POINCARÉ-BENDIXSON THEOREMS IN HOLOMORPHIC DYNAMICS

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THE CLASSICAL POINCARÉ-BENDIXSON THEOREM

THEOREM (POINCARÉ-BENDIXSON)

Let X be a smooth vector field on the unit sphere $S^2 \subset \mathbb{R}^3$.
Let $\gamma\colon [0,T) \to S^2$ be a maximal integral curve of X.
Then the ω-limit set of γ either cointans a singular point of X or is a periodic integral curve. Moreover, a recurrent integral curve is necessarily periodic.

A P-B theorem for meromorphic connections

Theorem (A.-Tovena, 2009)

Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \cong S^2$.
Let $\sigma\colon [0,T) \to \mathbb{P}^1(\mathbb{C}) \setminus \{poles\}$ be a maximal geodesic.

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σ geodesic iff $\nabla_{\dot\sigma}\dot\sigma = 0$ iff $\ddot\sigma + (k \circ \sigma)\dot\sigma^2 = 0$, with k meromorphic.

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The poles of ∇ are the poles of k. Residues: $\mathrm{Res}_p(\nabla)$.

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Closed does not mean periodic. Speed depends on

$$\sum_{\text{poles inside}} \operatorname{Im} \operatorname{Res}_p(\nabla) \,.$$

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3. *the ω-limit set of σ is a simple cycle of saddle connections; or*

Saddle connection: a geodesic connecting two poles.
Simple cycle of saddle connections: a Jordan curve composed by saddle connections.

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$$\sum_{\text{poles inside}} \operatorname{Re}\operatorname{Res}_p(\nabla) = -1 \;.$$

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$$\sum_{\text{poles inside a loop}} \operatorname{Re}\operatorname{Res}_p(\nabla) \in (-3/2,-1) \cup (-1,-1/2)\ .$$

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4. σ *intersects itself infinitely many times.*

We have examples of 1, 2 and 4, but not (yet?) of 3.

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A recurrent geodesic either is closed or intersects itself infinitely many times.

INGREDIENTS IN THE PROOF

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- Study of geodesics for holomorphic connections in simply connected surfaces.

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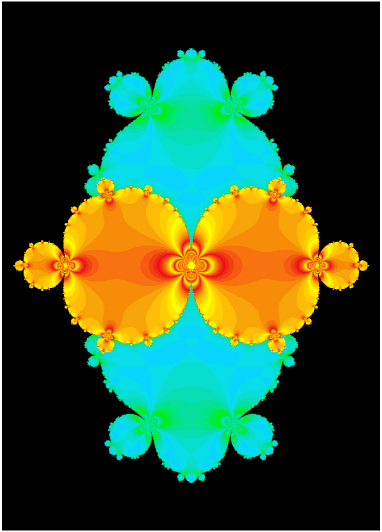
- A sort of Poincaré return map.

The Leau-Fatou flower theorem

A (germ of) holomorphic function tangent to the identity of order $\nu \geq 1$:

$$f(z) = z + a_{\nu+1} z^{\nu+1} + \cdots$$

THE LEAU-FATOU FLOWER THEOREM



Camacho's theorem

Theorem (Camacho, 1978)

Any holomorphic function tangent to the identity of order $\nu \geq 1$ is locally topologically conjugated to the time 1-map of the homogeneous vector field

$$Q = z^{\nu+1} \frac{\partial}{\partial z} .$$

HOMOGENEOUS VECTOR FIELDS

DEFINITION

A homogeneous vector field in $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$ is a vector field

$$Q = Q_1(z_1, z_2)\frac{\partial}{\partial z_1} + Q_2(z_1, z_2)\frac{\partial}{\partial z_2}$$

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CONJECTURE

Every generic map tangent to the identity of order ν is locally topologically conjugated to the time-1 map of a homogeneous vector field.

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Definition

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Remark

The dynamics inside characteristic lines is 1-dimensional.

INTEGRAL CURVES AND GEODESICS

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Theorem (A.-Tovena, 2009)

Let Q be a non-dicritical homogeneous vector field in $\mathbb{C}^2$, and let $\Omega \subset \mathbb{C}^2$ be the complement of the characteristic lines. Then there exists a meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C})$ whose poles are the characteristic directions of Q such that a curve $\gamma\colon [0,T) \to \Omega$ is an integral curve of Q if and only if $[\gamma]$ is a geodesic for ∇.

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COROLLARY

Let Q be a homogeneous vector field in $\mathbb{C}^2$. Then a recurrent maximal integral curve $\gamma\colon [0,T) \to \mathbb{C}^2$ either is periodic or $[\gamma]\colon [0,T) \to \mathbb{P}^1(\mathbb{C})$ intersects itself infinitely many times.

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- A canonical holomorphic ν-to-1 covering map $\chi_\nu\colon \mathbb{C}^2 \setminus \{O\} \to N_S^{\otimes\nu}$.
- A global geodesic field G on the total space of $N_S^{\otimes\nu}$.

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Local study around singularities

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- Give a formal (always) and holomorphic (for generic cases) classification of the singularities.
- Explain puzzling phenomena already known.
- Construct examples of unexpected phenomena.
- Give a complete description of the dynamics for large classes of homogenous vector fields (and thus of maps tangent to the identity).

THANKS!