J}(f)$ is not connected, we can write $\mathcal{J}(f) = J_0 \cup J_1$, where J_0, J_1 are disjoint not empty closed subsets. Since $\mathcal{J}(f)$ is perfect, both J_0 and J_1 are infinite sets. Let $U \subset \widehat{\mathbb{C}}$ be an open set intersecting J_0 but not intersecting J_1. Theorem 4.1.(ii) gives $k \in \mathbb{N}$ such that $f^k(U) \supset \mathcal{J}(f)$; the complete invariance of $\mathcal{J}(f)$ then implies $f^k(J_0) = \mathcal{J}(f) = J_0 \cup J_1$. Thus setting $J_{0j} = J_0 \cap f^{-k}(J_j)$ for $j = 0, 1$ we have $J_0 = J_{00} \cup J_{01}$, and thus J_0 too is a union of disjoint not empty closed subsets.

Arguing by induction on n, for every sequence $a_1 \ldots a_n \in \{0,1\}^n$ we can find $k_{a_1 \ldots a_{n-1}} \in \mathbb{N}$ and a not empty closed (and hence compact) set $J_{a_1 \ldots a_n} \subset \mathcal{J}(f)$ such that

$$
\begin{gathered}
J_{a_1 \ldots a_{n-1}} = J_{a_1 \ldots a_{n-1} 0} \cup J_{a_1 \ldots a_{n-1} 1} , \qquad J_{a_1 \ldots a_{n-1} 0} \cap J_{a_1 \ldots a_{n-1} 1} = \varnothing , \\
f^{k_{a_1 \ldots a_{n-1}}}(J_{a_1 \ldots a_{n-1}}) = \mathcal{J}(f) , \qquad J_{a_1 \ldots a_{n-1} j} = J_{a_1 \ldots a_{n-1}} \cap f^{-k_{a_1 \ldots a_{n-1}}}(J_j) .
\end{gathered}
$$

Then for each $\mathbf{a} = (a_1 a_2 \ldots) \in \{0,1\}^{\mathbb{N}^*}$ the infinite intersection

$$
J_{\mathbf{a}} = \bigcap_{n=1}^{\infty} J_{a_1 \ldots a_n}
$$

is closed and not empty, and $J_{\mathbf{a}} \cap J_{\mathbf{b}} = \varnothing$ if $\mathbf{a} \neq \mathbf{b}$; therefore each $J_{\mathbf{a}}$ contains at least one connected component of $\mathcal{J}(f)$, and being $\{0,1\}^{\mathbb{N}^*}$ uncountable we are done. □

5. Hyperbolic local dynamics

In this section we shall study the local dynamics about a fixed point, also discussing a few global consequences.

Definition 5.1: Given a Riemann surface X and $p \in X$, a (*one-dimensional, discrete, holomorphic*) *local dynamical system* on X at p is a holomorphic function $f\colon U \to X$ with $f(p) = p$, where $U \subseteq X$ is an open neighborhood of p. We shall denote by $\mathrm{End}(X, p)$ the family of local dynamical systems on X at p.

Definition 5.2: Let $f_j \in \mathrm{End}(X_j, p_j)$ be local dynamical systems for $j = 1, 2$. We say that f_1 and f_2 are *holomorphically locally conjugated* if there is a biholomorphism $\varphi\colon U_1 \to U_2$ with $\varphi(z_1) = z_2$ such that $f_2|_{U_2} = \varphi \circ f_1 \circ \varphi^{-1}$, where $U_j \subseteq X_j$ is a neighborhood of p_j contained in the domain of definition of f_j for $j = 1, 2$.

Remark 5.1: Using local coordinates, it is easy to see that every one-dimensional local dynamical system is holomorphically locally conjugated to an element of $\mathrm{End}(\mathbb{C}, 0)$; therefore it is enough to study $\mathrm{End}(\mathbb{C}, 0)$. Notice that the elements of $\mathrm{End}(\mathbb{C}, 0)$ can be identified with the converging power series at the origin withou constant terms; in particular, every $f \in \mathrm{End}(\mathbb{C}, 0)$ can be written in a unique way as

$$
f(z) = \lambda z + \sum_{j=2}^{\infty} a_j z^j ,
$$

where $\lambda = f'(0)$.

If $f \in \mathrm{End}(X, p)$ then the differential df_p is a linear endomorphism of the (complex) tangent space T_pX. Since T_pX has (complex) dimension 1, the action of df_p is given by multiplication by a complex number, the *derivative* $f'(p)$ of f at p. It is easy to check (exercise) that two holomorphically locally conjugated local dynamical systems have the same derivative at the fixed point, and that if $X = \mathbb{C}$ then this derivative coincides with the standard one. In particular, the classification of fixed points introduced in Definition 2.4 can be applied to local dynamical systems, and again we shall call *multiplier* the derivative at the fixed point.

We now start with the study of hyperbolic fixed points.

Remark 5.2: Notice that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}, 0)$ with non-zero multiplier, then f is locally invertible at 0, and the origin is a repelling fixed point for the inverse function $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$.

Theorem 5.1: (Kœnigs, 1884) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be with a hyperbolic fixed point of multiplier $\lambda \in \mathbb{C}^* \setminus S^1$. Then f is holomorphically locally conjugated to its linear part $g(z) = \lambda z$. The conjugation φ is unique up to a multiplicative constant.*

Proof: By Remark 5.2, we can assume $0 < |\lambda| < 1$; if $|\lambda| > 1$ it suffices to apply the same argument to $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$.

Choose $0 < \delta < 1$ such that $\delta^2 < |\lambda| < \delta$. Writing $f(z) = \lambda z + z^2 r(z)$ for a suitable holomorphic function r defined in a neighborhood of the origin, we can clearly find $\varepsilon > 0$ such that $|\lambda| + M\varepsilon < \delta$, where $M = \max\limits_{z \in \overline{\Delta_\varepsilon}} |r(z)|$, and Δ_ε is the open disk of center the origin and radius $\varepsilon > 0$ small enough. So we have

$$|f(z) - \lambda z| \le M|z|^2$$

for all $z \in \overline{\Delta_\varepsilon}$. Furthermore,

$$|f(z)| \le (|\lambda| + M\varepsilon)|z| < \delta|z| \; ;$$

in particular $f(\overline{\Delta_\varepsilon}) \subset \Delta_\varepsilon$, and by induction it easily follows that

$$|f^k(z)| \le \delta^k |z|$$

for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k/\lambda^k$; we claim that the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi\colon \Delta_\varepsilon \to \mathbb{C}$. Indeed we have

$$\begin{aligned}|\varphi_{k+1}(z) - \varphi_k(z)| &= \frac{1}{|\lambda|^{k+1}} \left| f\bigl(f^k(z)\bigr) - \lambda f^k(z) \right| \\ &\le \frac{M}{|\lambda|^{k+1}} |f^k(z)|^2 \le \frac{M}{|\lambda|} \left(\frac{\delta^2}{|\lambda|} \right)^k |z|^2\end{aligned}$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k (\varphi_{k+1} - \varphi_k)$ is uniformly convergent in Δ_ε to a holomorphic function ψ; since

$$\sum_{h=0}^{k-1} (\varphi_{h+1} - \varphi_h) = \varphi_k - \varphi_0 \; ,$$

it follows that $\varphi_k \to \varphi = \psi + \varphi_0$.

