Elements of Partial Differential Equations

Vladimir Georgiev

Dipartimento di Matematica "L.Tonelli",
Università di Pisa,
Largo Bruno Pontecorvo 5, I-56127, Pisa, Italy.
E-mail: georgiev@dm.unipi.it

Contents

Chapter 1

Preliminary chapter.

1.1 Introduction

In this Chapter we shall introduce two basic tools for the study of partial differential equations (PDE). Namely, we start with the Fourier transform. We shall avoid a complete and detailed representation of the theory of Fourier transform and Sobolev spaces. Nevertheless, we shall underline only the points, which are important for a further study of PDE.

The reader can use [46], [23] , [45] [7] for more detailed information about the space of distributions, Fourier transform and the convolution.

1.2 Gauss - Green formula for domains with smooth boundaries in $\mathbb{R}^n$.

Our goal is to recall the classical Gauss Green formula valid for any open domain Ω in $\mathbb{R}^n$ with smooth (or C^2) boundary $\partial\Omega$. Given any n continuous function

$$F(x) = (f_1(x), \cdots, f_n(x))$$

in the closure of Ω such that all first derivatives of $f_k(x)$ are well defined and continuos one has the formula

$$\int_\Omega \nabla.F(x)dx = \int_{\partial\Omega} N.F(x)dS_x, \tag{1.2.1}$$

where $F(x) = (f_1(x), \cdots, f_n(x))$,

$$\nabla.F = \partial_1 f_1(x) + \partial_2 f_2(x) + \cdots + \partial_n f_n(x).$$

Recall also that $N(x)$ is the unit outern normal at $x \in \partial\mathbf{O}$, while dS_x is the surface element on $\partial\mathbf{O}$.

Recall that for any surface $\partial\mathbf{O} \subset \mathbf{R}^n$ defined (even locally) by

$$\partial\Omega: \; x_n = \psi(x'), x' = (x_1, ..., x_{n-1}) \in \mathbf{R}^{n-1} \tag{1.2.2}$$

one can define the corresponding surface element

$$dS_x = \sqrt{1 + |\nabla\psi(x')|^2}\, dx'.$$

Remarks on Riemannian metric associated with the surface
The Riemannian metric

$$dl^2 = (dx')^2 + dx_n^2 \tag{1.2.3}$$

induces on $\partial\Omega$ a Riemannian metric. More precisely, the metric induced by the embedding $\partial\Omega \subset \mathbf{R}^n$ is

$$ds^2 = (dl_{|\partial\Omega})^2 = \sum_{i,j=1}^{n-1} g_{ij}\, dx^i dx^j, \tag{1.2.4}$$

where

$$g_{ij} = \delta_{ij} + \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}. \tag{1.2.5}$$

Problem 1.2.1. *Prove that*

$$g = \det(g_{ij}) = 1 + |\nabla\psi|^2.$$

The unit vector normal $N(x)$ to $\partial\mathbf{O}$ at the point $(x, \psi(x)) \in S$ is

$$N(x) = \frac{(1, -\nabla\psi(x))}{\sqrt{1 + |\nabla\psi|^2}}.$$

The second quadratic form on $\partial\mathbf{O}$ is

$$\sum_{i,j=1}^{n} b_{ij}\, dx^i dx^j,$$

where

$$b_{ij}(x) = \frac{\partial_{x_i}\partial_{x_j}\psi(x)}{\sqrt{1 + |\nabla\psi|^2}}.$$

Now the Gauss curvature $K(x)$ is

$$K(x) = \frac{\det b_{ij}}{\det g_{ij}} = \frac{\det(\nabla^2\psi)}{(1 + |\nabla\psi|^2)^{1+(n-1)/2}}. \tag{1.2.6}$$

Example 1. (Model of sphere of radius 1.) A special case of a surface is the sphere

$$S^{n-1}: \ x_n = \pm\sqrt{1-|x'|^2}.$$

Then one can solve the following

Problem 1.2.2. *Prove that*

$$K(x) = 1. \tag{1.2.7}$$

The coefficients of the Riemannian metric (1.2.4) are

$$g_{ij} = -\delta_{ij} + (1-|x|^2)^{-1} x_i x_j \tag{1.2.8}$$

according to (1.2.5) and (1.2.7). Using Problem 1.2.1. one can prove

Problem 1.2.3. *Prove that*

$$g = \det(g_{ij}) = \frac{1}{x^2 + a^2} \tag{1.2.9}$$

and find the matrix inverse to g_{ij}.

Applications: Set

$$E(u) = \int_{\mathbb{R}^n} \|\nabla u\|^2 dx.$$

This functional is well defined for $u \in C_0^\infty(\mathbb{R}^n)$. The critical points of the functional are such that for any $h \in C_0^\infty(\mathbb{R}^n)$ we have

$$\frac{d}{d\varepsilon}(E(u+\varepsilon h))\,|_{\varepsilon=0} = 0.$$

Since

$$E(u+\varepsilon h) = E(u) + 2\varepsilon \operatorname{Re} \int_{\mathbb{R}^n}\int \langle \nabla u, \nabla h\rangle dx + O(\varepsilon^2),$$

we see that u is a critical point, if and only if

$$\operatorname{Re} \int_{\mathbb{R}^n}\int \langle \nabla u, \nabla h\rangle dx = 0,$$

Applying the Gauss - Green formula we get

$$\operatorname{Re} \int_{\mathbb{R}^n}\int \Delta u h dx = 0, \ \forall h \in C_0^\infty(\mathbb{R}^n).$$

This relation implies

$$\Delta u = 0.$$

Problem 1.2.4. *Show that if* $u \in C_0^\infty$ *and* $\Delta u = 0$, *then* $u = 0$.

The functional $E(u)$ defined above satisfies some invariance properties

$$\text{(invariance by translation)}\ \ E(u) = E(\tau_v u),\ \ \tau_v u(x) = u(x+v),\ \ v \in \mathbb{R}^n$$

$$\text{(invariance by rotation)}\ \ E(u) = E(A^* u),\ \ A^* u(x) = u(Ax),\ A \in SO(n).$$

$$\text{(rescaling properties)}\ \ E(S_\lambda u) = \lambda^{(2-n)/2} E(u),\ \ S_\lambda u(x) = u(\lambda x),\ \lambda > 0.$$

The closure of C_0^∞ functions with respect to the norm

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{1/2}$$

is called Dirichlet space $\dot{H}^1(\mathbb{R}^n)$. he closure of C_0^∞ functions with respect to the norm

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} |u|^2 \right)^{1/2}$$

is called Sobolev space space $H^1(\mathbb{R}^n)$.

1.3 Interpolation

1.3.1 Preliminary facts about holomorphic functions

Let $\mathbf{C}$ be the complex plane and let $U \subseteq \mathbf{C}$ be an open domain in this plane. Any point $z \in U$ can be represented as

$$z = x + iy,$$

where x, y are real numbers. A function

$$f : U \to \mathbf{C}$$

is $C^1(U)$ if the partial derivatives

$$\partial_x f(x+iy), \partial_y f(x+iy)$$

exist and are continuous functions. Of special interest are the vector fields

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

If $f \in C^1(U)$, then f is called holomorphic in U, if satisfies the equation

$$\partial_{\bar{z}} f(z) = 0, \quad z \in U.$$

One can see that a function $f : U \to \mathbf{C}$ is holomorphic in U if and only if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for any $z \in U$.

The most important formula in the elementary theory of holomorphic functions is the Cauchy theorem and the Cauchy formula.

Let Γ be a closed path in U and let $z \in \mathbf{C}$ be a point such that Γ does not pass through z. Then the index of z with respect to Γ is

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}.$$

The Cauchy theorem states that if Γ is closed path in U such that $\mathrm{Ind}_\Gamma(w) = 0$ for any w outside U, then

$$\int_\Gamma f(\zeta) d\zeta = 0 \tag{1.3.1}$$

for any function holomorphic in $\mathbf{C}$. The corresponding Cauchy formula is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.3.2}$$

The condition $\mathrm{Ind}_\Gamma(w) = 0$ for w outside U is fulfilled for the case U is simply connected.

Also in the case of a simply connected domain U with smooth boundary ∂U for any function holomorphic in U and continuos in the closure of U we have the corresponding Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.3.3}$$

Applying for example the above formula for $\{z, |z - z_0| < \delta\} \subset U$, we obtain the estimate

$$|\partial_z^k f(z_0)| \le \frac{Mk!}{\delta^k}, \tag{1.3.4}$$

where

$$M = \sup_{|z - z_0| = \delta} |f(z)|.$$

This estimate guarantees that the formal Taylor series

$$\sum_{k=0}^{\infty} \partial_z^k f(z_0)(z-z_0)^k/k!$$

converges absolutely and uniformly for $|z-z_0|$ sufficiently small and moreover the series coincides with $f(z)$ for z sufficiently close to z_0. If $k=0$ the inequality in (1.3.4) gives the maximum principle

$$|f(z_0)| \le \sup_{|z-z_0|=\delta} |f(z)|,$$

so taking z_0 to vary in a fixed disc $|z|<R$ and choosing suitably $\delta>0$ one can solve the following.

