Chapter 3

Hyperbolicity

3.1 The stable manifold theorem

In this section we shall mix two typical ideas of hyperbolic dynamical systems: the use of a linear approximation to understand the behavior of nonlinear systems, and the examination of the behavior of the system along a reference orbit.

The latter idea is embodied in the notion of stable set.

Definition 3.1.1: Let (X,f) be a dynamical system on a metric space X. The *stable set* of a point $x\in X$ is

$$W^s_f(x)=\left\{y\in X \;\middle|\; \lim_{k\to\infty} d\big(f^k(y),f^k(x)\big)=0\right\}.$$

In other words, $y\in W^s_f(x)$ if and only if the orbit of y is asymptotic to the orbit of x. If no confusion will arise, we drop the index f and write just $W^s(x)$ for the stable set of x.

Clearly, two stable sets either coincide or are disjoint; therefore X *is the disjoint union of its stable sets.* Furthermore, $f\big(W^s(x)\big)\subseteq W^s\big(f(x)\big)$ for any $x\in X$.

The twin notion of unstable set for homeomorphisms is defined in a similar way:

Definition 3.1.2: Let $f\colon X\to X$ be a homeomorphism of a metric space X. Then the *unstable set* of a point $x\in X$ is

$$W^u_f(x)=\left\{y\in X \;\middle|\; \lim_{k\to\infty} d\big(f^{-k}(y),f^{-k}(x)\big)=0\right\}.$$

In other words, $W^u_f(x)=W^s_{f^{-1}}(x)$.

Again, unstable sets of homeomorphisms give a partition of the space. On the other hand, stable sets and unstable sets (even of the same point) might intersect, giving rise to interesting dynamical phenomena.

Remark 3.1.1. If f is not invertible, the definition of the unstable set is more complicated. Let (X,f) be a dynamical system. The *dynamical completion* of (X,f) is the dynamical system $(\hat X,\hat f)$, where

$$\hat X=\big\{\hat x=\{x_k\}\in X^{\mathbb Z}\mid x_{k+1}=f(x_k)\text{ for all }k\in\mathbb Z\big\},$$

and $\hat f$ is just the left shift $\hat f(\{x_k\})=\{f(x_k)\}$. Since f is continuous, $\hat X$ is a closed subset of $X^{\mathbb Z}$, and $\hat f$ is continuous with respect to the induced topology. More precisely, $\hat f$ is a *homeomorphism* of $\hat X$. Furthermore, the completion comes equipped with a canonical projection $\pi\colon\hat X\to X$ given by $\pi(\hat x)=x_0$, so that $f\circ\pi=\pi\circ\hat f$. In a very precise sense (see Exercise 3.1.1), the completion is the least possible invertible extension of (X,f). We also explicitely remark that if X is a metric space then $\hat X$ inherites a distance function too. Given $x\in X$, an *history* of x is a $\hat x\in\hat X$ such that $\pi(\hat x)=x$. Notice that if $\hat x\in\hat X$ then its positive half $\{x_k\}_{k\in\mathbb N}$ simply is the orbit of x_0; on the other hand, its negative half is just *a possible* backward orbit of x_0. In particular, π is a homeomorphism if and only if f is a homeomorphism (and it is onto if and only if f is, and injective if and only if f is). Finally, for every history $\hat x\in\hat X$ of $x\in X$ the *unstable set of x with respect to $\hat x$* is

$$W^u_f(x;\hat x)=\pi\big(W^u_{\hat f}(\hat x)\big).$$

In other words, $W^u_f(x;\hat x)$ is the set of $y\in X$ such that there exists a backward orbit of y which is asymptotic to the chosen backward orbit of x. So if f is not invertible there are in general as many unstable sets as there are backward orbits (and they do not form a partition of X).

Exercise 3.1.1. State the universal property enjoyed by the triple $(\hat{X}, \hat{f}, \pi)$, and use it to prove that the dynamical completion is unique up to topological conjugation.

For the sake of simplicity, in this section we shall limit ourselves to invertible maps only; however, using the dynamical completion it is possible to extend almost all the results we shall prove to non-invertible maps.

Exercise 3.1.2. If x_0 is a periodic point of period n of a homeomorphism $f\colon X \to X$ prove that

$$\bigcup_{k=0}^{n-1} W^s_f\big(f^k(x_0)\big) = \bigcup_{k=0}^{n-1} W^s_{f^n}\big(f^k(x_0)\big) \quad \text{and} \quad \bigcup_{k=0}^{n-1} W^u_f(x_0) = \bigcup_{k=0}^{n-1} W^u_{f^n}\big(f^k(x_0)\big).$$

If L is a hyperbolic linear map, then the stable and unstable sets of the origin are exactly the stable and unstable subspaces $E^{s/u}(L)$; see Theorem 1.3.5. In particular, they are smooth and intersect transversally. The main goal of this section is to prove that this is a general feature: if x_0 is a hyperbolic fixed point (see below for the definition) of a smooth dynamical system (M, f), where M is a smooth manifold, then the stable and unstable *sets* are actually *manifolds* as smooth as the map f, and intersecting transversally in x_0.

Definition 3.1.3: Let (M, f) be a C^1 dynamical system on a smooth manifold M. Then a periodic point p of period n is *hyperbolic* if the linear map $d(f^n)_p\colon T_pM \to T_pM$ is a hyperbolic linear map. Its orbit will be called a *hyperbolic periodic orbit.*

Definition 3.1.4: Let (X, f) be a dynamical system on a metric space X. Let p be a fixed point of f, and $B(p, \delta)$ the open ball of radius $\delta > 0$ centered in p. Then the *local stable set* for f at p of *radius* δ is

$$W^s_f(p, \delta) = \{x \in B(p, \delta) \mid x \in W^s_f(p) \text{ and } f^k(x) \in B(p, \delta) \text{ for all } k \in \mathbb{N}\}.$$

Similarly, if f is a homeomorphism the *local unstable set* for f at p of radius δ is

$$W^u_f(p, \delta) = \{y \in B(p, \delta) \mid y \in W^u_f(p) \text{ and } f^{-k}(y) \in B(p, \delta) \text{ for all } k \in \mathbb{N}\}.$$

Definition 3.1.5: Let $L\colon V \to V$ be a hyperbolic linear self-map of a vector space V on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *contraction rate* $\lambda(L)$ and the *expansion rate* $\mu(L)$ of L are defined by

$$\lambda(L) = r(L|_{E^s}) = \sup\{|\chi| \mid \chi \in \mathrm{sp}(L), |\chi| < 1\},$$
$$\mu(L) = r\big((L|_{E^u})^{-1}\big)^{-1} = \inf\{|\chi| \mid \chi \in \mathrm{sp}(L), |\chi| > 1\}.$$

We are now ready to state the main results of this section.

Theorem 3.1.1: (Stable manifold theorem) *Let $f\colon M \to M$ be a C^r diffeomorphism, with $r \geq 1$, of a Riemannian manifold M. Let $p \in M$ be a hyperbolic fixed point of f. Then there is $\delta > 0$ such that:*

(i) *the local stable set $W^s(p, \delta)$ is an embedded C^r submanifold such that $T_pW^s(p, \delta) = E^s(df_p)$;*

(ii) *$f\big(W^s(p, \delta)\big) \subseteq W^s(p, \delta)$;*

(iii) *for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$d\big(f^k(y), p\big) \leq C_\varepsilon \big[\lambda(df_p) + \varepsilon\big]^k d(y, p)$$

for all $y \in W^s(p, \delta)$ and $k \in \mathbb{N}$, where d is the distance induced by the Riemannian metric;

(iv) *$y \in W^s(p, \delta)$ if and only if $f^k(y) \in B(p, \delta)$ for all $k \in \mathbb{N}$;*

(v) *the global stable set is given by $\bigcup_{k\in\mathbb{N}} f^{-k}\big(W^s(p, \delta)\big)$, and thus it is a C^r immersed submanifold.*

Remark 3.1.2. If f is not invertible, the same statement holds except for the last assertion: the global stable set might not be a submanifold. For instance, if $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x, y) = \big(2x(y+1), x(y+1)\big)$ then the stable set of the origin is $\{x = 0\} \cup \{y = -1\}$.

Clearly, an analogous statement holds for the unstable sets.

Remark 3.1.3. Because of the previous theorem, the stable/unstable set of a hyperbolic fixed point is called the *stable/unstable manifold.* Notice that Exercise 3.1.2 implies that similar results hold for stable and unstable sets of hyperbolic periodic points.

We shall describe a proof of Theorem 3.1.1 due to M.C. Irwin, and working in the setting of Banach spaces.