Since $\varphi'_k(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrinking ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\bigl(f(z)\bigr) = \lim_{k \to +\infty} \frac{f^k\bigl(f(z)\bigr)}{\lambda^k} = \lambda \lim_{k \to +\infty} \frac{f^{k+1}(z)}{\lambda^{k+1}} = \lambda \varphi(z) \; ,$$

that is $g = \varphi \circ f \circ \varphi^{-1}$, as claimed.

If $\tilde{\varphi}$ is another local biholomorphism such that and $\tilde{\varphi} \circ f \circ \varphi^{-1} = g$, it follows that $\xi(\lambda z) = \lambda \xi$, where $\xi = \tilde{\varphi} \circ \varphi^{-1}$. Write

$$\xi(z) = \sum_{j=1}^{\infty} a_j z^j \; ;$$

then we must have

$$\sum_{j=1}^{\infty} a_j \lambda^j z^j = \lambda \sum_{j=1}^{\infty} a_j z^j \; .$$

The uniqueness of the coefficients in the expansion in power series thus implies that $\xi(z) = a_1 z$, that is $\tilde{\varphi} = a_1 \varphi$, and so φ is uniquely determined up to a multiplicative constant. $\square$

We shall now discuss a few global consequences of Kœnigs' theorem.

Corollary 5.2: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with an attracting fixed point z_0 with multiplier $\lambda \in \Delta^*$, and let Ω be the basin of attraction of z_0. Then there exists a unique $\varphi \in \mathrm{Hol}(\Omega, \mathbb{C})$ such that $\varphi(z_0) = 0$, $\varphi'(z_0) = 1$ and*

$$\forall z \in \Omega \qquad \varphi\big(f(z)\big) = \lambda\varphi(z) \; . \tag{5.1}$$

Furthermore, $\varphi(\Omega) = \mathbb{C}$.

Proof: Theorem 5.1 gives the existence of φ in a neighborhood U of z_0, and the uniqueness statement. Notice that $|\lambda| < 1$ implies that $f(U) \subset U$. Now let $z \in \Omega$, and choose $k \in \mathbb{N}$ such that $f^k(z) \in U$. Then we can define $\varphi(z)$ by setting

$$\varphi(z) = \frac{\varphi\big(f^k(z)\big)}{\lambda^k} \; .$$

To show that this is a good definition (i.e., independent of k), choose another $h \in \mathbb{N}$ such that $f^h(z) \in U$. Since (5.1) holds in U, assuming $h \geq k$ we get

$$\varphi\big(f^h(z)\big) = \varphi\big(f^{h-k}\big(f^k(z)\big)\big) = \lambda^{h-k}\varphi\big(f^k(z)\big) \; ,$$

and hence

$$\frac{\varphi\big(f^h(z)\big)}{\lambda^h} = \frac{\varphi\big(f^k(z)\big)}{\lambda^k} \; .$$

If $h < k$ the same argument works reversing the roles of h and k. So $\varphi: \Omega \to \mathbb{C}$ is well defined, satisfies (5.1) everywhere, and it is holomorphic because for each $z \in \Omega$ there are a neighborhood $V \subset \Omega$ and $k \in \mathbb{N}$ such that $f^k(w) \in U$ for all $w \in V$.

We are left to proving that $\varphi(\Omega) = \mathbb{C}$. By construction, $\varphi(\Omega)$ contains a neighborhood of the origin. Take $w \in \mathbb{C}$; then there exist $k \in \mathbb{N}$ and $z \in \Omega$ such that $\lambda^k w = \varphi(z)$. Since $f(\Omega) = \Omega$, there is $z' \in \Omega$ such that $f^k(z') = z$. Then

$$\lambda^k \varphi(z') = \varphi\big(f^k(z')\big) = \varphi(z) = \lambda^k w \; ;$$

hence $\varphi(z') = w$, and we are done. □

Definition 5.3: The *local basin* of an attracting fixed point z_0 of $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ is the connected component of the basin of attraction Ω containing z_0. The *Kœnigs map* of f at z_0 is the map $\varphi: \Omega \to \mathbb{C}$ given by Corollary 5.2. The *local basin* of an attracting periodic orbit of period p is the orbit of the local basin of any point of the orbit considered as an attracting fixed point of f^p.

Lemma 5.3: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with an attracting fixed point z_0 with multiplier $\lambda \in \Delta^*$, and let Ω_0 be the local basin of attraction of z_0. Then Ω_0 is a Fatou component, $f(\Omega_0) = \Omega_0$, and $\varphi(\Omega_0) = \mathbb{C}$, where φ is the Kœnigs map of f at z_0.*

Proof: If F_0 is Fatou component containing Ω_0, the sequence of iterates of f must converge to z_0 in F_0, by Vitali's theorem and the identity principle; therefore F_0 is contained in the basin of attraction of z_0, and thus $F_0 = \Omega$. Since $f(\Omega_0) \subseteq \Omega_0$, Lemma 3.8 implies $f(\Omega_0) = \Omega_0$. Finally, the same argument used in the proof of the last assertion of Corollary 5.2 yields $\varphi(\Omega_0) = \mathbb{C}$. □

Theorem 5.4: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with an attracting fixed point z_0 with multiplier $\lambda \in \Delta$, and let Ω_0 be the local basin of attraction of z_0. Assume that $\deg f \geq 2$. Then:*

(i) *Ω_0 contains at least a critical point of f;*
(ii) *if moreover $\lambda \neq 0$, then there is a largest $r > 0$ such that there exists a holomorphic $\psi: \Delta_r \to \Omega_0$ such that $\varphi \circ \psi = \mathrm{id}$, where $\varphi: \Omega_0 \to \mathbb{C}$ is the Kœnigs map of f at z_0. Furthermore, setting $U = \psi(\Delta_r)$ then:*
  (a) *$\overline{U} \subset \Omega_0$;*
  (b) *$\varphi(\partial U) = \partial\Delta_r$, and φ is a homeomorphism between $\overline{U}$ and $\overline{\Delta_r}$;*
  (c) *U does not contain any critical point of f whereas ∂U does.*

Proof: If $\lambda = 0$ then z_0 is a critical point of f, and we are done; assume then $\lambda \neq 0$. Since φ is invertible in a neighborhood of z_0, the ψ we are looking for must coincide (in a neighborhood of 0 and hence everywhere) with the unique branch of the inverse of φ sending 0 into z_0, and $r \in (0, +\infty]$ must be the radius of convergence of the series defining this inverse. Notice that $\psi(\Delta_r) \subseteq \Omega_0$: indeed $f\bigl(\psi(w)\bigr) = \psi(\lambda w)$ holds for $|w|$ small enough, and hence for all $w \in \Delta_r$, and thus $f^k\bigl(\psi(w)\bigr) = \psi(\lambda^k w) \to \psi(0) = z_0$. This shows that $\psi(\Delta_r)$ is contained in the basin of attraction of z_0; being connected and containing z_0, it must be contained in Ω_0.