Problem 1.3.1. *If $f(z)$ is holomorphic in the disc $|z|<R$ and continuous in the closure of the disc, then*

$$\max_{|z|\le R} |f(z)| = \max_{|z|=R} |f(z)|.$$

1.3.2 Il teorema di Riesz-Thorin

Lemma 1.3.1 (delle tre rette)**.** *Siano date due rette parallele in $\mathbb{C}$, siano esse l_0 e l_1 e sia Ω la striscia aperta di piano compresa tra le due rette. Sia $f:\overline{\Omega}\longrightarrow\mathbb{C}$ una funzione continua tale che $f|_\Omega$ sia olomorfa. Supponiamo che*

$$c_0 = \sup_{l_1} |f| < \infty, \qquad c_1 = \sup_{l_1} |f| < \infty$$

e supponiamo che esistano due costanti reali positive a e C tali che

$$|f(z)| \le Ce^{a|z|^2} \qquad \big(\forall z \in \overline{\Omega}\big).$$

Allora, per ogni retta l parallela a l_0 e l_1 e tutta contenuta in Ω, esiste un numero $\theta \in (0,1)$ tale che

$$\sup_{l} |f| \le c_0^{1-\theta} c_1^{\theta}.$$

Dimostrazione. Osserviamo che, a meno di diffeomorfismo, possiamo supporre

$$l_0 = \{\mathrm{Re}\, z = 0\} \quad \text{e} \quad l_1 = \{\mathrm{Re}\, z = 1\}.$$

Iniziamo col provare che, per ogni $\theta \in (0,1)$, posta $l_\theta = \{\mathrm{Re}\, z = \theta\}$,

$$c_\theta = \sup_{l_\theta} |f| \le \max\big(c_0, c_1\big).$$

Fissato $a' > a$, si ponga

$$g(z) = f(z)e^{a'z^2}.$$

Se $z = x + iy$ con $x \in [0,1]$ e $|y| \geq 2$, possiamo usare le stime

$$\text{Re}\,(a|z|^2 + a'z^2) = a(x^2 + y^2) + a'(x^2 - y^2) \leq C - (a' - a)y^2.$$

Prendendo $|y| \to \infty$, otteniamo

$$\lim_{|z|\to\infty} |g(z)| \leq \lim_{|z|\to\infty} Ce^{\text{Re}\,(a|z|^2 + a'z^2)} = 0.$$

Si ponga

$$\Omega_R = \left\{z \in \mathbb{C} : 0 \leq \text{ Re } z \leq 1, |\Im z| \leq R\right\}.$$

Denotiamo con Γ_R la frontiera di Ω_R. Poiché Ω_R è compatto, per il teorema di Weierstraß, esiste un punto z_0 appartenente a Ω_R tale che

$$|g(z_0)| = \max_{\Omega_R} |g|.$$

Supponiamo che z_0 non appartenga alla frontiera Γ_R di Ω_R, ossia $z_0 \in \mathring{\Omega}_R$. Allora esiste una palla $B = \overline{B(z_0, \varrho)}$ massima possibile (ossia tale che $B \cap \Gamma_R \neq \emptyset$) tale che $B \subset \Omega_R$. Per ogni $\bar{r} < \varrho$,

$$g(z) = \sum_{n\geq 0} c_n (z - z_0)^n$$

e, detto $z = z_0 + re^{i\theta}$,

$$g(z) = \sum_{n\geq 0} c_n r^n e^{in\theta}$$

e quindi, per l'identità di Parseval,

$$\sum_{n\geq 0} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z_0 + e^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z_0)|^2 d\theta = |g(z_0)|^2 = |c_0|^2.$$

Pertanto

$$\sum_{n\geq 0} |c_n|^2 r^{2n} \leq |c_0|^2$$

e, facendo il limite per $r \to \varrho$,

$$\sum_{n\geq 0} |c_n|^2 \varrho^{2n} \leq |c_0|^2$$

per cui deve necessariamente essere $c_n = 0$ per ogni $n \geq 1$ e quindi g deve essere costantemente c_0 su B cioè, essendo $B \cap \Gamma_R$, deve esistere $z_1 \in \Gamma_R$ tale che

$$g(z_1) = \max_{\Omega_R} |g|.$$

Da ciò segue che

$$|g(z)| \le \max_{w\in\Gamma_R} |g(w)| \qquad \big(\forall z \in \Omega_R\big)$$

e, al limite per $R \to \infty$,

$$|g(z)| \le \max\left(\sup_{l_0} |g|, \sup_{l_1} |g|\right)$$

ed in particolare

$$\sup_{l_\theta} |g| = \sup_{t\in\mathbb{R}} |g(\theta + it)| \le \max\left(\sup_{l_0} |g|, \sup_{l_1} |g|\right).$$

Ricordato, adesso, che $g(z) = f(z)e^{a'z^2}$, si ha

$$|f(\theta)| \le \max\big(c_0, c_1\big)$$

e lo stesso argomento per ogni $y \in \mathbf{R}$ ci da

$$|f(\theta + iy)| \le \max\big(c_0, c_1\big)$$

cosi abbiamo

$$\sup_{l_\theta} |f| = \sup_{t\in\mathbb{R}} |f(\theta + it)| \le \max\big(c_0, c_1\big).$$

Consideriamo, adesso, per ogni $\epsilon > 0$ e ogni $\lambda \in \mathbb{R}$,

$$f_{\epsilon,\lambda}(z) = e^{\epsilon z^2 + \lambda z} f(z).$$

Allora

$$\sup_{l_0} |f_{\epsilon,\lambda}| = \sup_{t\in\mathbb{R}} \left| e^{\epsilon(it)^2 + \lambda(it)} f(it) \right| = \sup_{t\in\mathbb{R}} e^{-\lambda t^2} |f(it)| \le c_0$$

ed inoltre,

$$\sup_{l_1} |f_{\epsilon,\lambda}| = \sup_{t\in\mathbb{R}} \left| e^{\epsilon(1+it)^2 + \lambda(1+it)} f(1+it) \right| = \sup_{t\in\mathbb{R}} e^{\epsilon - \epsilon t^2 + \lambda} |f(+it)| \le e^{\epsilon+\lambda} c_1.$$

Pertanto

$$|f_{\epsilon,\lambda}(\theta + it)| \le \max\left(c_0, c_1 e^{\epsilon+\lambda}\right)$$

ovvero

$$\left| e^{\epsilon(\theta+it)^2 + \lambda(\theta+it)} f(\theta + it) \right| \le \max\left(c_0, c_1 e^{\epsilon+\lambda}\right)$$

e, semplificando l'espressione,

$$e^{\epsilon\theta^2 - \epsilon t^2 + \lambda\theta} |f(\theta + it)| \le \max\left(c_0, c_1 e^{\epsilon+\lambda}\right)$$

che equivale a

$$|f(\theta+it)| \le e^{-\epsilon(\theta^2-t^2)} \max\Big(c_0 e^{-\lambda\theta}, c_1 e^{\epsilon+\lambda(1-\theta)}\Big).$$

Facendo il limite per $\epsilon \to 0$, otteniamo l'espressione,

$$|f(\theta+it)| \le \max\Big(c_0 e^{-\lambda\theta}, c_1 e^{\lambda(1-\theta)}\Big).$$

segliamo λ in modo che $c_1 e^{-\lambda\theta} = c_2 e^{\lambda(1-\theta)}$, ovvero, detto $\rho = e^{\lambda}$, vogliamo trovare ρ in modo tale che $c_0\rho^{-\theta} = c_1\rho^{1-\theta}$, ossia $\rho = \frac{c_0}{c_1}$ cioè $\lambda = \log\frac{c_0}{c_1}$. Per questo particolare valore di λ, il comune valore di $c_0\rho^{-\theta}$ e $c_1\rho^{1-\theta}$ è $c_0^{1-\theta}c_1^{\theta}$ e quindi

$$\sup_{l_\theta} |f| \le c_0^{1-\theta}c_1^{\theta}$$

che prova il lemma. □

Osservazione 1.3.1. *L'ipotesi centrale del lemma precedente è il fatto che $|f(z)| \le Ce^{a|z|^2}$. Questa ipotesi è essenziale. in effetti, se consideriamo $f(z) = e^{-\cosh z}$, sulle rette $\{\Im z = 0\}$ e $\{\Im z = 2\pi\}$ la funzione è limitata ma sulla retta $\{\Im z = \pi\}$ la funzione non è limitata.*

Problem 1.3.2. *If*

$$T : L^{p_0} \longrightarrow \mathbf{R}$$

and $T : L^{p_1} \to \mathbf{R}$, then $T : L^p \to \mathbf{R}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Hint Use Hölder inequality and duality of L^p.

Theorem 1.3.1 (di Riesz-Thorin)**.** *Siano $(E,\mathscr{E},\mu)$ e $(F,\mathscr{F},\nu)$ due spazi misurati e siano $p_0, p_1, q_0, q_1 \in [1,\infty]$. Sia T un operatore lineare definito sia su $L^{p_0}(E)$ che sue $L^{p_1}(E)$ a valori nello spazio delle funzioni misurabili di F e tale che*

$$\|Tf\|_{q_0} \le M_0\|f\|_{p_0} \quad e \quad \|Tg\|_{q_1} \le M_1\|g\|_{p_1}$$

per ogni $f \in L^{p_0}(E)$ e ogni $g \in L^{p_1}(E)$. Siano p e q due numeri reali definiti come segue:

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

dove $\theta \in (0,1)$. Allora T è definito sullo spazio di Lebesgue $L^p(E)$ e

$$\|Tf\|_q \le M_0^{1-\theta}M_1^{\theta}\|f\|_p.$$

Dimostrazione. Supponiamo, inizialmente che $p < \infty$. Lo spazio delle funzioni semplici su E è denso in $L^p(E)$. Pertanto

$$\|T\|_{p,q} = \sup_{f \text{ semplice}} \frac{\|Tf\|_q}{\|f\|_p}, \quad \|Tf\|_q = \sup_{g \text{ semplice}} \frac{\big|\langle Tf, g\rangle\big|}{\|g\|_{q'}}$$

per cui

$$\|T\|_{p,q} = \sup_{f,g \text{ semplici}} \frac{\big|\langle Tf, g\rangle\big|}{\|f\|_p \|g\|_{q'}}.$$

Ne segue che basta dimostrare la disuguaglianza

$$\left|\int_Y Tf(y)g(y)\nu(dy)\right| \leq M_0^{1-t} M_1^t \|f\|_p \|g\|_{q'} \tag{1.3.5}$$

quando f e g sono funzioni semplici su E e F rispettivamente, cioè

$$f(x) = \sum_{j\in\mathcal{J}} a_j I_{E_j}(x), \qquad g(y) = \sum_{k\in\mathcal{K}} b_k I_{F_k}(y).$$

Per $0 \leq \operatorname{Re} z \leq 1$ si definiscano p_z e q'_z come segue:

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'_z} = \frac{1-z}{q'_0} + \frac{z}{q'_1}.$$

Se, per ogni fissato indice $j \in \mathcal{J}$ e $k \in \mathcal{K}$, $a_j = |a_j| e^{i\theta_j}$ e $b_k = |b_k| e^{i\varphi_k}$, si ponga

$$f_z(x) = \sum_{j\in\mathcal{J}} |a_j|^{p/p_z} e^{i\theta_j} I_{E_j}(x), \qquad g_z(y) = \sum_{k\in\mathcal{K}} |b_k|^{q'/q'_z} e^{i\varphi_l} I_{F_k}(y).$$