Definition 3.1.6: Let $E = E_1 \oplus E_2$ be a splitting of a Banach space E (where here and in the rest of this section E_1 and E_2 are always *closed* subspaces); for $j = 1, 2$ we shall denote by $p_j : E \to E_j$ the linear projection of E onto E_j, and we shall endow E with the box norm $\|x\| = \max\{\|p_1(x)\|, \|p_2(x)\|\}$. Furthermore, if $r > 0$ we denote by $E_j(r)$ the closed ball of radius r centered at the origin in E_j; in particular, the closed ball of radius r centered at the origin in E (endowed with the box norm) can be identified with $E_1(r) \times E_2(r)$.

Definition 3.1.7: Let $T: E \to E$ be a linear automorphism of $E = E_1 \oplus E_2$ preserving the splitting, that is such that $T(E_1) = E_1$ and $T(E_2) = E_2$, and take $\lambda \in (0, 1)$. We say that T is *λ-hyperbolic* (with respect to the given splitting) if

$$\|T_1\| < \lambda \qquad \text{and} \qquad \|T_2^{-1}\| < \lambda,$$

where $T_j = T|_{E_j}$ for $j = 1, 2$.

Lemma 3.1.2: *Let $E = E_1 \oplus E_2$ be a splitting of a Banach space E. Given $r > 0$, let $f: E_1(r) \times E_2(r) \to E$ be a Lipschitz map such that* $\mathrm{Lip}(f - T) \le \varepsilon < 1 - \lambda$ *for a suitable λ-hyperbolic linear automorphism T. Then, setting $f_j := p_i \circ f$, if $x = (x_1, x_2)$, $y = (y_1, y_2) \in E_1(r) \times E_2(r)$ are such that $\|x_1 - y_1\| \le \|x_2 - y_2\|$, then*

$$\|f_1(x) - f_1(y)\| \le (\lambda + \varepsilon)\|x_2 - y_2\| < (\lambda^{-1} - \varepsilon)\|x_2 - y_2\| \le \|f_2(x) - f_2(y)\|.$$

Proof: First of all we have

$$\begin{aligned} \|f_1(x) - f_1(y)\| &\le \|p_1(f - T)(x) - p_1(f - T)(y)\| + \|T_1(x_1) - T_1(y_1)\| \\ &\le \varepsilon\|x - y\| + \lambda\|x_1 - y_1\| \le (\lambda + \varepsilon)\|x - y\| \\ &\le (\lambda + \varepsilon)\|x_2 - y_2\|, \end{aligned} \tag{3.1.1}$$

because we are using the box norm.

On the other hand,

$$\begin{aligned} \|f_2(x) - f_2(y)\| &\ge \|T_2(x_2) - T_2(y_2)\| - \|p_2(f - T)(x) - p_2(f - T)(y)\| \\ &\ge \lambda^{-1}\|x_2 - y_2\| - \varepsilon\|x - y\| \\ &= (\lambda^{-1} - \varepsilon)\|x_2 - y_2\|. \end{aligned}$$

To conclude, it suffices to observe that $\lambda + \varepsilon < 1 < \lambda^{-1} - \varepsilon$. □

Remark 3.1.4. Even without assuming that $\|x_1 - y_1\| < \|x_2 - y_2\|$, the computations in (3.1.1) yields

$$\|f_1(x) - f_1(y)\| \le (\lambda + \varepsilon)\|x - y\|. \tag{3.1.2}$$

Definition 3.1.8: Given $r > 0$ and a map $f: E_1(r) \times E_2(r) \to E$, we define the *local stable set* of f by

$$W^s(f, r) = \bigl\{x \in E_1(r) \times E_2(r) \mid \|f^k(x)\| \le r \text{ for all } k \ge 0\bigr\} = \bigcap_{k=0}^{\infty} f^{-k}\bigl(E_1(r) \times E_2(r)\bigr).$$

Clearly $W^s(f, r)$ is f-invariant and contains all fixed points of f in $E_1(r) \times E_2(r)$; however, it might be empty.

Remark 3.1.5. We shall see that if the origin is a hyperbolic fixed point for f and r is small enough, then the set $W^s(f, r)$ coincides with the local stable set of f at the origin of radius r introduced in the Definition 3.1.4.

Proposition 3.1.3: *Let $E = E_1 \oplus E_2$ be a splitting of a Banach space E. Given $0 < r < +\infty$, let $f: E_1(r) \times E_2(r) \to E$ be a Lipschitz map such that* $\mathrm{Lip}(f - T) \le \varepsilon < 1 - \lambda$ *for a suitable λ-hyperbolic linear automorphism T. Then the set $W^s(f, r)$ is the graph of a Lipschitz map $g: A \to E_2(r)$ with* $\mathrm{Lip}(g) \le 1$, *where $A = p_1\bigl(W^s(f, r)\bigr) \subseteq E_1(r)$. Furthermore, $f|_{W^s(f,r)}$ is a contraction. In particular, f has at most one fixed point which, if it exists, attracts exponentially all other points of $W^s(f, r)$.*

Proof: To prove the existence of g it suffices to show that for any pair $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of points of $W^s(f, r)$ we have $\|x_2 - y_2\| \le \|x_1 - y_1\|$. Indeed, if this happens then $p_1^{-1}(x_1)$ must consist of exactly one point $x_2 = g(x_1)$ for all $x_1 \in A$, and $g: A \to E_2(r)$ is Lipschitz of constant at most 1.

Let us assume, by contradiction, that we had $\|x_2 - y_2\| > \|x_1 - y_1\|$; using the previous lemma we get

$$\|f_2(x) - f_2(y)\| \geq (\lambda^{-1} - \varepsilon)\|x_2 - y_2\| > (\lambda + \varepsilon)\|x_2 - y_2\| \geq \|f_1(x) - f_1(y)\|,$$

and then, by induction,

$$\|p_2 \circ f^k(x) - p_2 \circ f^k(y)\| \geq (\lambda^{-1} - \varepsilon)^k \|x_2 - y_2\|$$

for all $k \in \mathbb{N}$. But, by definition of $W^s(f, r)$, we have $\|p_2 \circ f^k(x) - p_2 \circ f^k(y)\| \leq 2r$ for all $k \in \mathbb{N}$; since $(\lambda^{-1} - \varepsilon)^k \to +\infty$ as $k \to \infty$, we would have to conclude that $\|x_2 - y_2\| = 0$, against our assumption.

We are left to proving that $f|_{W^s(f,r)}$ is a contraction. We have seen that $x, y \in W^s(f, r)$ implies $\|x-y\| = \|x_1-y_1\|$. Moreover, since $W^s(f, r)$ is f-invariant, we also have $\|f(x)-f(y)\| = \|f_1(x)-f_1(y)\|$. But then (3.1.2) yields

$$\|f(x) - f(y)\| = \|f_1(x) - f_1(y)\| \leq (\lambda + \varepsilon)\|x - y\|;$$

so f is a contraction on $W^s(f, r)$, and the rest is a trivial consequence of Theorem 1.2.1. □

Of course, this is not enough to get a stable manifold theorem: we should at least prove that A is not empty. To do so we need a few ausiliary results.

Lemma 3.1.4: *Let $f, g: X \to F$ be continuous maps from a metric space X to a Banach space F. Let us suppose that f is injective with inverse f^{-1} Lipschitz, and that $\mathrm{Lip}(g - f)\,\mathrm{Lip}(f^{-1}) < 1$. Then g is injective, g^{-1} is Lipschitz and*

$$\mathrm{Lip}(g^{-1}) \leq \left\{ \left[\mathrm{Lip}(f^{-1})\right]^{-1} - \mathrm{Lip}(f - g) \right\}^{-1} = \frac{\mathrm{Lip}(f^{-1})}{1 - \mathrm{Lip}(g - f)\,\mathrm{Lip}(f^{-1})}.$$

Proof: To prove the assertion it is enough to observe that:

$$\begin{aligned}\|g(x) - g(y)\| &\geq \|f(x) - f(y)\| - \|(f - g)(x) - (f - g)(y)\| \\ &\geq \left\{ \left[\mathrm{Lip}\ \left(f^{-1}\right)\right]^{-1} - \mathrm{Lip}(g - f) \right\} d(x, y).\end{aligned}$$

□

Theorem 3.1.5: (Lipschitz inverse function theorem) *Let E and F be two Banach spaces, U an open set in E, V an open set in F, and $f: U \to V$ a homeomorphism with a Lipschitz inverse. Let, moreover, $h: U \to F$ be a Lipschitz map such that $\mathrm{Lip}(h)\,\mathrm{Lip}(f^{-1}) < 1$. Then $g = h + f$ is a homeomorphism, with a Lipschitz inverse, from U onto an open set of F. Moreover, if, in addition, $E = F$, the map f is a linear automorphism of the Banach space E, and h is of class C^r (with $r \geq 1$), then g^{-1} is of class C^r too.*

Proof: By the previous lemma we already know that g is injective with Lipschitz inverse; we need to prove that g is open, which is equivalent to saying that $g \circ f^{-1} = \mathrm{id} + h \circ f^{-1}$ is open, since f is a homeomorphism. Notice that if $v = h \circ f^{-1}$ and $\lambda = \mathrm{Lip}(v)$ we have

$$\lambda \leq \mathrm{Lip}(h)\,\mathrm{Lip}(f^{-1}) < 1.$$

Now let x be in V, and $r > 0$ such that $\overline{B_r}(x) \subseteq V$, where $\overline{B_r}(x)$ is the closed ball of center x and radius r. We claim that

$$(\mathrm{id} + v)\bigl(\overline{B_r}(x)\bigr) \supseteq \overline{B_{(1-\lambda)r}}\bigl((\mathrm{id} + v)(x)\bigr). \tag{3.1.3}$$

To prove this we can clearly assume $x = v(x) = O$. It suffices then to find a map $w: \overline{B_s}(O) \to F$ such that

$$(\mathrm{id} + w)\bigl(\overline{B_s}(O)\bigr) \subseteq \overline{B_r}(O) \qquad \text{and} \qquad (\mathrm{id} + v) \circ (\mathrm{id} + w) = \mathrm{id},$$

where $s = (1 - \lambda)r$. Notice that we must have $w = -v \circ (\mathrm{id} + w)$; so we try to get w as fixed point of a suitable operator.