If $r = +\infty$, we would then have defined a not constant holomorphic function $\psi: \mathbb{C} \to \Omega_0$; but this contradicts Liouville's Theorem 0.7, because Ω_0 is hyperbolic (its complement contains the Julia set, and so it is infinite). Thus $0 < r < +\infty$; we are left to prove assertions (a), (b) and (c).

From $\varphi \circ f = \lambda\varphi$ we deduce

$$(\varphi' \circ f)f' = \lambda\varphi' \; ; \tag{5.2}$$

in particular, critical points of f in Ω_0 are critical points of φ. Since $\psi \circ \varphi|_U = \mathrm{id}_U$ forces $\psi'\bigl(\varphi(z)\bigr)\varphi'(z) = 1$ for all $z \in U$, we obtain that U cannot contain critical points of f.

Now, $\varphi\bigl(f(U)\bigr) = \lambda\varphi(U) = \Delta_{|\lambda|r} \subset\subset \Delta_r = \varphi(U)$; therefore $f(U)$ is relatively compact in U. So $f(\overline{U}) \subset U$, and this implies that $\overline{U} \subset \Omega_0$. Now take $\hat{z} \in \partial U$, and $\{z_j\} \subset U$ a sequence converging to $\hat{z}$. This sequence has no limit points in U; therefore $\{\varphi(z_j)\}$ cannot have limit points in Δ_r. This implies that $|\varphi(z_j)| \to r$; since $\varphi(z_j) \to \varphi(\hat{z})$ we get $|\varphi(\hat{z})| = r$, and hence $\varphi(\partial U) \subseteq \partial\Delta_r$. Analogously, if $\hat{w} \in \partial\Delta_r$ the sequence $\{\psi\bigl((1 - 1/j)\hat{w}\bigr)\} \subset U$ has no limit points in U, and thus it contains a subsequence converging to some $\hat{z} \in \partial U$ such that $\varphi(\hat{z}) = \hat{w}$. Summing up, we have proved that $\varphi(\partial U) = \partial\Delta_r$. In particular, $\varphi\bigl(f(\partial U)\bigr) = \lambda\varphi(\partial U) = \partial\Delta_{|\lambda|r} \subset \varphi(U)$, that is $f(\partial U) \subset U$.

Assume now, by contradiction, that f has no critical points in ∂U. Take $\hat{w} \in \partial\Delta_r$, and $\hat{z} \in \partial U$ such that $\varphi(\hat{z}) = \hat{w}$. Since $\hat{z}$ is not a critical point of f, we can find a holomorphic branch g of f^{-1} defined in a neighborhood $V \subset U$ of $f(\hat{z}) \in U$ such that $g\bigl(f(\hat{z})\bigr) = \hat{z}$; in particular, $g(V)$ is a neighborhood of $\hat{z}$, and $\varphi\bigl(g(V)\bigr)$ is a neighborhood of $\hat{w}$. Furthermore $\varphi(V) = \varphi\Bigl(f\bigl(g(V)\bigr)\Bigr) = \lambda\varphi\bigl(g(V)\bigr)$; so ψ is defined on $\lambda\varphi\bigl(g(V)\bigr)$. Moreover, notice that $\varphi \circ g = \lambda^{-1}\varphi$. Then we can extend ψ holomorphically to $\varphi\bigl(g(V)\bigr)$ by setting $w \mapsto g\bigl(\psi(\lambda w)\bigr)$. Arguing in this way for all $\hat{w} \in \partial\Delta_r$ we then extend ψ holomorphically to a neighborhood of $\overline{\Delta_r}$, which is impossible because r is the radius of convergence of ψ.

To conclude the proof it suffices to show that $\varphi: \overline{U} \to \overline{\Delta_r}$ is a homeomorphism. We already know that it is surjective and open; so it suffices to show that it is injective. Assume that $z, z' \in \overline{U}$ are such that $\varphi(z) = \varphi(z') = w$, and choose sequences $\{z_j\}$, $\{z'_j\} \subset U$ converging to z, respectively z'; clearly both $\varphi(z_j)$ and $\varphi(z'_j)$ converge to $w \in \overline{\Delta_r}$. Let $L_j \subset \Delta_r$ be the segment joining $\varphi(z_j)$ and $\varphi(z'_j)$, and denote by $K \subset \overline{U}$ the set of accumulation points of the curves $\psi(L_j)$ as $j \to +\infty$; we claim that z and z' are contained in the same connected component of K. If not, it means we can write K as the union of two disjoint closed sets K_1 and K_2, with $z \in K_1$ and $z' \in K_2$. Since K_1 and K_2 are closed and disjoint, we can find two open sets V_1 and V_2 with $K_j \subset V_j$ and $V_1 \cap V_2 = \varnothing$. We then have $z_j \in V_1$ and $z'_j \in V_2$ for j large enough; this means that $H_j = \psi(L_j) \setminus \overline{V_1 \cup V_2} \neq \varnothing$ for j large enough, because $\psi(L_j)$ is a curve connecting z_j and z'_j. It follows that the accumulation points of the sequence $\{H_j\}$ do not belong to $V_1 \cup V_2 \supset K$, which is absurd because $H_j \subset \psi(L_j)$.

Then K has a compact connected component containing both z and z'. But since $\varphi|_K \equiv w$, and φ is holomorphic and not constant, this connected component must consists of one point only, that is $z = z'$. □

Corollary 5.5: *Take $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ with $\deg f \geq 2$. Then the immediate basin of any attracting periodic point of f contains a critical point of f. In particular, f has at most $2\deg f - 2$ attracting periodic orbits.*

Proof: Let $z_0 \in \widehat{\mathbb{C}}$ be an attracting periodic point of period p. Then z_0 is an attracting fixed point of f^p; by Theorem 5.4, the immediate basin of attraction of z_0 contains a critical point of f^p. But

$$(f^p)'(z) = \prod_{j=0}^{p-1} f'\bigl(f^j(z)\bigr) \; ;$$

so $z \in \mathrm{Crit}(f^p)$ implies $f^j(z) \in \mathrm{Crit}(f)$ for some $j = 0, \ldots, p-1$, and thus the immediate basin of attraction of z_0 contains a critical point of f.