La funzione g_z è ben definita finché $q' < \infty$. Se $q' = \infty$ allora $q'_0 = q'_1 = \infty$ e quindi $\frac{1}{q'_z} = 0$ per ogni z. In tal caso poniamo $\frac{q'}{q'_z} = 1$ per ogni z. Si definisca, infine, la funzione

$$F(z) = \int_Y Tf_z(y) g_z(y)\nu(dy) = \sum_{j\in\mathcal{J}, k\in\mathcal{K}} c_{jk} |a_j|^{p/p_z} |b_k|^{q'/q'_z}$$

dove si è posto

$$c_{jk} = e^{i\theta_j} e^{i\varphi_k} \int_Y T\big(I_{E_j}(y)\big) I_{F_k}(y)\nu(dy).$$

Vogliamo provare che la funzione F verifica le ipotesi del *lemma delle tre rette*. Anzitutto F è continua sulla striscia di piano $\overline{\Omega} = \{0 \leq \operatorname{Re} z \leq 1\}$. Inoltre,

$$|F(z)| \leq \sum_{j\in\mathcal{J}, k\in\mathcal{K}} |c_{jk}| \cdot |a_j|^{\operatorname{Re}(p/p_z)} |b_k|^{\operatorname{Re}(q'/q'_z)}$$

per cui F è limitata su $\overline{\Omega}$. Se $t \in \mathbb{R}$, allora

$$|F(it)| \le M_0 \|f_{it}\|_{p_0} \|g_{it}\|_{q_0'}.$$

Se $p_0 < \infty$, allora

$$\|f_{it}\|_{p_0} = \left(\sum_{j \in \mathscr{J}} |a_j|^p \mu(E_j) \right)^{p_0} = \left(\|f\|_p \right)^{p/p_0}$$

altrimenti, se $p_0 = \infty$, il primo e terzo membro sono eguali ad 1. Analogamente

$$\|g_{it}\|_{q_0'} = \left(\|g\|_{q'} \right)^{q'/q_0'}.$$

Pertanto

$$|F(it)| \le M_0 \left(\|f\|_p \right)^{p/p_0} \left(\|g\|_{q'} \right)^{q'/q_0'}$$

e, in maniera analoga,

$$|F(1+it)| \le M_1 \left(\|f\|_p \right)^{p/p_1} \left(\|g\|_{q'} \right)^{q'/q_1}.$$

Per il lemma delle tre rette, dunque,

$$|F(\theta)| \le M_0^{1-\theta} M_1^{\theta} \left(\|f\|_p \right)^{p\left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1} \right)} \left(\|g\|_{q'} \right)^{q'\left(\frac{1-\theta}{q_0'} + \frac{\theta}{q_1'} \right)} = M_0^{1-\theta} M_1^{\theta} \|f\|_p \|g\|_{q'}.$$

Siccome $f_\theta = f$ e $g_\theta = g$, si ha

$$F(\theta) = \int_Y Tf(y) g(y) \nu(dy)$$

che dimostra la (1.3.5) per $p < \infty$ e $q > 1$. Nel caso in cui $p = \infty$, allora $p_0 = p_1 = \infty$ e, per la disuguaglianza di Hölder,

$$\|Tf\|_q \le \left(\|Tf\|_{q_0} \right)^{1-\theta} \left(\|Tf\|_{q_1} \right)^{\theta} \le M_0^{1-\theta} M_1^{\theta} \|f\|_\infty$$

ovvero la tesi. □

Applicazione del teorema di Riesz - Torin implica la disequazione di Young (see [45])

(1.3.6) $$\|f * g\|_{L^q} \le \|f\|_{L^1} \|g\|_{L^q}$$

for $1 \le q \le \infty$. Qui

$$f * g(x) = \int f(x-y) g(y) dy.$$

In fatti, la stima (1.3.12) é ovvia per $q = \infty$ e per $q = 1$ e dopo l'interpolazione si ottiene (1.3.12). Si puo generealizzare la seguente versione di (1.3.12)

$$\|f * g\|_{L^s} \le \|f\|_{L^r} \|g\|_{L^p} \tag{1.3.7}$$

per $1/p + 1/r = 1 + 1/s$. Per $r = 1$ la stima e stata dimostrata in (1.3.12). In particolare, abbiamo

$$\|f * g\|_{L^q} \le \|f\|_{L^1} \|g\|_{L^q} \tag{1.3.8}$$

Per $s = \infty$ la disequazione di Hölder ci da

$$\|f * g\|_{L^\infty} \le \|f\|_{L^{q'}} \|g\|_{L^q}.$$

Interpolazione tra questa stima e (1.3.8) implica (1.3.8).

1.3.3 Density of continuous functions in L^1

Let L^q denote the Lebesgue space $L^q(\mathbf{R}^n)$.

In this subsection we'll show that continuous functions with compact support are dense in $L^1 = L^1(\mathbb{R}^n, m)$.

The support of a complex valued function f on $\mathbb{R}^n$ is the closure of $\{x \in \mathbb{R}^n : f(x) \neq 0\}$. We'll denote by $C_c(\mathbb{R}^n)$ the set of all complex valued continuous functions on $\mathbb{R}^n$ with compact support.

Osservazione 1.3.2. $C_c(\mathbb{R}^n) \subset L^1$

Our goal is to prove

Theorem 1.3.2. *For any $f \in L^1$ and any $\varepsilon > 0$ there is a g in $C_c(\mathbb{R}^n)$ such that $\int |f - g| dm < \varepsilon$.*

Thus integrable functions can be approximated by continuous functions. We'll need a special case of Urysohn's Lemma. The proof is deferred to Subsection 1.3.4.

Lemma 1.3.2. *Let X be any locally compact metric space, and let Let K and U be subsets of X with K compact, U open, and $K \subset U$. There is a function $\chi \in C_c(X)$ satisfying*

1. *$0 \le \chi(x) \le 1$ for every $x \in X$*
2. *$\chi(x) = 1$ for every $x \in K$*
3. *The support of χ is a subset of U.*

The proof of Theorem 1.3.2 begins with some elementary reductions to simpler special cases. First, since L^1 functions can be approximated in L^1 by integrable simple functions, it suffices to prove the theorem when f is a simple function. Next, since an integrable simple function is a linear combination of functions of the form χ_E, where E is a measurable set of finite measure, it suffices to prove the theorem when $f = \chi_E$. Therefore, Theorem 1.3.2 will be proved by establishing

Proposition 1.3.1. *Let E be a measurable subset of R^n with finite measure. Then for any $\varepsilon > 0$ there is a $g \in C_c(\mathbb{R}^n)$ with $\int |\chi_E - g| \, dm < \varepsilon$*

Proof. By regularity properties of Lebesgue measure, there are a compact set K and an open set U such that $K \subset E \subset U$ and $m(U \backslash K) < \varepsilon$. By Lemma 1.3.2, there is a $g \in C_c(\mathbb{R}^n)$ such that $0 \leq g \leq 1$ everywhere, $g = 1$ on K, and g vanishes outside of U. It follows that $|g - \chi_E|$ vanishes outside of $U \backslash K$, and that $|g - \chi_E| \leq 1$ on $U \backslash K$. Therefore

$$\int |g - \chi_E| \, dm = \int_{U \backslash K} |g - \chi_E| \, dm \leq m(U \backslash K) < \varepsilon$$

□

Let us recall Lusin's Theorem: Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable, , and suppose that f vanishes outside a set of finite measure. Then for any $\varepsilon > 0$ there is a measurable set E with $m(E) < \varepsilon$ such that $f|_{\mathrm{R}^n \backslash E}$ is continuous.

1.3.4 Proof of Urisohn lemma

In this subsection, we'll prove the version of Urysohn's Lemma that we used in the proof of Theorem 1.3.2. We'll work in a locally compact metric space (X, d) We'll use the notation $B_r(x)$ for the open ball in X with center x and radius r

Lemma 1.3.3. *Let $x_0 \in X$ and $r > 0$. There is a $\chi \in C_c(X)$ with*

1. $0 \leq \chi(x) \leq 1$ *for every* $x \in X$
2. $\chi(x) = 1$ *for every* $x \in B_r(x_0)$
3. *The support of* χ *is a subset of* $B_{2r}(x_0)$.

Proof. Let φ be the continuous function on $\mathbb{R}$ defined by $\varphi(t) = 1$ for $0 \leq t \leq r$ $\varphi(t) = 1 - \frac{2}{r}(t - r)$ for $r < t \leq \frac{3r}{2}$, and $\varphi(t) = 0$ for $t > \frac{3r}{2}$. Let $\chi(x) = \varphi(d(x, x_0))$
□ □

Proof of Lemma 1.3.2. Cover K by finitely many balls $B_j = B_{r_f}(x_j)$ such that $\overline{B_{2r_j}(x_j)}$ is a compact subset of U. Let χ_j be the function obtained from Lemma 1.3.3 with $x_0 = x_j$ and $r = r_j$. Let $\varphi = \sum \chi_j$. Then

1. φ is continuous on X;
2. The support of φ is a compact subset of U;
3. $\varphi \geq 0$ everywhere, and $\varphi \geq 1$ on K

Let $\chi = \min\{\varphi, 1\}$. Then χ has all the required properties. □

1.3.5 Basic interpolation theorems

The first important interpolation theorem is the Riesz-Thorin interpolation theorem. To state this theorem we start with some notations.

Given any positive real numbers p_0, p_1 with $1 \leq p_0 < p_1 \leq \infty$, we denote by $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ the linear space

$$\{f : f = f_0 + f_1, f_0 \in L^{p_0}(\mathbf{R}^n), f_1 \in L^{p_1}(\mathbf{R}^n)\}.$$

The norm in this space we define as follows

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}.$$

Here the infimum is taken over all representations $f = f_0 + f_1$, where $f_0 \in L^{p_0}(\mathbf{R}^n)$ and $f_1 \in L^{p_1}(\mathbf{R}^n)$.

It is easy to see that $L^{p_0} + L^{p_1}$ is a Banach space.

Theorem 1.3.3. *(Riesz - Torin) Suppose T is a linear bounded operator from $L^{p_0} + L^{p_1}$ into $L^{q_0} + L^{q_1}$ satisfying the estimates*

$$\begin{aligned} \|Tf\|_{L^{q_0}} &\leq M_0 \|f\|_{L^{p_0}}, \ f \in L^{p_0}, \\ \|Tf\|_{L^{q_1}} &\leq M_0 \|f\|_{L^{p_1}}, \ f \in L^{p_1}. \end{aligned} \tag{1.3.9}$$

Then for any $t \in (0,1)$ we have

$$\|Tf\|_{L^{q_t}} \leq M_0 \|f\|_{L^{p_t}}, \tag{1.3.10}$$

where

$$1/p_t = t/p_1 + (1-t)/p_0 \ , \ 1/q_t = t/q_1 + (1-t)/q_0. \tag{1.3.11}$$

Applying this interpolation theorem, one can derive (see [45]) the Young inequality

$$\|f * g\|_{L^q} \leq \|f\|_{L^1} \|g\|_{L^q} \tag{1.3.12}$$

for $1 \le q \le \infty$. Here

$$f * g(x) = \int f(x-y)g(y)dy.$$

It is not difficult to derive the following more general variant of (1.3.12)

$$\|f * g\|_{L^s} \le \|f\|_{L^r}\|g\|_{L^p} \tag{1.3.13}$$

for $1/p + 1/r = 1 + 1/s$.