Let

$$\mathcal{Z} := \left\{ w \in C^0\bigl(\overline{B_s}(O), F\bigr) \mid w(O) = O \text{ and } \mathrm{Lip}(w) \leq \frac{\lambda}{1 - \lambda} \right\},$$

which is complete with the sup-norm. Moreover it is easy to see that if $w \in \mathcal{Z}$ then $(\mathrm{id} + w)\bigl(\overline{B_s}(O)\bigr) \subseteq \overline{B_r}(O)$ and $-v \circ (\mathrm{id} + w) \in \mathcal{Z}$. So the operator $\Phi(w) := -v \circ (\mathrm{id} + w)$ sends $\mathcal{Z}$ into itself; furthermore

$$\|\Phi(w) - \Phi(w')\| = \| - v \circ (\mathrm{id} + w) + v \circ (\mathrm{id} + w')\| \leq \lambda\|w - w'\|,$$

and thus Φ is a contraction of $\mathcal{Z}$. So it has a unique fixed point in $\mathcal{Z}$, (3.1.3) is proved, and g is open.

The last assertion is well known. □

Corollary 3.1.6: *Let $U \subseteq E$ be an open subset of a Banach space E, and g a homeomorphism from U onto an open set of a Banach space F. If g^{-1} is Lipschitz and $\lambda \geq \mathrm{Lip}(g^{-1})$ then*

$$g\bigl(\overline{B_r}(x)\bigr) \supseteq \overline{B_{r/\lambda}}\bigl(g(x)\bigr)$$

for all $x \in U$ and $r > 0$ such that $\overline{B_r}(x) \subseteq U$.

Proof: We can again suppose $x = g(x) = 0$. Let $v \neq O$ be a point of the closed ball $\overline{B_{r/\lambda}}(O)$, and define

$$t_\infty := \sup\left\{t \geq 0 \mid [0,t]v \subset g\bigl(\overline{B_r}(O)\bigr)\right\}.$$

Since $g\bigl(\overline{B_r}(O)\bigr)$ contains a neighbourhood of the origin, we have that $t_\infty > 0$; moreover $[0, t_\infty)v \subset g\bigl(\overline{B_r}(O)\bigr)$. The continuity of g^{-1} then ensures that $\lim\limits_{t \to t_\infty} g^{-1}(tv) \in \overline{B_r}(O)$; and thus $t_\infty v \in g\bigl(\overline{B_r}(O)\bigr)$.

Now, to prove the assertion we have only to show that $t_\infty \geq 1$. To do this let us suppose $t_\infty < 1$; then

$$\|g^{-1}(t_\infty v)\| \leq \lambda\, t_\infty \|v\| < r,$$

and hence $t_\infty v \in g\bigl(B_r(O)\bigr)$. Thus it is possible to find $\varepsilon > 0$ such that $[t_\infty, t_\infty + \varepsilon)v$ is still contained in $g\bigl(B_r(O)\bigr)$, which contradicts the maximality of t_∞. □

We are finally ready to prove the

Theorem 3.1.7: (Stable Manifold Theorem for Banach spaces) *Let $E = E_1 \oplus E_2$ be a splitting of a Banach space E, and $T: E \to E$ be a λ-hyperbolic linear automorphism of E. Then there is an $\varepsilon > 0$, depending only on λ, such that for all $r > 0$ there exists a $\delta > 0$ so that if $f: E_1(r) \times E_2(r) \to E$ satisfies $\mathrm{Lip}(f - T) < \varepsilon$ and $\|f(O)\| < \delta$, then $W^s(f,r)$ is the graph of a Lipschitz function $g: E_1(r) \to E_2(r)$ with $\mathrm{Lip}(g) \leq 1$. Moreover, if f is of class C^k (with $k \geq 1$) then so is g.*

Proof: Observe preliminarily that for $x = (x_1, x_2) \in W^s(f,r)$, the sequence $\gamma_k = f^{k+1}(x)$ satisfies the following conditions:

(i) $\|\gamma_k\| \leq r$ for all $k \geq 0$;
(ii) $f(\gamma_k) - \gamma_{k+1} = O$ for all $k \geq 0$;
(iii) $f(x) - \gamma_0 = O$.

Conversely, if $x \in E(r)$ and $\{\gamma_k\}_{k \geq 0}$ satisfy the previous conditions, then $x \in W^s(f,r)$ and $\gamma_k = f^{k+1}(x)$.

Now we develop this idea. Let us consider the Banach space

$$\ell^\infty(E) = \left\{\gamma \in E^{\mathbb{N}} \mid \sup_{k \in \mathbb{N}} \|\gamma_k\| < \infty\right\},$$

endowed with the norm

$$\|\gamma\|_{\ell^\infty(E)} = \sup_{k \in \mathbb{N}} \|\gamma_k\|.$$

Let $\mathbf{B}(r)$ be the closed ball of radius r and center the origin in $\ell^\infty(E)$. Endow $E \times \ell^\infty(E)$ and $E_1 \times \ell^\infty(E)$ with the box norms induced by E, E_1, and $\ell^\infty(E)$.

Now let $\mathcal{F}: E_1(r) \times E_2(r) \times \mathbf{B}(r) \to E_1 \times \ell^\infty(E)$ be the map defined by

$$\mathcal{F}(x_1, x_2, \gamma) = \bigl(x_1, \mathcal{F}_{x_1}(x_2, \gamma)\bigr),$$

where $\mathcal{F}_{x_1}: E_2(r) \times \mathbf{B}(r) \to \ell^\infty(E)$ is given by

$$\bigl(\mathcal{F}_{x_1}(x_2,\gamma)\bigr)_k = \begin{cases} f(x_1, x_2) - \gamma_0, & \text{if } k = 0, \\ f(\gamma_{k-1}) - \gamma_k & \text{if } k \geq 1, \end{cases}$$

so that $\mathcal{F}_{x_1}(x_2, \gamma) = O$ if and only if $\gamma_k = f^{k+1}(x_1, x_2)$ for all $k \in \mathbb{N}$. If we show that $\mathcal{F}$ is invertible and that its image contains $E_1(r) \times \{O\}$, it would follow that for every $x_1 \in E_1(r)$ there exists a unique $x_2 \in E_2(r)$

such that $(x_1, x_2) \in W^s(f, r)$, and hence we can define g as $\pi_2 \circ \mathcal{F}^{-1}|_{E_1(r)\times\{O\}}$, where π_2 is the projection of $E_1 \times E_2 \times \ell^\infty(E)$ onto E_2. We shall of course use the Lipschitz inverse function theorem to do this.

Define $\mathcal{T}\colon E_1 \times E_2 \times \ell^\infty(E) \to E_1 \times \ell^\infty(E)$ as we did for $\mathcal{F}$, replacing f by T, i.e., $\mathcal{T}(x_1, x_2, \gamma) = (x_1, \nu)$ where

$$\nu_k = \mathcal{T}_{x_1}(x_2, \gamma)_k = \begin{cases} T(x_1, x_2) - \gamma_0 & \text{if } k = 0, \\ T(\gamma_{k-1}) - \gamma_k & \text{for } k \geq 1. \end{cases}$$

It is clear that $\mathcal{T}$ is a linear operator, and it is easy to see that $\|\mathcal{T}\| \leq 1 + \|T\|$, so $\mathcal{T}$ is also continuous. Moreover $\mathrm{Lip}(\mathcal{F} - \mathcal{T}) \leq \mathrm{Lip}(f - T)$. Consequently if $\mathcal{T}$ is invertible and $\mathrm{Lip}(f - T) \leq \|\mathcal{T}^{-1}\|^{-1}$, using the Lipschitz inverse function theorem we obtain the invertibility of $\mathcal{F}$.