Different attracting periodic orbits have disjoint basins of attraction; therefore the last assertion follows from the fact that f has at most $2\deg f - 2$ critical points (Remark 3.1 and Theorem 3.7). □

The equivalent of Theorem 5.1 for superattracting fixed points is the following:

Theorem 5.6: (Böttcher, 1904) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be with a superattracting fixed point z_0 of multiplicity $r \geq 2$. Then f is holomorphically locally conjugated to the map $g(z) = z^r$. The conjugation is unique up to multiplication by an $(r-1)$-root of unity.*

Proof: Write $f(z) = a_r z^r + O(z^{r+1})$. The first remark is that up to a linear conjugation $z \mapsto \mu z$ with $\mu^{r-1} = a_r$ we can assume $a_r = 1$.

Now write $f(z) = z^r h_1(z)$ for a suitable holomorphic function h_1 defined in a neighborhood of the origin with $h_1(0) = 1$. By induction, it is easy to see that we can analogously write $f^k(z) = z^{r^k} h_k(z)$ for a suitable holomorphic function h_k defined in a neighborhood of the origin with $h_k(0) = 1$. Furthermore, the equalities $f \circ f^{k-1} = f^k = f^{k-1} \circ f$ yield

$$h_{k-1}(z)^r h_1\bigl(f^{k-1}(z)\bigr) = h_k(z) = h_1(z)^{r^{k-1}} h_{k-1}\bigl(f(z)\bigr) . \tag{5.3}$$

Choose $0 < \delta < 1$. Then we can clearly find $1 > \varepsilon > 0$ such that $M\varepsilon < \delta$, where $M = \max\limits_{z \in \overline{\Delta_\varepsilon}} |h_1(z)|$; we can also assume that $h_1(z) \neq 0$ for all $z \in \overline{\Delta_\varepsilon}$. Since

$$|f(z)| \leq M|z|^r < \delta |z|^{r-1}$$

for all $z \in \overline{\Delta_\varepsilon}$, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$.

We also remark that (5.3) implies that each h_k is well-defined and never vanishing on $\overline{\Delta_\varepsilon}$. So for every $k \geq 1$ we can choose a unique ψ_k holomorphic in Δ_ε with $\psi_k(0) = 1$ such that $\psi_k(z)^{r^k} = h_k(z)$ on Δ_ε.

Set $\varphi_k(z) = z\psi_k(z)$, so that $\varphi_k'(0) = 1$ and $\varphi_k(z)^{r^k} = f^k(z)$ on Δ_ε. We claim that the sequence $\{\varphi_k\}$ converges to a holomorphic function φ on Δ_ε. Indeed, we have

$$\begin{aligned}\left|\frac{\varphi_{k+1}(z)}{\varphi_k(z)}\right| &= \left|\frac{\psi_{k+1}(z)^{r^{k+1}}}{\psi_k(z)^{r^{k+1}}}\right|^{1/r^{k+1}} = \left|\frac{h_{k+1}(z)}{h_k(z)^r}\right|^{1/r^{k+1}} = \bigl|h_1\bigl(f^k(z)\bigr)\bigr|^{1/r^{k+1}} \\ &= \bigl|1 + O\bigl(|f^k(z)|\bigr)\bigr|^{1/r^{k+1}} = 1 + \frac{1}{r^{k+1}} O\bigl(|f^k(z)|\bigr) = 1 + O\left(\frac{1}{r^{k+1}}\right) ,\end{aligned}$$

and so the telescopic product $\prod_k (\varphi_{k+1}/\varphi_k)$ converges to φ/φ_1 uniformly in Δ_ε.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrinking ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi_k\bigl(f(z)\bigr)^{r^k} = f(z)^{r^k} \psi_k\bigl(f(z)\bigr)^{r^k} = z^{r^{k+1}} h_1(z)^{r^k} h_k\bigl(f(z)\bigr) = z^{r^{k+1}} h_{k+1}(z) = \bigl[\varphi_{k+1}(z)^r\bigr]^{r^k} ,$$

and thus $\varphi_k \circ f = c[\varphi_{k+1}]^r$, where c is a r^k-th root of unity. Differentiating r times this equality and evaluating the result at the origin we get $\varphi_k'(0) f^{(r)}(0) = r! c \varphi_{k+1}'(0)$, that is $c = 1$, because $f^{(r)}(0) = r!$. Thus we have

$$\varphi_k \circ f = [\varphi_{k+1}]^r ;$$

passing to the limit we get $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local biholomorphism conjugating f with g, we must have $\psi \circ \varphi^{-1}(z^r) = \psi \circ \varphi^{-1}(z)^r$ for all z in a neighborhood of the origin; comparing the series expansions at the origin we get $\psi \circ \varphi^{-1}(z) = az$ with $a^{r-1} = 1$, and hence $\psi(z) = a\varphi(z)$, as claimed. □

6. Parabolic local dynamics

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be with a parabolic fixed point. Then we can write

$$f(z) = e^{2i\pi p/q} z + a_{r+1} z^{r+1} + a_{r+2} z^{r+2} + \cdots , \tag{6.1}$$

with $a_{r+1} \neq 0$.

Definition 6.1: The rational number $p/q \in \mathbb{Q} \cap [0,1)$ is the *rotation number* of f, and the number $r+1 \geq 2$ is the *valence* of f at the fixed point. If $p/q = 0$ (that is, if the multiplier is 1), we shall say that f is *tangent to the identity.*

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order:

Proposition 6.1: *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be with a parabolic fixed point z_0 with multiplier $\lambda = e^{2i\pi p/q}$. Then f is holomorphically locally conjugated to $g(z) = \lambda z$ if and only if $f^q \equiv \mathrm{id}$.*

Proof: If $\varphi^{-1} \circ f \circ \varphi(z) = e^{2\pi i p/q} z$ then $\varphi^{-1} \circ f^q \circ \varphi = \mathrm{id}$, and hence $f^q = \mathrm{id}$.