Further, we turn to a weighted variant of Young inequality. For simplicity, we consider only the continuous case. Let $w(x), w_1(x)$ and $w_2(x)$ be smooth positive functions satisfying the assumption

$$w(x+y) \le C w_1(x) w_2(y). \tag{1.3.14}$$

Then the argument of the proof of Young inequality leads to

$$\|w(f * g)\|_{L^q} \le C\|w_1 f\|_{L^1}\|w_2 g\|_{L^q} \tag{1.3.15}$$

Indeed, we have the inequality

$$|w(x)(f * g)(x)| \le C(|w_1 f| * |w_2 g|)(x)$$

and (1.3.15) follows from the classical Young inequality.

Two typical examples of weights satisfying the assumption (1.3.14) are considered below.

Example 1. let $w(x) =< x >^s$ with $s > 0$. Then we can choose $w_1 = w_2 = w$ and the assumption (1.3.14) is fulfilled.

Example 2. Let $w(x) =< x >^s$ with $s < 0$. Then we take $w_1(x) =< x >^{-s}$ and $w_2(x) =< x >^s$. Again (1.3.14) is fulfilled.

To prove the Sobolev inequality we need more fine interpolation theorems concerning the weak L^p spaces. To define these weak spaces we shall denote by μ the Lebesgue measure. Given any measurable function f we shall say that $f \in L^p_w$ if the quantity

$$\sup_t \left(t^p \mu\{x : |f(x)| > t\}\right)^{1/p} \tag{1.3.16}$$

is finite. Note that the quantity in (1.3.16) is not a norm. This quantity is equivalent to

$$\|f\|_{L^p_w} = \sup_{A,\ \mu(A)<\infty} \mu(A)^{-1/p'} \int_A |f(x)|dx,\ \frac{1}{p} + \frac{1}{p\prime} = 1. \tag{1.3.17}$$

Problem. Show that the quantities in (1.3.16) and (1.3.17) are equivalent.

We have the inclusion $L^p \subset L^p_w$ in view of the inequality

$$\|f\|^p_{L^p} \geq \int_{|x|\geq t} |f(x)|^p\, dx \geq t^p \mu\{x : |f(x)| > t\} \sim \|f\|^p_{L^p_w}.$$

Example. The function $|x|^{-n/p}$ is in L^p_w, but not in L^p.

The following two theorems play crucial role in the interpolation theory.

Theorem 1.3.4. *(**Marcinkiewicz interpolation theorem**) Suppose T is a linear operator satisfying the estimates*

$$\begin{aligned} \|Tf\|_{L^{q_0}_w} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}_w} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{1.3.18}$$

with $p_0 \neq p_1$, $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0 \neq q_1 \leq \infty$.

Then we have

$$\|Tf\|_{L^q} \leq M_0 \|f\|_{L^p}, \tag{1.3.19}$$

provided

$$1/p = t/p_1 + (1-t)/p_0 \ , \ 1/q = t/q_1 + (1-t)/q_0 \tag{1.3.20}$$

for some $t \in (0,1)$ and $p \leq q$.

Theorem 1.3.5. *(**Hunt interpolation theorem**) Suppose T is a linear operator satisfying the inequalities*

$$\begin{aligned} \|Tf\|_{L^{q_0}} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{1.3.21}$$

with $1 \leq p_1 < p_0 \leq \infty$ and $1 \leq q_1 < q_0 \leq \infty$. Then for any $t \in (0,1)$ we have

$$\|Tf\|_{L^{q_t}_w} \leq M_0 \|f\|_{L^{p_t}_w}, \tag{1.3.22}$$

where

$$1/p_t = t/p_1 + (1-t)/p_0 \ , \ 1/q_t = t/q_1 + (1-t)/q_0. \tag{1.3.23}$$

As an application of the above interpolation theorems one can prove (see [45]) the following generalization of the Young inequality

$$\|f * g\|_{L^s} \leq \|f\|_{L^p} \|g\|_{L^r_w} \tag{1.3.24}$$

for $1/p + 1/r = 1 + 1/s, 1 < p, r, s < \infty$.

After this preparation we can turn to the proof of the following Sobolev estimate.

Lemma 1.3.4. *Suppose* $0 < \lambda < n$, $f \in L^p(\mathbf{R}^n)$, $g \in L^r(\mathbf{R}^n)$, *where* $1/p + 1/r + \lambda/n = 2$ *and* $1 < p, r < \infty$. *Then we have*

$$\int\int \frac{|f(x)||g(y)|}{|x-y|^\lambda} dx dy \leq C \|f\|_{L^p} \|g\|_{L^r} \tag{1.3.25}$$

Proof of Lemma 1.3.4 We know that (1.3.24) is fulfilled. Then for the left hand side of the Sobolev inequality (1.3.25) we can apply the Hölder inequality so we get

$$\int\int \frac{|f(x)||g(y)|}{|x-y|^\lambda} dx dy \leq C \|f\|_{L^p} \|g * h\|_{L^{p'}} \tag{1.3.26}$$

with $h(x) = |x|^{-|\lambda|}$. Now the application of (1.3.24) yields

$$\|g * h\|_{L^{p'}} \leq \|g\|_{L^r} \|h\|_{L^l_w} \tag{1.3.27}$$

provided

$$\frac{1}{p'} + 1 = \frac{1}{r} + \frac{1}{l} \tag{1.3.28}$$

The example considered after the definition of the weak L^p spaces shows that the quantity $\|h\|_{L^l_w}$ is bounded when $\lambda l = n$. From this relation and (1.3.28) we see that for $2 = 1/p + 1/r + \lambda/n$ we have the Sobolev inequality.

1.3.6 Idea of abstract interpolation: Interpolation couples

Let A_0 and A_1 are Banach spaces. Set $\overline{A} = (A_0, A_1)$. We shall call A_0 and A_1 compatible if there exists separable topological vector space $\mathfrak{U}$, such that $A_0 \cup A_1 \subset \mathfrak{U}$. We can define

$$\Delta(\overline{A}) = A_0 \cap A_1, \tag{1.3.29}$$

while for any compatible couple $\overline{A} = (A_0, A_1)$ we can define

$$\Sigma(\overline{A}) = A_0 + A_1 = \{a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}. \tag{1.3.30}$$

Lemma 1.3.5. *We have the properties:*

- $\Delta(\overline{A})$ *is a Banach space with norm* $\|a\|_{A_0} + \|a\|_{A_1}$;
- $\Sigma(\overline{A})$ *is a Banach space with norm*

$$\|a\|_{\Sigma(\overline{A})} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1}; a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}.$$

Definizione. *Let A_0 and A_1 are Banach spaces and $\overline{A} = (A_0, A_1)$. The space $F(\bar{A})$ consists of all functions*

$$f(z) : S = \{0 \le \mathrm{Re} z \le 1\} \to \Sigma(\bar{A})$$

defined, continuous and bounded in the closed strip

$$S = \{0 \le \mathrm{Re} z \le 1\},$$

analytic in the open strip

$$S_0 = \{0 < \mathrm{Re} z < 1\},$$

and satisfying

$$\lim_{y\to\infty} f(j + iy) = 0, \quad j = 0, 1.$$

The interpolation space $(A_0, A_1)_\theta$ for $\theta \in [0, 1]$ consists of all $a \in \Sigma(\bar{A})$ such that $a = f(\theta)$ for some $f \in F(\bar{A})$. The corresponding norm is defined as follows

$$\|a\|_\theta = \inf\{\|f\|_F; a = f(\theta), f \in F(\bar{A})\}.$$

It is clear that $(A_0, A_1)_\theta$ is a Banach space.

One can show that

(1.3.31) $$A_0 \cap A_1 \text{ is dense in } (A_0, A_1)_\theta.$$

Moreover, for $f \in A_0 \cap A_1$ we have $f \in (A_0, A_1)_\theta$ and the following estimate

(1.3.32) $$\|f\|_{(A_0,A_1)_\theta} \le C\|f\|_{A_0}^{1-\theta}\|f\|_{A_1}^{\theta}$$

is fulfilled.

The next Theorem gives an estimate of the norm of a bounded operator with respect to interpolation space.

Theorem 1.3.6. *Let (A_0, A_1) and (B_0, B_1) be interpolation couples and let T be a bounded operator from $A_0 + A_1$ into $B_0 + B_1$, such that $T \in L(A_j, B_j)$ with norm $\|T\|_{L(A_j,B_j)}$ for $j = 0, 1$. Then for any $\theta, 0 < \theta < 1$ we have*

$$T \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta))$$

with

$$\|Tf\|_{(B_0,B_1)_\theta} \le \|T\|_{L(A_0,B_0)}^{1-\theta}\|T\|_{L(A_1,B_1)}^{\theta}\|f\|_{(A_0,A_1)_\theta}.$$

Proof. Let $f \in (A_0, A_1)_\theta$. Then there exists a function $f(z) \in F((A_0, A_1))$ so that $f = f(\theta)$. Consider the function

$$g(z) = \|T\|^{z-\theta}_{L(A_0,B_0)} \|T\|^{-z+\theta}_{L(A_1,B_1)} Tf(z).$$

Then $g(z) \in F(B_0, B_1)$. Since

$$\|g(it)\|_{B_0} \leq \|T\|^{-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|T\|_{L(A_0,B_0)} \|f(it)\|_{A_0}$$

and

$$\|g(1+it)\|_{B_0} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{-1+\theta}_{L(A_1,B_1)} \|T\|_{L(A_1,B_1)} \|f(it)\|_{A_1},$$

we see that

$$\|Tf\|_{(B_0,B_1)_\theta} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{(A_0,A_1)_\theta}.$$

This completes the proof.

A trivial modification in the above proof shows that we have the following.