Let us prove that $\mathcal{T}$ is invertible and compute its inverse. We need to express x_2 and γ in terms of x_1 and ν. Writing $\gamma_k = (\gamma_{k,1}, \gamma_{k,2}) \in E_1 \times E_2$ and $\nu_k = (\nu_{k,1}, \nu_{k,2}) \in E_1 \times E_2$, the definition of ν becomes:

$$\begin{cases} \nu_{0,1} = T_1(x_1) - \gamma_{0,1}, \\ \nu_{0,2} = T_2(x_2) - \gamma_{0,2}, \\ \nu_{k,1} = T_1(\gamma_{k-1,1}) - \gamma_{k,1} & \text{for } k \geq 1, \\ \nu_{k,2} = T_2(\gamma_{k-1,2}) - \gamma_{k,2} & \text{for } k \geq 1. \end{cases}$$

From the first and the third equations we obtain an expression for $\gamma_{k,1}$:

$$\gamma_{k,1} = T_1^{k+1}(x_1) - \sum_{j=0}^{k} T_1^{k-j}(\nu_{j,1}).$$

The fourth equation for $k \geq 0$ gives:

$$\gamma_{k,2} = T_2^{-1}(\nu_{k+1,2} + \gamma_{k+1,2}) = T_2^{-1}\big(\nu_{k+1,2} + T_2^{-1}(\nu_{k+2,2} + \gamma_{k+2,2})\big) = \cdots$$

so in the limit

$$\gamma_{k,2} = \sum_{j=1}^{\infty} T_2^{-j}(\nu_{k+j,2}).$$

Finally from the second equation

$$x_2 = \sum_{j=0}^{\infty} T_2^{-(j+1)}(\nu_{j,2}).$$

Obviously these series converge; thus we have proved that $\mathcal{T}$ is invertible, and it can be easily checked that $\|\mathcal{T}^{-1}\| \leq (1 - \lambda)^{-1}$.

Choose $\varepsilon \leq 1 - \lambda$. The Lipschitz inverse function theorem then implies that if $\mathrm{Lip}(f - T) < \varepsilon$ then $\mathcal{F}$ is invertible, because

$$\mathrm{Lip}(\mathcal{F} - \mathcal{T}) \leq \mathrm{Lip}(f - T) < \varepsilon \leq \|\mathcal{T}^{-1}\|^{-1}.$$

Now we prove that the image of $\mathcal{F}$ contains $E_1(r) \times \{O\}$ when ε and $f(O)$ are small enough. Now, the image of $\mathcal{F}$ contains (x_1, O) if and only if the image of $\mathcal{F}_{x_1}$ contains O. Furthermore $\mathcal{F}_{x_1}$ is a Lipschitz perturbation of $\mathcal{T}_{x_1}$ and $\mathrm{Lip}(\mathcal{F}_{x_1} - \mathcal{T}_{x_1}) \leq \mathrm{Lip}(f - T)$. Since $\mathcal{T}_{x_1}$ differs from $\mathcal{T}_O$ only by an additive constant, we have that $\mathrm{Lip}(\mathcal{F}_{x_1} - \mathcal{T}_O) \leq \mathrm{Lip}(f - T)$.

The above computations show that $\mathcal{T}_O$ is invertible, with inverse $\mathcal{T}_O^{-1}(\nu) = (x_2, \gamma)$ given by

$$x_2 = \sum_{j=0}^{\infty} T_2^{-(j+1)}(\nu_{j,2}),$$

$$\gamma_{k,1} = -\sum_{j=0}^{k} T_1^{k-j}(\nu_{j,1}),$$

$$\gamma_{k,2} = \sum_{j=1}^{\infty} T_2^{-j}(\nu_{k+j,2}),$$

so that $\|\mathcal{T}_O^{-1}\| \le (1-\lambda)^{-1}$.

If $\mathrm{Lip}(f-T) < \varepsilon \le 1-\lambda$, we get $\mathrm{Lip}(\mathcal{T}_O^{-1}\mathcal{F}_{x_1} - \mathrm{id}) \le \varepsilon/(1-\lambda) < 1$ and using Lemma 3.1.4 we obtain

$$\mathrm{Lip}\left[\left(\mathcal{T}_O^{-1}\mathcal{F}_{x_1}\right)^{-1}\right] \le \frac{1}{1-\frac{\varepsilon}{1-\lambda}};$$

then Corollary 3.1.6 yields

$$\mathcal{T}_O^{-1}\mathcal{F}_{x_1}\left[E_2(r)\times \mathbf{B}(r)\right] \supseteq \mathcal{T}_O^{-1}\mathcal{F}_{x_1}(O,O) + E_2(s)\times \mathbf{B}(s),$$

where $s = r(1-\varepsilon/(1-\lambda))$.

Now we compute $\|\mathcal{T}_O^{-1}\mathcal{F}_{x_1}(O,O)\|$. First of all we have $\mathcal{F}_{x_1}(O,O) = \nu \in \ell^\infty(E)$, with $\nu_0 = f(x_1,O)$ and $\nu_k = f(O,O)$ for $k \ge 1$. Consequently, $\mathcal{T}_O^{-1}\mathcal{F}_{x_1}(O,O) = (x_2,\gamma)$, where

$$\begin{aligned} x_2 &= T_2^{-1}\big(f_2(x_1,O)\big) + \sum_{j=2}^{\infty} T_2^{-j}\big(f_2(O,O)\big),\\ \gamma_{k,1} &= -T_1^{k}\big(f_1(x_1,O)\big) - \sum_{j=1}^{k} T_1^{k-j}\big(f_1(O,O)\big),\\ \gamma_{k,2} &= \sum_{j=1}^{\infty} T_2^{-j}\big(f_2(O,O)\big). \end{aligned}$$

Now we have

$$\begin{aligned} \|f_1(x_1,O)\| &\le \|p_1(f-T)(x_1,O)\| + \|T_1(x_1)\|\\ &\le \|(f-T)(x_1,O)\| + \lambda\|x_1\|,\\ \|f_2(x_1,O)\| &\le \|p_2(f-T)(x_1,O)\| \le \|(f-T)(x_1,O)\|, \end{aligned}$$

and further

$$\begin{aligned} \|(f-T)(x_1,O)\| &\le \|(f-T)(O)\| + \|(f-T)(x_1,O) - (f-T)(O)\|\\ &\le \|f(O)\| + \varepsilon\|x_1\|. \end{aligned}$$

Putting all together we get

$$\|\mathcal{T}_O^{-1}\mathcal{F}_{x_1}(O)\| \le (\lambda+\varepsilon)r + \frac{1}{1-\lambda}\|f(O)\|.$$

Therefore the image of $\mathcal{T}_O^{-1}\mathcal{F}_{x_1}$ contains the ball centered at $\mathcal{T}_O^{-1}\mathcal{F}_{x_1}(O)$ of radius $s = r(1-\varepsilon/(1-\lambda))$; hence it will contain O whenever $(\lambda+\varepsilon)r + \frac{1}{1-\lambda}\|f(O)\| < s$. We can rewrite this inequality as

$$\|f(O)\| < (1-\lambda)r\left(1-\lambda-\varepsilon-\frac{\varepsilon}{1-\lambda}\right) = r[(1-\lambda)^2 - \varepsilon(2-\lambda)].$$

Suppose then $\varepsilon < (1-\lambda)^2/(2-\lambda) < 1-\lambda$ and take $\delta > 0$ so that $\delta < r[(1-\lambda)^2 - \varepsilon(2-\lambda)]$; if $\mathrm{Lip}(f-T) < \varepsilon$ and $\|f(O)\| < \delta$, we have shown that $\mathcal{T}_O^{-1}\mathcal{F}_{x_1}$ contains O. Since $\mathcal{T}_O$ is linear, the image of $\mathcal{F}_{x_1}$ contains O as well, and so the image of $\mathcal{F}$ contains $E_1(r)\times\{O\}$, as desired.

In conclusion we have proved the following:

If $\mathrm{Lip}(f-T) < \varepsilon < (1-\lambda)^2/(2-\lambda)$ *and* $\|f(O)\| < \delta < r[(1-\lambda)^2 - \varepsilon(2-\lambda)]$, *then* $W^s(f,r)$ *is the graph of a Lipschitz map* $g = \pi_2 \circ \mathcal{F}^{-1}|_{E_1(r)\times\{O\}} \colon E_1(r) \to E_2(r)$ *with* $\mathrm{Lip}(g) \le 1$.

To finish the proof of the stable manifold theorem we only need to show that g is C^k whenever f is C^k.

By definition we have that g is C^k if $\mathcal{F}^{-1}$ is so. We saw that $\mathrm{Lip}(\mathcal{F}-\mathcal{T}) < \|\mathcal{T}^{-1}\|^{-1}$; therefore, by the Lipschitz inverse function theorem, $\mathcal{F}^{-1}$ is C^k whenever $\mathcal{F}$ is.

We are reduced then to showing that $\mathcal{F}$ is C^k when f is C^k. Unfortunately, $\ell^\infty(E)$ is too large for this to be possible, but we can easily bypass this difficulty.