Conversely, assume that $f^q \equiv \mathrm{id}$ and set

$$\varphi(z) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{f^j(z)}{\lambda^j}.$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda \varphi(z)$, and so f is holomorphically locally conjugated to λz. □

In particular, if f is tangent to the identity then it *cannot* be locally conjugated to the identity (unless it was the identity to begin with, which is not a very interesting case dynamically speaking). To have an idea of the dynamics of such a dynamica system, let us first consider a function of the form

$$f(z) = z(1 + a z^r)$$

for some $a \neq 0$. Let $v \in S^1 \subset \mathbb{C}$ be such that av^r is real and positive. Then for any $c > 0$ we have

$$f(cv) = c(1 + c^r a v^r) v \in \mathbb{R}^+ v;$$

moreover, $|f(cv)| > |cv|$. In other words, the half-line $\mathbb{R}^+ v$ is f-invariant and repelled from the origin. Conversely, if av^r is real and negative then the segment $[0, |a|^{-1/r}]v$ is f-invariant and attracted by the origin.

This example suggests the following definition:

Definition 6.2: Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity at z_0 of valence $r+1 \geq 2$, written as in (6.1). Then a unit vector $v \in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at the origin if $a_{r+1} v^r$ is real and negative (respectively, positive).

Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions: if $a_{r+1} = |a_{r+1}| e^{i\alpha}$, then $v = e^{i\theta}$ is attracting (respectively, repelling) if and only if

$$\theta = \frac{2k+1}{r}\pi - \frac{\alpha}{r} \qquad \left(\text{respectively, } \theta = \frac{2k}{r}\pi - \frac{\alpha}{r}\right).$$

Furthermore, a repelling (attracting) direction for f is attracting (repelling) for $f^{-1} \in \mathrm{End}(\mathbb{C}, 0)$.

It turns out that to every attracting direction is associated a connected component of $K_f \setminus \{0\}$.

Definition 6.3: Let $v \in S^1$ be an attracting direction for an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity at the origin. The *basin* centered at v is the set of points z in the domain of f such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that $f(z) \neq 0$ for all $z \neq 0$ close enough to the origin). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

A slightly more specialized (but quite useful) object is the following:

Definition 6.4: An *attracting petal* centered at an attracting direction v of an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity at the origin is an open simply connected f-invariant set $P \subseteq \widehat{\mathbb{C}} \setminus \{0\}$ such that a point z in the domain of f belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

We can now state and prove the important *Leau-Fatou flower theorem:*

Theorem 6.2: (Leau, 1897; Fatou, 1919-20) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity at the origin with valence $r + 1 \geq 2$. Let $v_1^+, \ldots, v_r^+ \in S^1$ be the r attracting directions of f at the origin, and $v_1^-, \ldots, v_r^- \in S^1$ the r repelling directions. Then:*

(i) *for each attracting (repelling) direction $v_j^\pm$ there exists an attracting (repelling) petal $P_j^\pm$, so that the union of these $2r$ petals forms a pointed neighborhood of the origin. Furthermore, the $2r$ petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is π/r.*

(ii) *the orbit of a point z_0 converges to the origin if and only if z_0 belongs to the (disjoint) union of the basins centered at the r attracting directions.*

(iii) *If B is a basin centered at one of the attracting directions, then there is a function $\varphi: B \to \mathbb{C}$such that*

$$\varphi \circ f(z) = \varphi(z) + 1 \tag{6.2}$$

for all $z \in B$. Furthermore, if P is the corresponding petal constructed in part (i)*, then $\varphi|_P$ is a biholomorphism with an open subset of the complex plane — and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.*

Proof: Up to a linear conjugation, we can assume that $a_{r+1} = -1$, so that the attracting directions are the r-th roots of unity. For any $\delta > 0$, the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$ has exactly r connected components, each one symmetric with respect to a different r-th root of unity; it will turn out that, for δ small enough, these connected components are attracting petals of f, even though to get a pointed neighbourhood of the origin we shall need larger petals.

For $j = 1, \ldots, r$ let $\Sigma_j \subset \mathbb{C}^*$ denote the sector centered about the attractive direction v_j^+ and bounded by two consecutive repelling directions, that is

$$\Sigma_j = \left\{ z \in \mathbb{C}^* \;\middle|\; \frac{2j-3}{r}\pi < \arg z < \frac{2j-1}{r}\pi \right\} .$$

Notice that each Σ_j contains a unique connected component $P_{j,\delta}$ of $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$; moreover, $P_{j,\delta}$ is tangent at the origin to the sector centered about v_j of amplitude π/r.

The main technical trick in this proof consists in transfering the setting to a neighbourhood of infinity in the Riemann sphere $\widehat{\mathbb{C}}$. Let $\psi: \mathbb{C}^* \to \mathbb{C}^*$ be given by

$$\psi(z) = \frac{1}{rz^r} ;$$

it is a biholomorphism between Σ_j and $\mathbb{C}^* \setminus \mathbb{R}^-$, with inverse $\psi^{-1}(w) = (rw)^{-1/r}$, suitably choosing the r-th root. Furthermore, $\psi(P_{j,\delta})$ is the right half-plane

$$H_\delta = \{w \in \mathbb{C} \mid \operatorname{Re} w > 1/(2r\delta)\} .$$

When $|w|$ is so large that $\psi^{-1}(w)$ belongs to the domain of definition of f, the composition $F = \psi \circ f \circ \psi^{-1}$ makes sense, and we have

$$F(w) = w + 1 + O(w^{-1/r}) . \tag{6.3}$$

Thus to study the dynamics of f in a neighbourhood of the origin in Σ_j it suffices to study the dynamics of F in a neighbourhood of infinity.

The first observation is that when $\operatorname{Re} w$ is large enough then

$$\operatorname{Re} F(w) > \operatorname{Re} w + \frac{1}{2} ;$$

this implies that for δ small enough H_δ is F-invariant (and thus $P_{j,\delta}$ is f-invariant). Furthermore, by induction one has

$$\operatorname{Re} F^k(w) > \operatorname{Re} w + \frac{k}{2} \tag{6.4}$$

for all $w \in H_\delta$, which implies that $F^k(w) \to \infty$ in H_δ (and $f^k(z) \to 0$ in $P_{j,\delta}$) as $k \to \infty$.

Now we claim that the argument of $w_k = F^k(w)$ tends to zero. Indeed, (6.3) and (6.4) yield

$$\frac{w_k}{k} = \frac{w}{k} + 1 + \frac{1}{k} \sum_{l=0}^{k-1} O(w_l^{-1/r}) ;$$

hence Cesaro's theorem on the averages of a converging sequence implies

$$\frac{w_k}{k} \to 1 , \tag{6.5}$$

and so $\arg w_k \to 0$ as $k \to \infty$. Going back to $P_{j,\delta}$, this implies that $f^k(z)/|f^k(z)| \to v_j^+$ for every $z \in P_{j,\delta}$. Since furthermore $P_{j,\delta}$ is centered about v_j^+, every orbit converging to 0 tangent to v_j^+ must intersect $P_{j,\delta}$, and thus we have proved that $P_{j,\delta}$ is an attracting petal.