Theorem 1.3.7. *Let (A_0, A_1) and (B_0, B_1) be interpolation couples and let $T(z)$ be a holomorphic in S_0 operator-valued function defined in the strip S and continuous there. Suppose that for $z \in S$ we have that $T(z)$ is a linear bounded operator from $A_0 + A_1$ into $B_0 + B_1$, such that $T(j+it) \in L(A_j, B_j)$ with norm*

$$\sup_{t\in\mathbf{R}} \|T(j+it)\|_{L(A_j,B_j)} < \infty$$

for $j = 0, 1$. Then for any $\theta, 0 < \theta < 1$ we have

$$T(\theta) \in L((A_0, A_1)_\theta, (B_0, B_1)_\theta)).$$

1.4 Idea to define Distributions, why they are needed?

The purpose of this section is to recall some basic notions and properties of the spaces , where the solutions of the PDE are defined.

The first important space is $C_0^\infty(\mathbf{R}^n)$. This space consists of all smooth functions with compact support.

The space $C_0^\infty(\mathbf{R}^n)$ is nonempty linear vector space. Indeed, we can first construct a smooth function $f(x)$, such that $f(x) = 0$ for $x \leq 0$ and $f(x) > 0$ for $x > 0$. For the purpose take

$$f(x) = e^{-1/x}$$

for $x > 0$. Then $f(x)f(1-x)$ is a smooth function with support in the interval $[0,1]$.

Recall that the support of function $f(x)$ defined for $x \in \mathbf{R}^n$ is the closure of the set

$$\{x : f(x) \neq 0\}.$$

Sometimes this space is called space of test functions and is denoted by $D(\mathbf{R}^n)$. This space can be equipped with infinite number of semi norms. In fact, given any integer $N \geq 1$ we define

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |\alpha| \leq N\}, \tag{1.4.33}$$

where here and below $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

In order to work with a complete space, i.e. space where any Cauchy sequence converges to an element of the space, we have to define the topology on $C_0^\infty(\mathbf{R}^n)$ using all the collection of semi norms $\|.\|_N$. To define a complete topology the simplest way is to define the convergence of a sequence of functions $\{f_k\}_{k=1}^\infty$ to zero. Recall that this sequence converges to zero if there exists a compact set K such that $\mathrm{supp} f_k \subseteq K$ for any integer $k \geq 1$ and $\|f_k\|_N$ tends to 0 as $k \to \infty$ for any integer $N \geq 0$. Applying the Arzela-Ascoli compactness theorem one can check that the topology determined by this convergence is complete.

We refer to [46] for a complete discussion of the topology on this space.

In a similar way one can consider the space $C^\infty(\mathbf{R}^n)$ consisting of all smooth functions. Now we have the following family of **semi norms**.

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |x| \leq N, |\alpha| \leq N\},$$

The above family of semi norms enables one directly to introduce a complete topology (even one can introduce a complete metric).

The weakest space, where we shall look for solutions of the nonlinear partial differential equations, is the space of distributions $D'(\mathbf{R}^n)$ consisting of all linear continuous functionals on $C_0^\infty(\mathbf{R}^n)$. Given any distribution Λ we shall denote by

$$< \Lambda, f >$$

the action of the distribution (the linear functional) Λ on the test function $f \in C_0^\infty(\mathbf{R}^n)$. It is clear that

$$C_0^\infty(\mathbf{R}^n) \subset D'(\mathbf{R}^n)$$

and

$$< \Lambda, f >= \int \Lambda(x) f(x) dx$$

for $\Lambda \in C_0^\infty(\mathbf{R}^n)$.

A typical example of a distribution, which is not a test function, is the Dirac delta function δ defined by

$$< \delta, f >= f(0).$$

Since the space of distributions is the dual space to the space of test functions, we choose the topology on the space of distributions to be the weak topology on this dual space. This means that a sequence of distributions $\{\Lambda_k\}_{k=1}^\infty$ tends to zero if for any test function f we have $< \Lambda_k, f >$ tends to zero.

The space $D'(\mathbf{R}^n)$ equipped with this weak topology is a complete space.

Another useful characterization of the distributions is the following one. A linear functional Λ on $C_0^\infty(\mathbf{R}^n)$ is bounded if for any compact $K \subseteq \mathbf{R}^n$ there exist integer $k \geq 0$ and a positive real number C so that for any smooth function $\varphi(x)$ with compact support in K we have

$$| < \Lambda, \varphi > | \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi(x)|.$$

Example. Let $\varphi(x)$ be smooth non-negative function such that $\varphi(0) > 0$. Given any $\varepsilon > 0$, we can define the function

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon). \tag{1.4.34}$$

Then it is easy to see that

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = c\delta,$$

where

$$c = \int \varphi(x) dx > 0.$$

Another fact for distributions is the meaning of the identity

$$\Lambda = 0$$

in the sense of distributions. This means

$$< \Lambda, \varphi >= 0$$

for any test function φ.

A natural operation in the space of distribution is the differentiation defined by

$$< \partial^\alpha \Lambda, f >= (-1)^{|\alpha|} < \Lambda, \partial^\alpha f > .$$

Then $\partial^\alpha \Lambda$ is a bounded linear functional on $C_0^\infty(\mathbf{R}^n)$ provided $\Lambda \in D'(\mathbf{R}^n)$.

Problem 1.4.1. *Given a distribution Λ on $\mathbf{R}$ with*

$$\Lambda' = 0$$

in the sense of distributions, find a constant c such that

$$\Lambda = c$$

also in the sense of distributions.

A distribution $u \in S'(\mathbb{R}^n)$ is called non negative ($u \geq 0$) if $< u, \varphi >\geq 0$ for any $\varphi \in S(\mathbb{R}^n)$.

Chapter 2

Equazioni del prim'ordine e metodo delle caratteristiche

2.1 Trasporto lungo un campo vettoriale.

Sia $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Siamo interessati per ora alla seguente tipologia molto particolare di E.D.P. del primo ordine:

$$\sum_{j=1}^{n} a_j(x)\,\partial_{x_j} u(x) = 0 \tag{2.1.1}$$

detta anche E.D.P. del trasporto lungo il campo vettoriale $(a_1(x), \dots, a_n(x))$.

Osserviamo che se l'operatore

$$u(x) \to L(u)(x) = \sum_{j=1}^{n} a_j(x)\,\partial_{x_j} u(x)$$

é lineare rispetto u. Cio" implica che (2.1.1) é lineare come E.D.P., ossia

$$\mathscr{L}(u_1 + u_2)(x) = \mathscr{L}(u_1)(x) + \mathscr{L}(u_2)(x)\mathscr{L}(cu)(x) = c\mathscr{L}(u)(x)$$

dove u, u_1, u_2 sono funzioni della variabile indipendente $x \in \mathbb{R}^n$, $c \in \mathbb{R}\mathscr{L}(u)(x) = \sum_{j=1}^{n} a_j(x)\partial_{x_j} u(x)$.

Possiamo effetuare cambiamento di variabili

$$x \to y = F(x)$$

con matrice Jacobiana

$$J(f) = \left(\partial_{x_j} F_k(x)\right)_{j,k=1}^{n}$$

invertibile e quindi localmente possiamo definire la funzione

$$y \to x = G(y) = F^{-1}(y).$$

Usando le relazioni

$$\partial_{x_j} = \sum_{k=1}^{n} \partial_{x_j} F_k \partial_{y_k},$$

si puovedere che se $u(x)$ e C^1 soluzione del problema (2.1.1),

$$v(y) = u(G(y))$$

e soluzione di

$$\sum_{j=1}^{n} b_k\left(y\right)\partial_{y_k} v(y) = 0, \tag{2.1.2}$$

dove

$$b_k(y) = \sum_{j=1}^{n} a_j(G(y))\left(\partial_{x_j} F_k\right)(G(y)).$$

Definition 2.1.1. *Curva caratteristica e ogni soluzione* $y(s) = \left(y_1(s), \ldots, y_n(s)\right)$ *del sistema E.D.O.*

$$\begin{cases} \frac{d}{ds} y_1(s) = a_1\left(y_1(s), \ldots, y_n(s)\right) \\ \cdots \\ \frac{d}{ds} y_n(s) = a_n\left(y_1(s), \ldots, y_n(s)\right) \\ y_1(0) = \alpha_1, \\ \cdots \\ y_n(0) = \alpha_n \end{cases} \tag{2.1.3}$$

Ponendo $a(x) = (a_1(x), \cdots, a_n(x))$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ possiamo rescrivere (2.1.5) come

$$\begin{cases} \frac{d}{ds} y(s) = a(y(s)) \\ y(0) = \alpha, \end{cases} \tag{2.1.4}$$

Il problema e nonlineare e quindi in generale abbiamo solo soluzioni locali in s. Useremo la notazione $y(s;\alpha)$ per la soluzione (locale) del problema (2.1.4).

Abbiamo la seguente proprietá

Lemma 2.1.1. *Sia* $y(s) = \big(y_1(s),\dots,y_n(s)\big) \in \mathbb{R}^n$ *una soluzione del seguente sistema di E.D.O.*

$$\frac{d}{ds}y(s) = a(y(s)) \tag{2.1.5}$$

ed $u(x) \in \mathbb{R}$ *soluzione di* (2.1.1). *Allora*

$$\frac{d}{ds}u(y(s)) = 0$$

Quindi la soluzione u e'costante lungo la traiettoria descritta dall'immagine di $y(s)$ (le curve caratteristiche). In generale si e' interessati non solo a cercare soluzioni (osserviamo che ad esempio nel caso dell" E.D.P. (2.1.1) le funzioni costanti sono certamente soluzioni), ma si studia il seguente problema di Cauchy con dati su una superficie S in $\mathscr{R}^n$

$$\begin{cases} \sum_{j=1}^n a_j(x)\,\partial_{x_j}u(x) = 0, \\ u(x) = \varphi(x), x \in S. \end{cases} \tag{2.1.6}$$

Possiamo rescrivere il problema come segue.

$$\begin{cases} \langle a(x), \nabla u(x)\rangle = 0, \\ u|_S = \varphi \end{cases} \tag{2.1.7}$$

Per avere unica soluzione del problema (2.1.7) si puo prendere $\alpha_0 \in S$ e un piccolo intorno $U \subset \mathbb{R}^n$ di α_0. Per trovare una soluzione $u \in C^1(U)$ del

$$\begin{cases} \langle a(x), \nabla u(x)\rangle = 0, x \in U \\ u|_{S\cap U} = \varphi \end{cases} \tag{2.1.8}$$

possiamo usare le curve caratteristiche ponendo ipotesi

(H1) per $\forall\ \alpha \in S \cap U$ la superficie S e trasversale alla curva $y(s;\alpha)$

L'ipotesi (H1) é cruciale e significa che il vettore $a(\alpha)$ non é tangente alla S.