The obvious candidate for the derivative of $\mathcal{F}$ at the point

$$(x,\gamma)=(x_1,x_2,\gamma)\in E_1(r)\times E_2(r)\times \mathbf{B}(r)$$

is the linear map $\mathcal{L}\colon E_1\times E_2\times \ell^\infty(E)\to E_1\times\ell^\infty(E)$ given by $\mathcal{L}(y_1,y_2,\nu)=(y_1,\zeta)$, where

$$\zeta_k=\begin{cases} df_x(y)-\nu_0 & \text{for } k=0,\\ df_{\gamma_{k-1}}(\nu_{k-1})-\nu_k & \text{for } k\ge 1.\end{cases}$$

We have $\mathcal{F}(x+y,\gamma+\nu)-\mathcal{F}(x,\gamma)-\mathcal{L}(y,\nu)=(O,\varphi)$, where

$$\begin{cases} \varphi_0=f(x+y)-f(x)-df_x(y)=\int_0^1\left(df_{x+ty}-df_x\right)(y)\,dt,\\ \varphi_k=f(\gamma_{k-1}+\nu_{k-1})-f(\gamma_{k-1})-df_{\gamma_{k-1}}(\nu_{k-1})\\ \quad\ =\int_0^1\left(df_{\gamma_{k-1}+t\nu_{k-1}}-df_{\gamma_{k-1}}\right)(\nu_{k-1})\,dt, & \text{for } k\ge 1.\end{cases}$$

This shows that the quotient

$$\frac{\|\mathcal{F}(x+y,\gamma+\nu)-\mathcal{F}(x,\gamma)-\mathcal{L}(y,\nu)\|}{\|(y,\nu)\|}$$

is bounded from above by

$$\max\left(\int_0^1\|df_{x+ty}-df_x\|\,dt,\ \sup_{k\ge 1}\int_0^1\left\|df_{\gamma_{k-1}+t\nu_{k-1}}-df_{\gamma_{k-1}}\right\|\,dt\right).$$

It is clear that the supremum of all these integrals does not necessarily tend to zero with $\|(y,\nu)\|$, since $\ell^\infty(E)$ is not locally compact and df is not necessarily uniformly continuous. If we knew that $\{\gamma_k\}$ were contained in a compact subset of $E(r)$, for instance if γ_k were convergent, then the supremum would go to zero with $\|(y,\nu)\|$. This leads us to consider the subspace $\mathbf{C}$ of convergent sequences:

$$\mathbf{C}=\left\{\gamma\in\ell^\infty(E)\mid \lim_{k\to\infty}\gamma_k \text{ exists}\right\}.$$

Since E is complete, $\mathbf{C}$ is closed in $\ell^\infty(E)$, and thus $\mathbf{C}$ itself is a Banach space.

Notice that, obviously, $\mathcal{F}\big(E_1(r)\times E_2(r)\times\mathbf{C}(r)\big)\subseteq E_1(r)\times\mathbf{C}$; so it make sense to consider the restriction $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $E_1(r)\times E_2(r)\times\mathbf{C}(r)$ and to define $\tilde{\mathcal{T}}$ in the same way as the mapping induced by $\mathcal{T}$. The previous argument then shows that $\tilde{\mathcal{F}}$ is C^1, and by induction it can be proved that $\tilde{\mathcal{F}}$ is C^r when f is C^r.

Now, we can repeat our previous construction of g with $\tilde{\mathcal{F}}$, $\tilde{\mathcal{T}}$ and $\mathbf{C}$ in place of $\mathcal{F}$, $\mathcal{T}$ and $\ell^\infty(E)$, provided that we know $\tilde{\mathcal{T}}$ is invertible, i.e., that $\mathcal{T}^{-1}(E_1\times\mathbf{C})\subset E_1\times E_2\times\mathbf{C}$. We proceed to check this.

Let (x,ν) be in $E_1\times\mathbf{C}$ and recall that $\mathcal{T}^{-1}(x,\nu)=(x_1,x_2,\gamma)$ is given by

$$x_2=\sum_{j=0}^{\infty}T_2^{-(j+1)}(\nu_{j,2}),$$

$$\gamma_{k,1}=T_1^{k+1}(x_1)-\sum_{j=0}^{k}T_1^{k-j}(\nu_{j,1}),$$

$$\gamma_{k,2}=\sum_{j=1}^{\infty}T_2^{-j}(\nu_{k+j,2}).$$

To show that γ belongs to $\mathbf{C}$ we shall prove that γ_1 and γ_2 are Cauchy. Let us begin with γ_1:

$$\|\gamma_{h,1}-\gamma_{k,1}\|\le\left\|\sum_{j=0}^{h}T_1^{h-j}(\nu_{j,1})-\sum_{j=0}^{k}T_1^{k-j}(\nu_{j,1})\right\|+\|T_1^{h+1}(x_1)-T_1^{k+1}(x_1)\|.$$

Since $\|T_1\| < \lambda < 1$, the term $\|T_1^h(x_1) - T_1^k(x_1)\|$ goes to zero as h and k go to ∞.

For the other term, setting $N = \min\{h, k\}$, we have

$$\left\| \sum_{j=0}^{h} T_1^{h-j}(\nu_{j,1}) - \sum_{j=0}^{k} T_1^{k-j}(\nu_{j,1}) \right\| = \left\| \sum_{j=0}^{h} T_1^{j}(\nu_{h-j,1}) - \sum_{j=0}^{k} T_1^{j}(\nu_{k-j,1}) \right\|$$
$$\leq \left(\sum_{j=0}^{\infty} \lambda^j \right) \sup_{\substack{r\geq h-N \\ s\geq k-N}} \|\nu_{r,1} - \nu_{s,1}\| + \left(\sum_{k=N}^{\infty} \lambda^k \right) \|\nu_1\|.$$

Now, for any $\varepsilon > 0$ we can find an N_0 large enough such that

$$\sup_{r,s\geq N_0} \|\nu_{r,1} - \nu_{s,1}\| \leq \frac{1-\lambda}{2}\varepsilon \quad \text{and} \quad \sum_{k=N_0}^{\infty} \lambda^k < \frac{\varepsilon}{2\|\nu_1\|},$$

which tells us that for h, $k \geq 2N_0$ we have

$$\left\| \sum_{j=0}^{h} T_1^{h-j}(\nu_{j,1}) - \sum_{j=0}^{k} T_1^{k-j}(\nu_{j,1}) \right\| \leq \left(\sum_{k=0}^{\infty} \lambda^k \right) \frac{1-\lambda}{2}\varepsilon + \frac{\varepsilon\|\nu_1\|}{2\|\nu_1\|} = \varepsilon,$$

and thus γ_1 is a Cauchy sequence.

The same is true for γ_2. In fact we have

$$\|\gamma_{h,2} - \gamma_{k,2}\| \leq \left(\sum_{j=1}^{\infty} \lambda^j \right) \sup_{j\geq 1} \|\nu_{h+j,2} - \nu_{k+j,2}\|;$$

thus, since the sequence ν_2 is Cauchy, so is γ_2.

Therefore g can be defined as $g = \tilde{\pi}_2 \circ \tilde{\mathcal{F}}^{-1}|_{E_1(r)\times\{O\}}$ where $\tilde{\pi}_2$ is the projection of $E_1 \times E_2 \times \mathbf{C}$ onto E_2. Since we have shown above that such a g is as smooth as $\tilde{\mathcal{F}}$, which in turn is as smooth as f, we are done. □

Corollary 3.1.8: *Let $E = E_1 \oplus E_2$ be a splitting of a Banach space E. Then:*

(i) *Let $f: E_1(r)\times E_2(r) \to E$ be a C^1 map fixing the origin and such that df_O is a λ-hyperbolic automorphism of E. Assume that* $\mathrm{Lip}(f - df_O) < \varepsilon$, *where $\varepsilon = \varepsilon(\lambda)$ is given by the previous theorem; this can be achieved by choosing r small enough. Then the Lipschitz map $g: E_1(r) \to E_2(r)$ given by the previous theorem satisfies $g(O) = O$ and $Dg_O = O$. Consequently, $W^s(f, r)$ is tangent to E_1 at O.*

(ii) *Given a λ-hyperbolic automorphism T of E, let*

$$N^k_{\varepsilon,\delta} := \big\{ f: E_1(r) \times E_2(r) \to E \mid \mathrm{Lip}(f - T) < \varepsilon, \|f(O)\| < \delta,\ f \text{ is } C^k$$
$$\text{and } d^k f \text{ is bounded and uniformly continuous in the } C^k \text{ topology} \big\},$$

where ε and δ are given by Theorem 3.1.7. Then the map $\Xi: N^k_{\varepsilon,\delta} \to C^k\big(E_1(r), E_2(r)\big)$ sending f to g is continuous.