Arguing in the same way with f^{-1} we get repelling petals; unfortunately, the petals obtained so far are too small to form a full pointed neighbourhood of the origin. In fact, as remarked before each $P_{j,\delta}$ is contained in a sector centered about v_j^+ of amplitude π/r; therefore the repelling and attracting petals obtained in this way do not intersect but are tangent to each other. We need larger petals.

So our aim is to find an f-invariant subset P_j^+ of Σ_j containing $P_{j,\delta}$ and which is tangent at the origin to a sector centered about v_j^+ of amplitude strictly greater than π/r. To do so, first of all remark that there are R, $C > 0$ such that

$$|F(w) - w - 1| \le \frac{C}{|w|^{1/r}} \tag{6.6}$$

as soon as $|w| > R$. Choose $\varepsilon \in (0,1)$ and select $\delta > 0$ so that $4r\delta < R^{-1}$ and $\varepsilon > 2C(4r\delta)^{1/r}$. Then $|w| > 1/(4r\delta)$ implies

$$|F(w) - w - 1| < \varepsilon/2 .$$

Set $M_\varepsilon = (1+\varepsilon)/(2r\delta)$ and let

$$\tilde H_\varepsilon = \{ w \in \mathbb{C} \mid |\operatorname{Im} w| > -\varepsilon \operatorname{Re} w + M_\varepsilon \} \cup H_\delta .$$

If $w \in \tilde H_\varepsilon$ we have $|w| > 1/(2r\delta)$ and hence

$$\operatorname{Re} F(w) > \operatorname{Re} w + 1 - \varepsilon/2 \qquad \text{and} \qquad |\operatorname{Im} F(w) - \operatorname{Im} w| < \varepsilon/2 ; \tag{6.7}$$

it is then easy to check that $F(\tilde H_\varepsilon) \subset \tilde H_\varepsilon$ and that every orbit starting in $\tilde H_\varepsilon$ must eventually enter H_δ. Thus $P_j^+ = \psi^{-1}(\tilde H_\varepsilon)$ is as required, and we have proved (i).

To prove (ii) we need a further property of $\tilde H_\varepsilon$. If $w \in \tilde H_\varepsilon$, arguing by induction on $k \ge 1$ using (6.7) we get

$$k\left(1 - \frac{\varepsilon}{2}\right) < \operatorname{Re} F^k(w) - \operatorname{Re} w$$

and

$$\frac{k\varepsilon(1-\varepsilon)}{2} < |\operatorname{Im} F^k(w)| + \varepsilon \operatorname{Re} F^k(w) - \big(|\operatorname{Im} w| + \varepsilon \operatorname{Re} w\big) .$$

This implies that for every $w_0 \in \tilde H_\varepsilon$ there exists a $k_0 \ge 1$ so that $F^{k_0}(w) \ne w_0$ for all $w \in \tilde H_\varepsilon$. Coming back to the z-plane, this says that any inverse orbit of f must eventually leave P_j^+. Thus every (forward) orbit of f must eventually leave any repelling petal. So if $z \ne 0$ is such that its orbit is completely contained in the neighborhood of the origin given by the union of repelling and attracting petals (together with the origin), then its orbit must eventually land in an attracting petal, and thus z belongs to a basin centered at one of the r attracting directions — and (ii) is proved.

To prove (iii), first of all we notice that we have

$$|F'(w) - 1| \le \frac{2^{1+1/r} C}{|w|^{1+1/r}} \tag{6.8}$$

in $\tilde{H}_\varepsilon$. Indeed, (6.6) says that if $|w| > 1/(2r\delta)$ then the function $w \mapsto F(w) - w - 1$ sends the disk of center w and radius $|w|/2$ into the disk of center the origin and radius $C/(|w|/2)^{1/r}$; inequality (6.8) then follows from the Cauchy estimates on the derivative.

Now choose $w_0 \in H_\delta$, and set $\tilde{\varphi}_k(w) = F^k(w) - F^k(w_0)$. Given $w \in \tilde{H}_\varepsilon$, as soon as $k \in \mathbb{N}$ is so large that $F^k(w) \in H_\delta$ we can apply Lagrange's theorem to the segment from $F^k(w_0)$ to $F^k(w)$ to get a $t_k \in [0,1]$ such that

$$\begin{aligned}\left|\frac{\tilde{\varphi}_{k+1}(w)}{\tilde{\varphi}_k(w)} - 1\right| &= \left|\frac{F\big(F^k(w)\big) - F^k\big(F^k(w_0)\big)}{F^k(w) - F^k(w_0)} - 1\right| = \left|F'\big(t_k F^k(w) + (1-t_k)F^k(w_0)\big) - 1\right| \\ &\le \frac{2^{1+1/r}C}{\min\{|\operatorname{Re} F^k(w)|, |\operatorname{Re} F^k(w_0)|\}^{1+1/r}} \le \frac{C'}{k^{1+1/r}} ,\end{aligned}$$

where we used (6.8) and (6.5), and the constant C' is uniform on compact subsets of $\tilde{H}_\varepsilon$ (and it can be chosen uniform on H_δ).

As a consequence, the telescopic product $\prod_k \tilde{\varphi}_{k+1}/\tilde{\varphi}_k$ converges uniformly on compact subsets of $\tilde{H}_\varepsilon$ (and uniformly on H_δ), and thus the sequence $\tilde{\varphi}_k$ converges, uniformly on compact subsets, to a holomorphic function $\tilde{\varphi}\colon \tilde{H}_\varepsilon \to \mathbb{C}$. Since we have

$$\begin{aligned}\tilde{\varphi}_k \circ F(w) &= F^{k+1}(w) - F^k(w_0) = \tilde{\varphi}_{k+1}(w) + F\big(F^k(w_0)\big) - F^k(w_0) \\ &= \tilde{\varphi}_{k+1}(w) + 1 + O\big(|F^k(w_0)|^{-1/r}\big) ,\end{aligned}$$

it follows that

$$\tilde{\varphi} \circ F(w) = \tilde{\varphi}(w) + 1$$

on $\tilde{H}_\varepsilon$. In particular, $\tilde{\varphi}$ is not constant; being the limit of injective functions, by Hurwitz's Theorem 0.11 it is injective, and hence a biholomorphism with its image.