Visto che cerchiamo soluzioni locali (solo in U) possiamo supporre che S e definita in U con l'equazione

$$f(x) = 0.$$

Lemma 2.1.1 suggerisce a cercare la soluzione di (2.1.8) nella forma

$$u(y(s,\alpha)) = \varphi(\alpha), \alpha \in S.$$

Usando il teorema della funzione inversa si puo vedere che

Lemma 2.1.2. *L'ipotesi* (H1) *implica che la mappa*

$$s \in (-\varepsilon, \varepsilon), \alpha \in S \cap U \ \rightarrow \ x = y(s, \alpha)$$

sia localmente invertibile.

Proof. Il fatto che l'equazione di trasporto rimane equazione di trasporto dopo cambiamento di variabili (2.1.2) ci premette a supporre S definita localmente con $x_n = 0$. L'ipotesi di trasversalita implica $a_n(\alpha) \neq 0$ e

$$\alpha \in S \iff \alpha_n = 0.$$

Quindi dobbiamo vedere che la mappa

$$s \in (-\varepsilon, \varepsilon), \alpha' = (\alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{R}^{n-1} \to x = y(s; \alpha', 0)$$

e invertibile. La sua matrice Jacobiana e

$$\begin{pmatrix} I_{n-1} & 0 \\ a'(\alpha) & a_n(\alpha) \end{pmatrix}$$

dove

$$a' = (a_1, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$$

Ovviamente $a_n(\alpha) \neq 0$ implica invertibilita. □

2.2 Hamilton - Jacobi equation

L'equazione di Hamilton – Jacobi é un'equazione differenziale alle derivate parziali non lineare del primo ordine che ha la forma

$$H(x, \nabla_x S) = 0, x \in \mathbb{R}^n. \tag{2.2.9}$$

We are trying to solve this equation imposing

$$S(y) = \alpha(y), \quad y \in \Sigma, \tag{2.2.10}$$

where Σ is a surface in$\mathbb{R}^n$.

The Hamiltonian $H : \mathbb{R}^n_{\mathrm{p}} \times \mathbb{R}^n_{\mathrm{x}} \to \mathbb{R}$, enables us to define the bicharacteristic curves as solutions to the Hamiltonian equations

$$\begin{cases} \dot{\xi}(s) = -\nabla_{\mathbf{x}} H(\xi(s), \mathbf{x}(s)) \\ \dot{\mathbf{x}}(s) = \nabla_{\xi} H(\xi(s), \mathbf{x}(s)) \end{cases} \tag{2.2.11}$$

since H is given, this is an system of $2n$ ordinary differential equations, with $\xi(s)$ and $\mathbf{x}(s)$ as the unknowns. If initial data is given (for example, if $\xi(0) = \xi_0$ and $\mathbf{x}(0) = \mathbf{x}_0$), then the ODE can be solved and we can describe the motion of the system. We shall denote by $\rho = (x,\xi)$ the points in $T^*(\mathrm{R}^n) = R^{2n}$. Then

$$\rho(t) = \rho(s,\rho_0),\ \rho_0 = (x_0,\xi_0)$$

is the solution to the system (2.2.11) of the bicharacteristics.

Example 2.2.1. *If*

$$H(x,\xi) = \frac{1}{2}|\xi|^2,\ \xi = (\xi_1,\cdots,\xi_n) \in \mathrm{R}^n, n \geq 2,$$

then

$$\xi(s) = \xi_0, x(s) = x_0 + \xi_0 s.$$

Example 2.2.2. *If*

$$H(t,x,\tau,\xi) = \tau - c|\xi|^2,\ \xi = (\xi_1,\cdots,\xi_n) \in \mathrm{R}^n, n \geq 2,$$

then

$$\tau(s) = \tau_0, \xi(s) = \xi_0, t(s) = t_0 + s,\ x(s) = x_0 - 2c\xi_0 s,\ \ \tau_0 = c|\xi_0|^2.$$

These bi characteristic shall be used to solve the propblem

$$\begin{aligned} &\partial_t S - c(\partial_x S)^2 = 0 \\ &S(0,y) = \alpha(y). \end{aligned} \tag{2.2.12}$$

Note that characteristics$(t(s),x(s))$ *are usually the projection of the bicharacteristics on the* t,x *space.*

Example 2.2.3. *If we use rectangular coordinates, the Hamiltonian for a particle of mass* m *in a force field is*

$$H(\xi,\mathbf{x}) = \frac{1}{2m}|\xi|^2 + V(\mathbf{x})$$

where $V : \mathbb{R}^n_x \to \mathbb{R}$ *is the potential energy. Then equations* (2.2.11) *reduce to* $\dot{\xi} = -\nabla_x V(\mathbf{x})$ *and* $\dot{\mathrm{x}} = \frac{1}{m}\xi$. *The first equation is a statement of Newton's second law* $\mathbf{F} = ma$. *The second equation relates the classical position and momentum vectors.*

Remark 2.2.1. *Observe that in Example 2.2.3, the Hamiltonian is equal to the total energy of the system. This is no coincidence. To see why, fix a Hamiltonian* $H : \mathbb{R}^n_{\mathrm{p}} \times \mathbb{R}^n_x \to \mathbb{R}$, *and let* $(\mathrm{p}(t),\mathrm{x}(t))$ *be a solution to Hamilton's equations. Then set* $E(t) = H(\xi(t),\mathrm{x}(t))$. *We have*

$$\dot{E}(t) = \nabla_\xi H(\xi(t),\mathbf{x}(t)) \cdot \dot{\xi}(t) + \nabla_{\mathbf{x}} H(\xi(t),\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) = 0 \tag{2.2.13}$$

where the equality on the right is due to (2.2.11). *Thus, the quantity* $E(t)$ *is indeed conserved, and the Hamiltonian admits an interpretation as energy.*

To be specific, if S is a solution of (2.2.9) then the graph of the gradient (or differential) of the function S

$$\Gamma = \{(x, S_x(x))\}$$

(where $S_x = \partial S/\partial x$) is invariant with respect to the flow defined by the system (2.2.11), since if $(x(t), \xi(t))$ is a solution of (2.2.11) and $\xi(0) = S_x(x(0))$ then

$$\begin{aligned}\frac{d}{dt}(\xi(t) - S_x(x(t))) &= \dot{\xi}(t) - S_{xx}(x(t))\dot{x}(t) \\ &= -H_x(x(t)) - S_{xx}(x(t))H_\xi(x(t), S_x(x(t))) \\ &= -\frac{d}{dx}H(x, S_x(x))\Big|_{x=x(t)} = 0.\end{aligned}$$

Lemma 2.2.1. *If $S(x)$ is a solution to the Hamilton - Jacobi equation* (2.2.9) *and $\rho(t) = (x(t), \xi(t))$, $t \in I$ is a bi characteristic in the interval I, then*

$$\frac{d}{dt}(\xi(t) - S_x(x(t))) = 0, \ \forall t \in I.$$

Example 2.2.4. *Turning to the example 2.2.2, we can apply the above Lemma and deduce*

$$\xi_0 = \nabla_x S(x_0) \implies \xi_0 = \nabla_x S(x_0 - 2c\xi_0 s), \ \forall s. \tag{2.2.14}$$

We note that in the case of most importance for us, when the function $H = H(x, \xi)$ is positive-homogeneous in ξ (of any degree m), the solution $S = S(x)$ of the Hamilton-Jacobi equation (3.33) is constant along the projections $x(t)$ of the trajectories $(x(t), \xi(t))$ of the Hamiltonian system (2.2.11) lying on the graph of the gradient of the function S, since by Euler's identity

$$\frac{d}{dt}S(x(t)) = S_x(x(t)) \cdot \dot{x}(t) = \xi(t) \cdot H_\xi(x(t)) = mH(x(t), \xi(t)) = 0$$

Lemma 2.2.2. *If $S(x)$ is a solution to the Hamilton - Jacobi equation* (2.2.9)*, $H(x, \xi)$ is homogeneous of order m in ξ and $\rho(s) = (x(s), \xi(s))$, $t \in I$ is a bi characteristic in the interval I, then*

$$\frac{d}{ds}S(x(s)) = 0, \ \forall s \in I.$$

Now we turn to the general solution of the Hamilton Jacobi equation, knowing the Hamltonian flow $\rho = \rho(t, \rho_0)$.

First we consider the surface $\Sigma \subset \mathbb{R}^n$ that is transversal to $H(x, \xi)$ at point $(x, \xi) \in \Sigma \times \mathbb{R}^n$, such that

$$H(x, \xi) = 0. \tag{2.2.15}$$

Definition 2.2.1. *We shall say that* $\Sigma \subset \mathbb{R}^n$ *that is transversal to* H *at the point* $(x,\xi) \in \Sigma \times \mathbb{R}^n$ *satisfying* (2.2.17) *if*

$$N(x).\nabla_\xi H(x,\xi) \neq 0 \tag{2.2.16}$$

with $N(x)$ *being the unit normal vector at* $x \in \Sigma$.

One can easily see that if $\Sigma \subset \mathbb{R}^n$ that is transversal to H at the point $(\widetilde{x},\widetilde{\xi}) \in \Sigma \times \mathbb{R}^n$ satisfying

$$H(\widetilde{x},\widetilde{\xi}) = 0, \tag{2.2.17}$$

then the map defined through Hamiltonian flow

$$t \in (-\varepsilon,\varepsilon) \times y \in \Sigma \cap \{|y-\widetilde{x}| \le \varepsilon\} \to x = x(t,y,\eta) \in \mathbb{R}^n \tag{2.2.18}$$

is invertible for $\eta \in R^n$ close to $\widetilde{\xi}$.

Indeed, the proof is similar to the proof of Lemma 2.1.2 and we omit it.

Our next step is to choose appropriate initial data

$$x(0) = y, \;\; \xi(0) = \eta$$

for the Hamiltonian flow (2.2.11) that is compatible with the data (2.2.10) on the surface Σ.