Proof: (i) Since $f(O) = O$, it is obvious that $O \in W^s(f, r)$ and that $g(O) = O$. Moreover,

$$dg_O = \tilde{\pi}_2 \circ d(\tilde{\mathcal{F}}^{-1}|_{E_1(r)\times\{0\}})_O,$$

and $d(\tilde{\mathcal{F}}^{-1})_O = d\tilde{\mathcal{F}}_O^{-1} = \tilde{\mathcal{T}}^{-1}$, where we are setting $T = df_O$. So for $v_1 \in E_1$ we have

$$dg_O(v_1) = \tilde{\pi}_2 \circ \tilde{\mathcal{T}}^{-1}(v_1, O) = \sum_{j=1}^{\infty} T_2^{-j}(O) = O.$$

Thus the tangent space to $W^s(f, r)$ at the point O is nothing else but E_1.

(ii) It is a consequence of the following two facts, which follows easily arguing as at the end of the previous proof:

(a) If $f \in N^k_{\varepsilon,\delta}$ then $d^k\tilde{\mathcal{F}}$ is uniformly continuous, bounded, and the map $f \longmapsto \tilde{\mathcal{F}}$ is continuous in the C^k topology.

(b) The map $\tilde{\mathcal{F}} \longmapsto \tilde{\mathcal{F}}^{-1}$ is continuous in the C^k topology on the set of $\tilde{\mathcal{F}}$ whose k-th derivative is uniformly continuous and bounded. □

We are finally ready for the

Proof of Theorem 3.1.1: Let $E = T_pM$, and set $E_1 = E^s(df_p)$ and $E_2 = E^u(df_p)$. Since the statements (i)-(v) are local and (vi) is an obvious consequence of the previous ones, we can identify a neighbourhood of p in M with a neighbourhood of the origin in E, and replace $f\colon M \to M$ by a map, still denoted by the same letter, $f\colon E_1(r) \times E_2(r) \to E$, with $r > 0$ small enough. Furthermore, shrinking r if necessary, we can assume that $\mathrm{Lip}(f - df_p) < \varepsilon$, where $\varepsilon > 0$ is given by Theorem 3.1.7. Then Theorem 3.1.1 is a consequence of Theorem 3.1.7, Proposition 3.1.3, Corollary 3.1.8 and Theorem 1.2.1. □

3.2 Hyperbolic sets

In this section we collect some of the most important properties of hyperbolic sets.

Definition 3.2.1: Let M be a smooth manifold, and $f\colon M \to M$ a C^1 diffeomorphism. A compact completely f-invariant set $\Lambda \subseteq M$ is a *hyperbolic set* for the map f (and the dynamical system $(\Lambda, f|_\Lambda)$ is a *hyperbolic dynamical system*) if for any Riemannian metric on M there exist $0 < \lambda < 1 < \mu$ and numbers $0 < c < C$ such that for any $x \in \Lambda$ there is a splitting $T_xM = E^s_x \oplus E^u_x$ so that

(i) $df_x(E^s_x) = E^s_{f(x)}$ and $df_x(E^u_x) = E^u_{f(x)}$ for every $x \in \Lambda$;
(ii) $\|df^k_x(v)\| \le c\lambda^k \|v\|$ for every $x \in \Lambda$, $k \in \mathbb{N}$ and $v \in E^s_x$;
(iii) $\|df^k_x(v)\| \ge C\mu^k \|v\|$ for every $x \in \Lambda$, $k \in \mathbb{N}$ and $v \in E^u_x$.

Definition 3.2.2: An *Anosov diffeomorphism* is a C^1 diffeomorphism f of a compact manifold M such that M is a hyperbolic set for f.

Clearly a hyperbolic periodic orbit is an example of hyperbolic set. On the other hand, the hyperbolic automorphism F_L of the 2-torus discussed in Section 1.5 is an Anosov diffeomorphism: the invariant splitting is obtained by translating to each point the eigenspaces of the matrix L.

Another example of hyperbolic set is the set Λ inside the horseshoe (see Section 2.3); in this case the splitting is given by the vertical and horizontal directions.

Proposition 3.2.1: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$. Then the subspaces $E^{s/u}_x$ depend continuously on x and have locally constant dimensions.*

Corollary 3.2.2: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$. Then the subspaces E^s_x and E^u_x are uniformly transverse, that is there is $\alpha_0 > 0$ such that for any $x \in \Lambda$, $u \in E^s_x$ and $v \in E^u_x$ the angle between u and v is at least α_0.*

Possibly the main feature of hyperbolic sets is the following stable manifold theorem:

Theorem 3.2.3: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^r diffeomorphism $f\colon M \to M$ of a smooth manifold M, with $r \ge 1$. Then there is $\delta > 0$ such that for each $x \in \Lambda$ we have:*

(i) *the local stable and unstable sets, given by $W^s(x,\delta) = W^s(x) \cap B(x,\delta)$ and $W^u(x,\delta) = W^u(x) \cap B(x,\delta)$, are embedded C^r disks intersecting only at x and such that $T_xW^{s/u}(x) = E^{s/u}_x$;*
(ii) *$f\big(W^s(x,\delta)\big) \subseteq W^s\big(f(x),\delta\big)$ and $f^{-1}\big(W^u(x,\delta)\big) \subseteq W^u\big(f^{-1}(x),\delta\big)$;*
(iii) *for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$d\big(f^k(x), f^k(y)\big) \le C_\varepsilon(\lambda+\varepsilon)^k\, d(x,y) \qquad \text{for all } y \in W^s(x,\delta),$$
$$d\big(f^{-k}(x), f^{-k}(y)\big) \le C_\varepsilon(\mu-\varepsilon)^{-k}\, d(x,y) \qquad \text{for all } y \in W^u(x,\delta),$$

for all $k \in \mathbb{N}$, where $0 < \lambda < 1 < \mu$ are the costants in Definition 3.2.1;
(iv) *a point $y \in M$ belongs to $W^s(x,\delta)$ if and only if $d\big(f^k(x), f^k(y)\big) \le \delta$ for all $k \in \mathbb{N}$, and to $W^u(x,\delta)$ if and only if $d\big(f^{-k}(x), f^{-k}(y)\big) \le \delta$ for all $k \in \mathbb{N}$;*
(v) *the global stable and unstable sets are given by $W^s(x) = \bigcup_{k\in\mathbb{N}} f^{-k}\big(W^s(f^k(x),\delta)\big)$ and $W^u(x) = \bigcup_{k\in\mathbb{N}} f^k\big(W^u(f^{-k}(x),\delta)\big)$, and thus are C^r immersed smooth manifolds.*

We explicitely remark that, locally, stable and unstable manifolds intersect in at most one point:

Proposition 3.2.4: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then:*

(i) *if $\delta > 0$ is small enough then for any $x, y \in \Lambda$ the intersection $W^s(x,\delta) \cap W^u(y,\delta)$ consists of at most one point;*

(ii) *there is $\varepsilon > 0$ so that if $x, y \in \Lambda$ are such that $d(x,y) < \varepsilon$ then $W^s(x,\delta) \cap W^u(y,\delta) \neq \varnothing$.*

In particular, for $\delta > 0$ small enough we have a map

$$(x,y) \mapsto [x,y]_\delta = W^s(x,\delta) \cap W^u(y,\delta)$$

defined on all pairs such that $d(x,y) < \delta$; it can be proven that this map is continuous.

Definition 3.2.3: We say that a hyperbolic set Λ has a *local product structure* if there are δ, $\varepsilon > 0$ such that $[x,y]_\delta \in \Lambda$ for all $x, y \in \Lambda$ with $d(x,y) < \varepsilon$.

Definition 3.2.4: Let (X,f) be a dynamical system on a metric space X, and $\varepsilon > 0$. An *ε-pseudo orbit* is a sequence $\{x_k\}_{k\in\mathbb{Z}} \subset X$ such that $d\big(f(x_k), x_{k+1}\big) < \varepsilon$ for all $k \in \mathbb{Z}$. A *segment* of ε-pseudo orbit is a finite sequence $\{x_0,\ldots,x_m\} \subset X$ such that $d\big(f(x_{k-1}), x_k\big) < \varepsilon$ for all $k = 0,\ldots,m-1$. If $\{x_0,\ldots,x_m\}$ is a segment of ε-pseudo orbit such that $x_m = x_0$, we say that $\{x_0,\ldots,x_m\}$ is a *periodic* ε-pseudo orbit, and that x_0 is *ε-pseudo periodic.*

Theorem 3.2.5: (Anosov closing lemma) *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then there exist an open neighbourhood $V \supseteq \Lambda$ and constants $C > 1$ and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any periodic ε-pseudo orbit $\{x_0,\ldots,x_m\} \subset V$ there is a periodic point $y \in M$ of period m and such that $d\big(f^k(y), x_k\big) < C\varepsilon$ for $k = 0,\ldots,m-1$.*

A particular instance of periodic ε-pseudo orbit is an orbit segment x_0, $f(x_0),\ldots,f^{m-1}(x_0)$ such that $d(f^m(x_0), x_0) < \varepsilon$. For instance, such a segment exists if x_0 is recurrent, and thus a consequence of the previous Theorem is that close to any recurrent point $x_0 \in \Lambda$ there is a periodic point. Unfortunately, in general this periodic point might not belong to Λ. A notable exception is the case of locally maximal hyperbolic sets, that we now discuss.