We now prove that the image of $\tilde{\varphi}$ contains a right half-plane. First of all, we claim that

$$\lim_{\substack{|w|\to+\infty \\ w\in H_\delta}} \frac{\tilde{\varphi}(w)}{w} = 1 . \tag{6.9}$$

Indeed, choose $\eta > 0$. Since the convergence of the telescopic product is uniform on H_δ, we can find $k_0 \in \mathbb{N}$ such that

$$\left|\frac{\tilde{\varphi}(w) - \tilde{\varphi}_{k_0}(w)}{w - w_0}\right| < \frac{\eta}{3}$$

on H_δ. Furthermore, we have

$$\left|\frac{\tilde{\varphi}_{k_0}(w)}{w - w_0} - 1\right| = \left|\frac{k_0 + \sum_{j=0}^{k_0-1} O(|F^j(w)|^{-1/r}) + w_0 - F^{k_0}(w_0)}{w - w_0}\right| = O(|w|^{-1})$$

on H_δ; therefore we can find $R > 0$ such that

$$\left|\frac{\tilde{\varphi}(w)}{w - w_0} - 1\right| < \frac{\eta}{3}$$

as soon as $|w| > R$ in H_δ. Finally, if R is large enough we also have

$$\left|\frac{\tilde{\varphi}(w)}{w - w_0} - \frac{\tilde{\varphi}(w)}{w}\right| = \left|\frac{\tilde{\varphi}(w)}{w - w_0}\right| \left|\frac{w_0}{w}\right| < \frac{\eta}{3} ,$$

and (6.9) follows.

By (6.9) it exists $R > 1/(2r\delta)$ so that $|w| > R$ implies $|\tilde\varphi(w) - w| < |w|/3$. We claim that the right half-plane $H = \{\mathrm{Re}\, w > 2R\}$ is contained in the image of $\tilde\varphi$. Take $w^o \in H$, and consider the closed disk $D \subset H_\delta$ of center w^o and radius $|w^o|/2$. For every $w \in D$ we have $R < |w| \le 3|w^o|/2$; in particular

$$\forall w \in \partial D \qquad \big|\big(\tilde\varphi(w) - w^o\big) - (w - w^o)\big| = |\tilde\varphi(w) - w| < \frac{|w|}{3} \le \frac{|w^o|}{2} = |w - w^o| \,.$$

Rouché's Theorem 0.5 then implies that $\tilde\varphi - w^o$ and $w - w^o$ have the same number of zeroes inside that circle, and thus $w^o \in \tilde\varphi(H_\delta)$, as required.

So setting $\varphi = \tilde\varphi \circ \psi$, we have defined a function φ with the required properties on P_j^+. To extend it to the whole basin B it suffices to put

$$\varphi(z) = \varphi\big(f^k(z)\big) - k \,, \tag{6.10}$$

where $k \in \mathbb{N}$ is the first integer such that $f^k(z) \in P_j^+$. □

Remark 6.1: Notice that changing base point w_0 in the construction of the map $\tilde\varphi$ amounts to changing $\tilde\varphi$ (and hence φ) by an additive constant. Actually, it is possible to prove that the solution of (6.2) on a given basin is unique up to an additive constant.

Definition 6.5: Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be tangent to the identity at a point $z_0 \in \widehat{\mathbb{C}}$, and $v \in S^1$ an attracting direction. The *parabolic basin* $B(v)$ of f at z_0 along v is the set of $z \in \widehat{\mathbb{C}}$ whose orbit converges to z_0 tangentially to v. Clearly, $B(v) = \bigcup_{k\ge 0} f^{-k}(P)$, where P is an attracting petal at z_0 centered at v. The *local parabolic basin* at z_0 along v is the only forward invariant connected component of $B(v)$, that is the only connected component of $B(v)$ containing z_0 in the boundary. A *Fatou map* of f at z_0 is a map $\varphi\colon B \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in B$ constructed as in Theorem 6.2. A *local parabolic basin* of a parabolic periodic orbit of period p with multiplier $e^{2\pi i q/r}$ is the orbit of a local parabolic basin of any point of the orbit considered as an parabolic fixed point tangent to the identity of f^{pr}.

Remark 6.2: The flower Theorem 6.2 implies that every point whose orbit converges to z_0 must belong to one and exactly one of the parabolic basins.

Lemma 6.3: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be tangent to the identity at $z_0 \in \widehat{\mathbb{C}}$, and let B_0 a local parabolic basin at z_0. Then B_0 is a Fatou component, $f(B_0) = B_0$, and $\varphi(B_0) = \mathbb{C}$, where φ is a Fatou map of f at z_0.*

Proof: Clearly $B_0 \subseteq \mathcal{F}(f)$. Now take $\hat z \in \partial B_0$. If $\hat z \in GO(z_0)$, Proposition 2.2.(ii) implies $\hat z \in \mathcal{J}(f)$. Assume, by contradiction, that $\hat z \in \mathcal{F}(f) \setminus GO(z_0)$; then there is a neighborhood U of $\hat z$ where the sequence of iterates of f converges to z_0, by Vitali's theorem and the identity principle. But this would imply that $\hat z$ belong to some parabolic basin, which is impossible because the parabolic basins are open. Therefore we have shown that $\partial B_0 \subseteq \mathcal{J}(f)$, and thus B_0 is a Fatou component. Then $f(B_0) \subseteq B_0$ implies, by Lemma 3.8, that $f(B_0) = B_0$.

Finally, since $\varphi(B_0)$ contains a right-half plane, an argument very similar to the one used in the proof of the last assertion of Corollary 5.2 yields $\varphi(B_0) = \mathbb{C}$. □

Proposition 6.4: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be tangent to the identity at $z_0 \in \widehat{\mathbb{C}}$, and let B_0 be a local parabolic basin at z_0. Assume that $\deg f \ge 2$. Then B_0 contains a critical point of f.*

Proof: Let $\varphi\colon B_0 \to \mathbb{C}$ be a Fatou map; we already know that $\varphi(B_0) = \mathbb{C}$. Assume, by contradiction, that φ has no critical points in B_0. Then, by analytical continuation, we could construct a $\psi\colon \mathbb{C} \to B_0$ such that $\varphi \circ \psi = \mathrm{id}$, and this is against Liouville's theorem, because B_0 is a hyperbolic domain.

Let then $z_0 \in B_0$ a critical point of φ. For k large, $f^k(z_0)$ belongs to a petal, where φ has no critical points; so up to replacing z_0 by an iterate, we can assume that z_0 is a critical point of φ while $f(z_0)$ is not. Differentiating $\varphi\big(f(z)\big) = \varphi(z) + 1$ we get $\varphi'\big(f(z)\big)f'(z) = \varphi'(z)$; evalutaing this in z_0 we get $f'(z_0) = 0$, that is z_0 is a critical point of f. □

Corollary 6.5: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with $\deg f \ge 2$. Then f has at most $2d - 2$ attracting or parabolic periodic orbits.*

Proof: Proposition 6.4 and the argument used in the proof of Corollary 5.5 show that every local parabolic basin must contain a critical point of f. Since local attracting or parabolic basins are disjoint, the number of attracting or parabolic periodic orbits is bounded by the number of critical points of f, that is by $2\deg f - 2$. □

7. Chaos on the Julia set

In this section we shall prove that every $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ has only a finite number of non-repelling periodic points. Then Propositions 4.4 and 2.2 will imply that repelling periodic points are dense in $\mathcal{J}(f)$, and so, by Corollary 4.5, f restricted to $\mathcal{J}(f)$ is chaotic.