First we choose

$$x(0) = y \in \Sigma. \tag{2.2.19}$$

Further, we note that for any $y \in \Sigma$ and for any vector $\eta \in R^n$ we may use decomposition of η into tangential and normal parts respectively

$$\eta = \eta^{(\tau)} + \eta^{(\nu)} N(y), \;\; \eta^{(\nu)} = \eta.N(y), \tag{2.2.20}$$

where $N(y)$ being the unit normal vector at $y \in \Sigma$. From the data (2.2.10) on the surface we have

$$S(y) = \alpha(y), \;\; y \in \Sigma, \tag{2.2.21}$$

so taking into account Lemma 2.2.1 we can make the tangential projections of

$$\xi(t) - \nabla_x S(x(t))$$

at the point $x(0) = y \in \Sigma$ and find

$$\eta^{(\tau)} - \nabla_y \alpha(y) = 0. \tag{2.2.22}$$

To define the normal part $\eta^{(\nu)}$ of η we impose the condition

$$H(y,\eta)=0. \tag{2.2.23}$$

One can solve this equation locally near a point $y_0 \in \Sigma$, $\eta_0 \in \mathbb{R}^n$, such that

$$\eta_0^{(\tau)} - \nabla_y \alpha(y_0) = 0 \tag{2.2.24}$$

and

$$H(y_0,\eta_0)=0. \tag{2.2.25}$$

imposing the transversality of Σ at (y_0,η_0) by using the implicit function theorem.

Indeed, if Σ is defined locally by $x_n=0$, then we use $y=(y_1,\cdots,y_{n-1})$ as parametrization of Σ. For any $\eta\in\mathbb{R}^n$ we see that its tangential part is

$$\eta^{(\tau)} = (\eta_1,\cdots,\eta_{n-1},0)$$

while the normal part is η_n. Let

$$(y_0,\eta_0),\, y_0=0,\;\; \eta_0=(\eta_1^{(0)},\cdots,\eta_{n-1}^{(0)},\eta_n^{(0)})$$

are chosen so that (2.2.24) and (2.2.23) hold, i.e.

$$\eta_j^{(0)} - \partial_{y_j}\alpha(0)=0,\;\; j=1,\cdots,n-1 \tag{2.2.26}$$

and

$$H(0,\eta_0)=0. \tag{2.2.27}$$

Then to solve (2.2.24) and (2.2.23) with $(y,\eta)\in\Sigma\times\mathbb{R}^n$ close to $(0,\eta_0)$ we have to find only η_n that solves

$$H(y,\nabla_y\alpha,\eta_n)=0.$$

The transversality condition at $(0,\eta_0)$ reads as

$$\partial_{\xi_n} H(0,\eta_0)\neq 0,$$

so implicit function theorem guarantees local solvability of

$$H(y,\nabla_y\alpha,\eta_n)=0$$

with respect to η_n.

Next, we define $z(t,y,\eta)$ to be the solution to

$$\begin{cases} z' = \xi(t).H_\xi(x(t),\xi(t)) \\ z(0)=\alpha(y). \end{cases} \tag{2.2.28}$$

Lemma 2.2.3. *If*

$$x = x(t, y, \eta), \xi = \xi(t, y, \eta)$$

is the Hamiltonian flow determined by (2.2.11) *with initial data* $(x(0), \xi(0)) = (y, \eta)$*, satisfying* (2.2.22) *and* (2.2.23)*, then if* $S(x) = z(t(x), y(x))$*, where* $(y(x, \eta), t(x, \eta))$ *is the inverse of* (2.2.18) *and z is a solution of* (7.0.43)*, then we have the properties*

a)

$$\xi(t, y, \eta) = \nabla_x S(x(t, y, \eta)). \tag{2.2.29}$$

b) *the function* $S(x)$ *is a solution to the Hamilton - Jacobi equation*

$$H(x, \nabla_x S(x)) = 0.$$

Proof. We have the relations

$$\frac{d}{dt} S(x(t, y, \eta)) = \frac{d}{dt} z(t, y) = \nabla_\xi H(x(t, y, \eta), \xi(t, y, \eta)).\xi(t, y, \eta) = x'(t, y, \eta)).\xi(t, y, \eta)$$

and since

$$\frac{d}{dt} S(x(t, y, \eta)) = \nabla_x S(x(t, y, \eta)).x'(t, y, \eta)),$$

we conclude that

$$\left(\nabla_x S(x(t, y, \eta)) - \xi(t, y, \eta)\right).x'(t, y, \eta)) = 0. \tag{2.2.30}$$

Without loss of generality we can assume that $\Sigma : x_n = 0$ and then we shall show

$$\left(\nabla_x S(x(t, y, \eta)) - \xi(t, y, \eta)\right).\partial_{y_j} x(t, y, \eta)) = 0, \ j1, \cdots, (n-1)). \tag{2.2.31}$$

Indeed, we have the relations

$$\partial_{y_j} S(x(t, y, \eta)) = \partial_{y_j} z(t, y)$$

and

$$\partial_{y_j} S(x(t, y, \eta)) = \nabla_x S(x(t, y, \eta)).\partial_{y_j} x(t, y, \eta)),$$

therefore we need to show that

$$\partial_{y_j} z(t, y) = \xi(t, y, \eta).\partial_{y_j} x(t, y, \eta). \tag{2.2.32}$$

For the purpose we use the relations

$$\partial_{y_j} z'(t, y) = \nabla^2_{\xi, x} H(x(t, y, \eta), \xi(t, y, \eta))(\partial_{y_j} x(t, y, \eta)).\xi(t, y, \eta) +$$

$$+\nabla^2_{\xi,\xi}H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}\xi(t,y,\eta)).\xi(t,y,\eta)+$$
$$+\nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi'(t,y,\eta)).$$

Moreover, we have

$$\frac{d}{dt}\Big(\xi(t,y,\eta).\partial_{y_j}x(t,y,\eta)\Big) = -\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).\partial_{y_j}x(t,y,\eta)+$$
$$+\xi(t,y,\eta).\partial_{y_j}x'(t,y,\eta) = -\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).\partial_{y_j}x(t,y,\eta)+$$
$$+\nabla^2_{\xi,x}H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}x(t,y,\eta)).\xi(t,y,\eta)+$$
$$+\nabla^2_{\xi,\xi}H(x(t,y,\eta),\xi(t,y,\eta))(\partial_{y_j}\xi(t,y,\eta)).\xi(t,y,\eta)$$

and we arrive at

$$\frac{d}{dt}\Big(\partial_{y_j}z(t,y)-\Big(\xi(t,y,\eta).\partial_{y_j}x(t,y,\eta)\Big)\Big)=$$
$$=\nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi(t,y,\eta))+\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}x(t,y,\eta)).$$

Since we are working on null bi characteristics $H(x(t,y,\eta),\xi(t,y,\eta)) = 0$ so we have also

$$\nabla_x H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}x(t,y,\eta))+\nabla_\xi H(x(t,y,\eta),\xi(t,y,\eta)).(\partial_{y_j}\xi(t,y,\eta)) = 0$$

so we arrive at

$$\frac{d}{dt}\Big(\partial_{y_j}z(t,y)-\Big(\xi(t,y,\eta).\partial_{y_j}x(t,y,\eta)\Big)\Big)=0.$$

For $t = 0$ we have

$$\partial_{y_j}z(0,y)-\xi(0,y,\eta).\partial_{y_j}x(0,y,\eta)=0$$

so we arrive at (2.2.32). This relation implies (2.2.31) and combining with (2.2.30), we find (2.2.29). Therefore we have the property a). The property b) follows from a) and (2.2.13). □

Example 2.2.5. *We continue to study the Example 2.2.2. Recall that the null bi -characteristics are defined by*

$$\begin{aligned}\tau(s)=\tau_0,\xi(s)=\xi_0,\ \ \tau_0=c|\xi_0|^2,\\ t(s)=t_0+s,\ x(s)=x_0-2c\xi_0 s,\end{aligned} \tag{2.2.33}$$

We want to solve the problem (2.2.12)*, i.e.*

$$\begin{aligned}\partial_t S - c(\partial_x S)^2 = 0\\ S(0,y)=\alpha(y).\end{aligned}$$

so we take $x_0 = y$, $t_0 = 0$ *with* $y \in \mathbb{R}$.
Note that the relation (2.2.29) *with* $t = 0$ *implies*

$$\xi_0 = \nabla_y \alpha(y). \tag{2.2.34}$$

As in (7.0.43) *we define the function*

(2.2.35)
$$\begin{cases} z'(s) = \xi(s, y, \xi_0).H_\xi(t(s), x(s), \tau(s), \xi(s)) + \tau(s) H_\tau(t(s), x(s), \tau(s), \xi(s)) \\ z(0) = \alpha(y) \end{cases}$$

From (2.2.33) *we find*

$$\xi(s, y, \xi_0).H_\xi(t(s), x(s), \tau(s), \xi(s)) + \tau(s) H_\tau(t(s), x(s), \tau(s), \xi(s)) = -2c|\xi_0|^2 + \tau_0 = -c|\xi_0|^2.$$

Hence

$$z(s, y, \xi_0) = \alpha(y) - sc|\xi_0|^2.$$

Then

$$S(t(s, y, \xi_0), x(s, y, \xi_0)) = z(s, y, \xi_0)$$

shall be the solution to our problem so

$$S(s, y - 2c\xi_0 s) = \alpha(y) - sc|\xi_0|^2.$$

We have to make change of variables

$$t = s,\ x = y - 2c\nabla_y \alpha(y) s$$

and we get

$$S(t, x) = \alpha(y(t, x)) - tc|\nabla_y \alpha(y(t, x))|^2.$$

Moreover, from (2.2.29) *we have*

$$\nabla_x S(t(s, y, \xi_0), x(s, y, \xi_0)) = \xi(s, y, \xi_0) = \xi_0 = \nabla_y \alpha(y(t, x)).$$

Therefore

$$\nabla_x S(s, y - 2cs\nabla_y \alpha(y)) = \nabla_y \alpha(y). \tag{2.2.36}$$

Example 2.2.6. *For given initial data* $g(x)$, *we consider a classical solution to the* PDE

$$\partial_t u + \frac{1}{2}(\partial_x u)^2 = 0 \quad on\, \mathbb{R}_x \times (0, \infty)_t \tag{2.2.37}$$

$$u = g \quad on\, \mathbb{R}_x \times \{0\}_t$$

(By classical, we simply mean that u is differentiable, so that the PDE can be satisfied everywhere on $\mathbb{R}_x \times (0,\infty)_t$ (2.2.37). *Characteristic lines. Suppose u is a classical solution to* (2.2.37). *Furthermore, suppose u is twice differentiable in the variable x. since u is differentiable, we can let* $v(x,t) = \partial_x u(x,t)$. *Then v satisfies the equation*

$$\partial_t v + v \cdot \partial_x v = 0 \quad on\, \mathbb{R}_x \times (0,\infty)_t \tag{2.2.38}$$

with initial data

$$v = h \quad on\, \mathbb{R}_x \times \{0\}_t$$

where $h(x) := g'(x)$ *The PDE* (2.2.38) *is known as the inviscid Burgers ' equation and plays an important role in fluid mechanics. We have the following property for solutions of Burgers' equation.*

Lemma 2.2.4. *If v is solution to* (2.2.38), *then v is a constant along the lines* $x = x_0 + th(x_0)$ *for each* $x_0 \in \mathbb{R}$.