Clearly, every closed completely invariant subset of a hyperbolic set is still a hyperbolic set. On the other hand, a hyperbolic set might be a subset of a larger hyperbolic set:

Proposition 3.2.6: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$. Then there exists an open neighbourhood $V \supset \Lambda$ such that for any C^1 diffeomorphism $g: M \to M$ sufficiently C^1-close to f the completely g-invariant set $\Lambda_V^g = \bigcap_{m\in\mathbb{Z}} g^m(\overline{V})$ is hyperbolic for g, if not empty. In particular, $\Lambda_V^f \supseteq \Lambda$ is hyperbolic.*

Remark 3.2.1. We have not yet proved that Λ_V^g is not empty. This will be a consequence of Theorem 3.2.17.

Corollary 3.2.7: *The family of Anosov diffeomorphisms of a compact manifold M is open with respect to the C^1 topology.*

Definition 3.2.5: Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$. Then Λ is *locally maximal* (or *basic*) if there is an open neighbourhood V of Λ such that $\Lambda_V^f = \Lambda$.

For instance, the discussion in Section 2.3 implies that the hyperbolic set in the horseshoe is locally maximal; and it is not difficult to prove that a hyperbolic fixed point (and thus a hyperbolic periodic orbit) is locally maximal.

Theorem 3.2.8: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then Λ is locally maximal if and only if it has a local product structure.*

Proposition 3.2.9: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then the periodic points are dense in $NW(f|_\Lambda)$.*

Corollary 3.2.10: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M, and assume that f is topologically transitive on $NW(f|_\Lambda)$. Then f is chaotic on $NW(f|_\Lambda)$.*

Definition 3.2.6: A C^1-diffeomorphism $f: M \to M$ is *Axiom A* if $NW(f)$ is hyperbolic and periodic points are dense in it.

Remark 3.2.2. In general it is possible, even for locally maximal hyperbolic sets, to have $NW(f|_\Lambda) \neq \Lambda$; see the Exercise 3.2.1.

Remark 3.2.3. It is not known whether any hyperbolic set can be imbedded into a locally maximal one.

Exercise 3.2.1. Let $f\colon \tilde{X} \to \tilde{X}$ be the horseshoe map defined in Section 2.3, and let $\Lambda = NW(f) \setminus \{p_0\}$. Let Λ_0^N be the subset of Λ given by the points whose coding contains at least N 0's between any pair of 1's, and let Λ_0 be the subset of points whose coding contains at most a single 1. Prove that every Λ_0^N is a locally maximal hyperbolic set for f, and that Λ_0 is a hyperbolic set for f which is not locally maximal.

Exercise 3.2.2. Let $f\colon \tilde{X} \to \tilde{X}$ be the horseshoe map defined in Section 2.3, and let $\Lambda = NW(f) \setminus \{p_0\}$. Find a locally maximal hyperbolic subset of Λ where periodic points are not dense.

In general, f is not topologically transitive on $NW(f|_\Lambda)$, but we can decompose the nonwandering set in a finite disjoint union of completely invariant closed subsets where f is topologically transitive — and thus chaotic.

Lemma 3.2.11: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$ of a smooth manifold M. Then $NW(f|_\Lambda)$ is closed under the local product structure of Λ. In particular, $NW(f|_\Lambda)$ is a locally maximal hyperbolic set for f.*

And then:

Theorem 3.2.12: (Spectral decomposition theorem) *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$ of a manifold M. Then there is a decomposition $NW(f|_\Lambda) = P_1 \cup \cdots \cup P_r$ into disjoint closed sets such that:*

(i) *every P_j is a completely f-invariant, locally maximal hyperbolic set for f, and $f|_{P_j}$ is chaotic;*
(ii) *for every $j = 1, \ldots, r$ there is a decomposition $P_j = X_{1,j} \cup \cdots \cup X_{s_j,j}$ into disjoint closed subsets such that $f(X_{i,j}) = X_{i+1,j}$ for $i = 1, \ldots, s_j - 1$ and $f(X_{s_j,j}) = X_{1,j}$;*
(iii) *every $X_{i,j}$ is locally maximal for f^{s_j}, and f^{s_j} is topologically mixing on it;*
(iv) *the sets P_j and $X_{i,j}$ are unique up to indexing.*

Furthermore, every $X_{i,j}$ is of the form $\overline{W^u(p) \cap NW(f|_\Lambda)}$ for a suitable periodic point $p \in NW(f|_\Lambda)$.

Corollary 3.2.13: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$ of a smooth manifold M. Assume that $f|_\Lambda$ is topologically mixing; then periodic points are dense in Λ, and the unstable manifold of every periodic point is dense in Λ.*

Corollary 3.2.14: *Let $f\colon M \to M$ be an Anosov diffeomorphism of a connected smooth manifold M. Then f is chaotic on M if and only if $NW(f) = M$.*

Remark 3.2.4. It is not known whether $NW(f) = M$ for any Anosov diffeomorphism, though it is conjectured to be true.

The Anosov closing lemma shows that every periodic ε-pseudo orbit is closely followed by an actual periodic orbit. A striking feature of hyperbolic sets is that this is true even for non-periodic orbits.

Definition 3.2.7: Let (X, f) be a dynamical system on a metric space X. An ε-pseudo orbit $\{x_k\}_{k\in\mathbb{Z}}$ is *δ-shadowed* by the orbit of $x \in X$ if $d\big(x_k, f^k(x)\big) < \delta$ for all $k \in \mathbb{Z}$.

Theorem 3.2.15: (Shadowing Lemma) *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$ of a smooth manifold M. Then there is a neighbourhood V of Λ such that for every $\delta > 0$ there is $\varepsilon > 0$ so that every ε-pseudo orbit in V is δ-shadowed by an orbit of f. Furthermore, there is $\delta_0 > 0$ such that if $\delta < \delta_0$ then the orbit of f shadowing the given pseudo orbit is unique.*

The idea behind shadowing is that the orbits of a perturbation of a dynamical system are ε-pseudo orbits of the original system; since they are shadowed by actual orbits, this might give a way to conjugate the perturbated and the original systems. But for this to work we need some kind of continuous dependence of the shadowing orbits on the ε-pseudo orbit. This is the rationale behind the

Theorem 3.2.16: (Shadowing Theorem) *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then there exist a neighbourhood V of Λ and a neighbourhood W of f in $C^1(M, M)$ such that for all $\delta > 0$ there is $\varepsilon > 0$ so that: for all topological spaces X, homeomorphisms $g: X \to X$ and continuous maps $h_0: X \to V$, if $\tilde{f} \in W$ is a C^1 diffeomorphism so that $d_0(h_0 \circ g, \tilde{f} \circ h_0) < \varepsilon$, then there is a continuous $h_1: X \to V$ such that $h_1 \circ g = \tilde{f} \circ h_1$ and $d_0(h_0, h_1) < \delta$. Furthermore, h_1 is locally unique, in the sense that there is $\delta_0 > 0$ (depending only on Λ and f) such that if $h': X \to V$ is a continuous map satisfying $h' \circ g = \tilde{f} \circ h'$ and $d_0(h_1, h') < \delta_0$ then $h' = h_1$. Finally, h_1 depends continuously on $\tilde{f}$.*

To better understand this statement, let us first show how to prove Theorem 3.2.15 using Theorem 3.2.16.

Proof of Theorem 3.2.15: Take $X = \mathbb{Z}$, endowed with the discrete topology; $g: X \to X$ given by $g(k) = k + 1$; $h_0: X \to V$ given by $h_0(k) = x_k$; and $\tilde{f} = f$. Then Theorem 3.2.16 yields $h_1: X \to V$ such that $h_1 \circ g = f \circ h_1$ and $d_0(h_0, h_1) < \delta$, that translated means $h_1(k + 1) = f\big(h_1(k)\big)$ for all k — that is $h_1(k) = f^k(x)$, where $x = h_1(0)$ —, and $d\big(x_k, f^k(x)\big) < \delta$ for all $k \in \mathbb{Z}$, as requested. $\square$

It is interesting to notice that the Anosov closing lemma is a consequence of Theorem 3.2.16 too: it suffices to choose $X = \mathbb{Z}_m$, $g(k) = k + 1 \pmod{m}$, $h_0(k) = x_k$ and $\tilde{f} = f$.