Everything will follow from the following

Lemma 7.1: *Let $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ be with $d = \deg f \geq 2$. Then f has at most $4d-4$ periodic orbits with multiplier $\lambda \in S^1 \setminus \{1\}$.*

Proof: Let us write $f(z) = P(z)/Q(z)$, where P and Q are polynomials without common factors, and for $t \in \mathbb{C}$ define

$$f_t(z) = \frac{(1-t)P(z) + tz^d}{(1-t)Q(z) + t} \ .$$

In particular, $f_0 = f$ and $f_1(z) = z^d$; notice that z^d has only two (super)attracting fixed points, and all others periodic points are repelling.

We recall that, given two polynomials P and Q, there is a polynomial in the coefficients of P and Q (the *resultant* of P and Q) vanishing only if P and Q have a common zero. The resultant of $(1-t)P(z) + tz^d$ and $(1-t)Q(z) + t$ is then a polynomial in t not vanishing in $t = 0, 1$; therefore it has only a finite number of zeroes, which means that $(1-t)P(z) + tz^d$ and $(1-t)Q(z) + t$ have no common factors for all but a finite number of values of $t \in \mathbb{C}$. In particular, $\deg f_t = d$ for all $t \in \mathbb{C} \setminus E$, where E is the finite set of exceptions, and $f_t(z)$ depends holomorphically on z and t.

Assume, by contradiction, that $f = f_0$ has $4d-3$ distinct periodic orbits with multiplier in $S^1 \setminus \{1\}$. If $z_j \in \mathrm{Per}(f)$ is one of these, with period m_j and multiplier λ_j, we have

$$f_0^{m_j}(z_j) - z_j = 0 \quad \text{and} \quad (f_0^{m_j})'(z_j) - 1 = \lambda_j - 1 \neq 0 \ .$$

The implicit function theorem then implies that if $|t| < \varepsilon$ with $\varepsilon > 0$ small enough there is a holomorphic function $z_j(t)$ with $z_j(0) = z_j$ and $f_t^{m_j}\big(z_j(t)\big) = z_j(t)$. We can also assume that $z_h(t) \neq z_k(t')$ if $h \neq k$ and $t, t' \in \Delta_\varepsilon$; furthermore, the multiplier $\lambda_j(t)$ of $z_j(t)$ depends holomorphically on t and $|\lambda_j(0)| = |\lambda_j| = 1$.

Assume, by contradiction, that for some j the function $t \mapsto \lambda_j(t)$ is constant (necessarily equal to $\lambda_j \neq 1$) in a neighborhood of $t = 0$. Let $\tau\colon [0,1] \to \mathbb{C} \setminus E$ be a smooth curve with $\tau(0) = 0$ and $\tau(1) = 1$. We claim that $f_{\tau(s)}^{m_j}$ has a fixed point of multiplier λ_j for all $s \in [0,1]$. To prove the claim, let $s_0 \in [0,1]$ be the supremum of s such that $f_{\tau(s)}^{m_j}$ has a fixed point as required for all $s < s_0$. First of all, $s_0 > 0$ by assumption. Since the limit of fixed points with equal multiplier is a fixed point with the same multiplier, $f_{\tau(s_0)}^{m_j}$ has a fixed point as required. If $s_0 < 1$ then the implicit function theorem says that $f_{\tau(s)}^{m_j}$ has again a fixed point as required (the multiplier, being holomorphic and constant on a curve, is constant everywhere) for $|s - s_0|$ small enough, against the definition of s_0. Hence $s_0 = 1$; but this means that $f_1^{m_j}$ has a fixed point with multiplier in $S^1 \setminus \{1\}$, impossible.

Therefore no multiplier $\lambda_j(t)$ is constant in a neighborhood of $t = 0$. Hence we can write

$$\lambda_j(t) = \lambda_j(0)[1 + a_j t^{n(j)} + o(t^{n(j)})]$$

for suitable $a_j \neq 0$ and $n(j) \geq 1$. In particular,

$$\log|\lambda_j(t)| = \log|1 + a_j t^{n(j)} + o(t^{n(j)})| = \mathrm{Re}(a_j t^{n(j)}) + o(t^{n(j)}) \ .$$

Now, there are $n(j)$ open sectors with vertex at the origin of equal amplitude where $\mathrm{Re}(a_j t^{n(j)}) > 0$, and $n(j)$ open sectors with vertex at the origin of the same amplitude where $\mathrm{Re}(a_j t^{n(j)}) < 0$. It follows that the function

$$\sigma_j(\theta) = \lim_{r \to 0^+} \mathrm{sgn} \log|\lambda_j(re^{2\pi i\theta})|$$

is well-defined in $[0,1]$ with values ± 1, a finite number of discontinuities, and it has 0 average.

It follows that the function $\sigma_1 + \cdots + \sigma_{4d-3}$ is well-defined on $[0,1]$ with a finite number of discontinuities, it has 0 average, and odd values. It follows that there is $\theta_0 \in [0,1]$ such that

$$\sigma_1(\theta_0) + \cdots + \sigma_{4d-3}(\theta_0) \leq -1 \ .$$

But then for r small enough we have $\log|\lambda_j(re^{2\pi i\theta_0})| < 0$ for at least $2d-1$ values of j, which means that $f_{re^{2\pi i\theta_0}}$ has at least $2d-1$ attracting periodic orbits, against Corollary 5.5. □

Corollary 7.2: (Fatou) *Let* $f \in \mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$ *be with* $d = \deg f \geq 2$. *Then* f *has at most* $6d - 6$ *non-repelling periodic points. In particular, repelling periodic points are dense in the Julia set* $\mathcal{J}(f)$, *and* f *is chaotic on* $\mathcal{J}(f)$.

Proof: It follows from Lemma 7.1, Corollary 6.5, Propositions 4.4 and 2.2, and Corollary 4.5. □

Remark 7.1: Shishikura has shown that actually f has at most $2d - 2$ non-repelling periodic points. This estimate is sharp: for instance z^2 has exactly $2 = 2 \cdot 2 - 2$ non-repelling periodic points, 0 and ∞.

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