The Shadowing Theorem is a wonderful tool for proving structural stability. In particular, we are able to prove the strong structural stability of hyperbolic sets:

Theorem 3.2.17: *Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then for every open neighbourhood V of Λ and every $\eta > 0$ there exists a neighbourhood W of f in $C^1(M, M)$ such that for all diffeomorphisms $\tilde{f} \in W$ there is a hyperbolic set $\tilde{\Lambda} \subset V$ for $\tilde{f}$ and a homeomorphism $h: \Lambda \to \tilde{\Lambda}$ with $h \circ f|_\Lambda = \tilde{f}|_{\tilde{\Lambda}} \circ h$ and $d_0(\mathrm{id}, h) + d_0(\mathrm{id}, h^{-1}) < \eta$. Furthermore, h is unique if δ is small enough.*

Proof: The proof consists in three applications of the Shadowing Theorem 3.2.16. First of all, we apply Theorem 3.2.16 taking $\delta < \min\{\delta_0/2, \eta/2\}$, $X = \Lambda$, $h_0 = \mathrm{id}_\Lambda$ and $g = f$ to get a neighbourhood $V_1 \subset V$ of Λ, a neighbourhood W_1 of f (such that $d_0(\tilde{f}, f) < \varepsilon$ for all $\tilde{f} \in W_1$) and a unique $h_1: \Lambda \to V_1$ such that $h_1 \circ f = \tilde{f} \circ h_1$ and $d_0(\mathrm{id}_\Lambda, h_1) < \delta$. In particular, $\tilde{\Lambda} = h_1(\Lambda)$ is completely $\tilde{f}$-invariant and hyperbolic (up to shrinking W_1 if necessary) by Proposition 3.2.6.

To prove that h_1 is injective, we apply Theorem 3.2.16 taking δ as before, $X = \tilde{\Lambda}$, $h_0 = \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f}$; we explicitely remark that from the proof of Theorem 3.2.16 we infer that we get the same neighbourhood W_1 as soon as ε is small enough. Then we have a unique $h_2: \tilde{\Lambda} \to V$ such that $h_2 \circ \tilde{f} = f \circ h_2$ and $d_0(\mathrm{id}_{\tilde{\Lambda}}, h_2) < \delta$.

To end the proof it suffices to show that $h_2 \circ h_1 = \mathrm{id}_\Lambda$. We apply again Theorem 3.2.16 with $X = \Lambda$, $h_0 = \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f} = f$. Since

$$d_0(\mathrm{id}_\Lambda, h_2 \circ h_1) \leq d_0(\mathrm{id}_\Lambda, h_1) + d_0(h_1, h_2 \circ h_1) = d_0(\mathrm{id}_\Lambda, h_1) + d_0(\mathrm{id}_{\tilde{\Lambda}}, h_2) < 2\delta < \delta_0,$$

we can apply the uniqueness statement in Theorem 3.2.16 to get $h_2 \circ h_1 = \mathrm{id}_\Lambda$, because they both commute with f and are close to h_1. $\square$

Corollary 3.2.18: *Anosov diffeomorphisms of a compact manifold M are strongly structurally stable.*

Proof: The idea is to apply the previous proof with $\Lambda = M$. We get a homeomorphism $h_1: M \to \tilde{\Lambda} \subseteq M$ close to the identity and such that $h_1 \circ f = \tilde{f} \circ h_1$. We then apply the second step of the previous proof with $\Lambda = M$ again to get another map $h_2: M \to M$ close to the identity and such that $h_2 \circ \tilde{f} = f \circ h_2$. But then the third step in the previous proof shows that $h_1 \circ h_2 = \mathrm{id}_M$, and thus h_1 is surjective, proving that $\tilde{\Lambda} = M$ and that h is a homeomorphism of M, as required. $\square$

We can now give give an explicit model (up to semiconjugacy) for the dynamics on locally maximal hyperbolic sets:

Theorem 3.2.19: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f: M \to M$ of a smooth manifold M. Then $f|_\Lambda$ is semiconjugated a topological Markov chain σ_A. Furthermore, $f|_\Lambda$ is topologically conjugated to a topological Markov chain if and only if Λ is totally disconnected.*

The Spectral Decomposition Theorem (Theorem 3.2.12) says that given a smooth manifold M and a locally maximal hyperbolic set $\Lambda \subseteq M$ for a C^1 diffeomorphism $f: M \to M$, we can find a finite family of disjoint sets $X_i \subseteq NW(f|_\Lambda)$, $i = 1, \ldots, r$ with the following properties:

(i) the sets X_i are closed and open in $NW(f|_\Lambda)$;
(ii) for every $i = 1, \dots, r$ there exists a $k_i \geq 0$ such that X_i is a completely invariant topologically mixing locally maximal hyperbolic set for f^{k_i}.

We shall now see why the topological mixing condition on the partition of $NW(f|_\Lambda)$ is crucial.

If the diffeomorphism f is topologically mixing on Λ, Corollary 3.2.13 implies that periodic points are dense in Λ, and that the unstable manifold of every periodic point is dense in Λ. Actually, it turns out that all unstable manifolds are dense. More precisely:

Proposition 3.2.20: *Let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for a C^1 diffeomorphism $f\colon M \to M$ of a smooth manifold M. Assume that $f|_\Lambda$ is topologically mixing. Then for every $\alpha > 0$ there is an $N \in \mathbb{N}$ such that for any $x, y \in \Lambda$ and all $n \geq N$ we have $f^n\big(W^u(x,\alpha)\big) \cap W^s(y,\alpha) \neq \varnothing$. In particular, $W^u(x)$ is dense in Λ for every $x \in \Lambda$.*

We are now ready to define the *specification property* for a discrete dynamical system.

Definition 3.2.8: Let $f\colon X \to X$ be a bijection of a set X. A *specification* $\mathcal{S} = (\tau, P)$ consists of a finite collection $\tau = \{I_1, \dots, I_m\}$ of finite intervals $I_j = [a_j, b_j] \subset \mathbb{Z}$ together with a map $P\colon \bigcup_{j=1}^m I_j \to X$ such that for all $1 \leq j \leq m$ and all $t_1,\ t_2 \in I_j$ we have $f^{t_2 - t_1}\big(P(t_1)\big) = P(t_2)$. We say that $\mathcal{S}$ *parametrizes* the collection $\{P|_{I_j} \mid j = 1, \dots, m\}$ of orbit segments of f. The set $T(\mathcal{S}) = \bigcup_{j=1}^m I_j$ is the *domain* of the specification, while $\ell(\mathcal{S}) = b_m - a_1$ is its *length*. A specification $\mathcal{S}$ is said to be *n-spaced* if $a_{j+1} - b_j > n$ for all $j = 1, \dots, m-1$; the minimum n verifying this condition is the *spacing* of the specification. Finally, if (X, d) is a metric space, then we say that a specification $\mathcal{S}$ is *ε-shadowed* by a point $x \in X$ if $d\big(f^n(x), P(n)\big) < \varepsilon$ for all $n \in T(S)$.

Definition 3.2.9: Let (X, d) be a metric space and $f\colon X \to X$ a homeomorphism. Then f is said to have the *specification property* if for any $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that any M-spaced specification $\mathcal{S}$ is ε-shadowed by a point of X, and such that for any $q \geq M + \ell(\mathcal{S})$ there is a q-periodic orbit ε-shadowing $\mathcal{S}$.

Our aim is to show that topologically mixing locally maximal hyperbolic set have the specification property. So this further underlines the abundance of periodic orbits in a hyperbolic set showing that, roughly speaking, we can construct true orbits of the dynamical system from any finite collection of finite orbit segments, with an approximation depending only on the spacing and not on the length of the segments.

Theorem 3.2.21: (Specification Theorem) *Let $f\colon M \to M$ be a C^1 diffeomorphism of a smooth manifold M, and let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for f. Then $f|_\Lambda$ is topologically mixing if and only if $f|_\Lambda$ has the specification property.*

Proposition 3.2.22: *Any homeomorphism $f\colon X \to X$ of a compact metric space X with the specification property is topologically mixing.*

Theorem 3.2.23: *Let $f\colon M \to M$ be a C^1 diffeomorphism of a smooth manifold M, and let $\Lambda \subseteq M$ be a locally maximal hyperbolic set for f such that $f|_\Lambda$ is topologically transitive. Then there exists $N \in \mathbb{N}$ such that for any $\varepsilon > 0$ there is an $M = M(\varepsilon) \in \mathbb{N}$ such that for every finite collection $\mathcal{C}$ of orbit segments of f there is an M-spaced specification $\mathcal{S}$ parametrizing $\mathcal{C}$ which is ε-shadowed by a point of Λ and ε-shadowed by a qN-periodic orbit as soon as $qN \geq M + \ell(\mathcal{S})$.*

We end this section quoting the Grobman-Hartman linearization theorem:

Theorem 3.2.24: (Grobman-Hartman) *Let $f\colon M \to M$ be a C^1 diffeomorphism of a smooth manifold M, and $p \in M$ a hyperbolic fixed point of f. Then f is topologically conjugate to df_p in a neighborhood of p.*