Example 2.2.7. Non-existence of a classical solution.

Suppose we have initial data $g(x) = -|x|$ *to the Hamilton-Jacobi equation* (2.2.37). *Then v has initial data*

$$h(x) = \begin{cases} 1, & if\, x < 0 \\ -1, & if\, x > 0 \end{cases}$$

However, we immediately see a problem with our given initial data. Take, for example, the characteristic lines $x = 1 - t$ *and* $x = -1 + t$ *must be constant along these two lines. This is fine for all* $t < 1$, *but the two characteristics intersect at* $(x,t) = (0,1)$. *(This intersection is called a shock. since the values of v along the two lines are different, there cannot be a differentiable solution).*

2.3 Symplectic manifolds

Symplectic manifold is a smooth manifold, M , equipped with a closed nondegenerate differential 2-form ω called the symplectic form.

Example 2.3.1. *Let* $\{v_1, \ldots, v_{2n}\}$ *be a basis for* R^{2n} *We define our symplectic form* ω *on this basis as follows:*

$$\omega(v_i, v_j) = \begin{cases} 1 & j - i = n \text{ with } 1 \leqslant i \leqslant n \\ -1 & i - j = n \text{ with } 1 \leqslant j \leqslant n \\ 0 & otherwise \end{cases}$$

In this case the symplectic form reduces to a simple quadratic form. If I_n denotes the $n\times n$ identity matrix then the matrix, Ω of this quadratic form is given by the $2n\times 2n$ block matrix:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Example 2.3.2. *If (x,ξ) are local coordinates, then one such form is*

$$\omega = dx\wedge d\xi.$$

So if we have vector field

$$v_1 = \sum_{j=1}^{n} a_j^{(1)}\partial_{x_j} + b_j^{(1)}\partial_{\xi_j},\ v_2 = \sum_{j=1}^{n} a_j^{(2)}\partial_{x_j} + b^{(2)})_j\partial_{\xi_j},$$

then

$$\omega(v_1,v_2) = a^{(1)}.b^{(2)} - a^{(2)}.b^{(1)},$$

where

$$a^{(1)} = (a_1^{(1)},a_2^{(1)},\cdots,a_n^{(1)}), a^{(2)} = (a_1^{(2)},a_2^{(2)},\cdots,a_n^{(2)}),$$
$$b^{(1)} = (b_1^{(1)},b_2^{(1)},\cdots,b_n^{(1)}), b^{(2)} = (b_1^{(2)},b_2^{(2)},\cdots,b_n^{(2)}).$$

and $a.b$ denotes the scalar product of the vectors a,b.

The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds. For example, in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field, the set of all possible configurations of a system is modeled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

Chapter 3

1 - D Maximum principle

3.1 Introduction

Let the operator L is defined by

$$L(u)(x) = -u'' + V(x)u'(x).$$

The we can consider the Sturm problem

$$L(u)(x) = f(x), a < x < b. \tag{3.1.1}$$

Here and below $V(x)$ is $C[a,b]$ function. Given any $f \in C(a,b)$ we look for classical solutions to (3.1.1) $u \in C[a,b] \cap C^2(a,b)$.

3.2 Easy and weak maximum principles

Lemma 3.2.1. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (3.1.1), then we have the following properties*

a) *(EASY MIN principle) if $f(x)$ is continuous POSITIVE function ($f(x) > 0$ for any $x \in (a,b)$), then*

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

b) *(EASY MAX principle) if $f(x)$ is continuous NEGATIVE function ($f(x) > 0$ for any $x \in (a,b)$), then*

$$\max(u(a), u(b)) = \max_{[a,b]} u(x).$$

Proof. We shall prove b) only. If

$$u(c) = \max_{[a,b]} u(x)$$

for some $c \in (a,b)$, then in the point c we have

$$u'(c) = 0, u''(c) \leq 0.$$

□

Lemma 3.2.2. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (3.1.1), then we have the following properties*

a) *(WEAK MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in (a,b)$), then*

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

b) *(WEAK MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in (a,b)$), then*

$$\max(u(a), u(b)) = \max_{[a,b]} u(x).$$

Remark 3.2.1. *A function satisfying $L(u)(x) \leq 0$ is called a subsolution. We are thus asserting a subsolution attains its maximum on the boundary of $[a,b]$. Similarly, if $L(u)(x) \leq 0$ holds, u is a supersolution and attains its minimum on the boundary of $[a,b]$.*

Proof. We shall prove only the assertion a). We shall modify $u(x)$ as follows

$$w_\varepsilon(x) = u(x) - \varepsilon z(x),$$

where

$$z(x) = e^{Rx}.$$

Then w_ε tends uniformly to u, so $\min_{[x_1,x_2]} w_\varepsilon(x)$ tends to $\min_{[x_1,x_2]} u(x)$ for any interval $[x_1, x_2] \subseteq [a,b]$.

Choose $R > 0$ so large that

$$L(w_\varepsilon) \geq -\varepsilon z'' + \varepsilon V(x) z'(x) = \varepsilon(R^2 - VR)e^{Rx} > 0$$

for any $x \in (a,b)$. Then the Easy MIN principle implies

$$\min(w_\varepsilon(a), w_\varepsilon(b)) = \min_{[a,b]} w_\varepsilon(x).$$

Taking the limit as $\varepsilon > 0$ tends to zero we get

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

□

3.3 Hopf lemma and strong maximum principle

In order to prepare strong maximum principle we shall start with Hopf Lemma

Lemma 3.3.1. *(Hopf's Lemma). Assume* $u \in C^2(a,b)) \cap C^1([a,b])$, *and*

$$Lu = f \le 0 \quad \textit{in } (a,b).$$

Then we have

a) *if*

$$u(b) > u(x) \quad \textit{for all } x \in (a,b) \tag{3.3.2}$$

then

$$u'(b) > 0.$$

b) *if*

$$u(a) > u(x) \quad \textit{for all } x \in (a,b) \tag{3.3.3}$$

then

$$u'(a) < 0.$$

Proof. We shall prove only a). Without loss of generality we can assume

$$a < 0 < b.$$

Then we take

$$w_\varepsilon(x) = u(x) + \varepsilon z(x),$$

where

$$z(x) = e^{-Rx^2} - e^{-Rb^2}$$

Then $z(b) = 0$ and

$$L(z)(x) = e^{-Rx^2}\left(-4R^2x^2 + 2R - VxR\right) < 0$$

for R big enough and $x \in (b/2, b)$. In view of (3.3.3) we can find $\varepsilon > 0$ so that

$$w_\varepsilon(x_0) = u(x_0) > u(b/2) + \varepsilon z(b/2) = w_\varepsilon(b/2).$$

We have also

$$w_\varepsilon(x_0) = u(x_0) \ge u(b) + \varepsilon z(b) = w_\varepsilon(b).$$

Applying the weak maximum principle for $w_\varepsilon(x) - w_\varepsilon(b)$ and the interval $[b/2, b]$ and we find

$$\max_{[b/2,b]} w_\varepsilon - w_\varepsilon(b) \le 0$$

and hence

$$w'_\varepsilon(b) \ge 0.$$

In this way we find

$$u'(b) \ge -\varepsilon z'(b) = \varepsilon R b e^{-Rb^2} > 0.$$

□

Lemma 3.3.2. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (3.1.1), then we have the following properties*

a) *(STRONG MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \ge 0$ for any $x \in (a,b)$) and f has minimum in internal point $x_0 \in (a,b)$, then is a constant;*

b) *(STRONG MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \le 0$ for any $x \in (a,b)$)and f has maximum in internal point $x_0 \in (a,b)$, then is a constant.*

Proof. We shall prove only b). Set

$$M := \max_{[a,b]} u$$

and assuming u is not a constant we can decompose (a,b) as

$$(a,b) = C \cup V,$$

where

$$C := \{x \in (a,b) \mid u(x) = M\},$$

$$V := \{x \in (a,b) \mid u(x) < M\}.$$

Since V is an open set, it is a union of open intervals and we can take such interval $(\alpha, \beta) \subset V$ so that $\beta \in C$. In this way we can apply Hopf lemma and deduce $u'(\beta) > 0$ and this is a contradiction with the fact that $\beta \in (a,b)$ is maximum point. The contradiction shows that V is empty. □

Problem 3.3.1. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of*

(3.3.4) $$u'' + V(x)u'(x) + W(x)u(x) = f(x), a < x < b,$$

$V(x)$ and $W(x)$ are $C[a,b]$ functions, $W(x) \le 0$, $f(x)$ is any bounded NON-NEGATIVE function and if

$$u(c) = \max_{[a,b]} u(x)$$

is POSITIVE for some $c \in (a,b)$, then $u(x)$ is a constant.

Problem 3.3.2. *Show that the condition* $W(x) \leq 0$ *in the Problem (3.3.1) can not be removed.*

Problem 3.3.3. *Show that the condition* $u(c)$ *is POSITIVE in the Problem (3.3.1) can not be removed.*

Problem 3.3.4. *If* $u \in C[a,b] \cap C^2(a,b)$ *is a solution of (4.1.1)* $V(x)$ *and* $W(x)$ *are bounded functions,* $W(x) \leq 0$, $f(x)$ *is any bounded NON-NEGATIVE function and if*

$$u(a) = u(b) = 0,$$

then $u(x) < 0$ *in* (a,b) *or* $u(x) = 0$.

Chapter 4

Maximum principle in domains

4.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain with boundary $\partial\Omega$.

Let the operator L be defined by

$$L(u)(x) = -\Delta u(x) + \sum_{j=1}^{n} V_j(x)\partial_{x_j} u(x).$$

Then we can consider the problem

$$L(u)(x) = f(x), x \in \Omega. \tag{4.1.1}$$

Here and below $V_j(x)$ are $C(\overline{\Omega})$ function. Given any $f \in C(\overline{\Omega})$ we look for classical solutions to (4.1.1) $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

For simplicity we shall concentrate in this chapter to the case $n = 3$.

4.2 Easy and weak maximum principles

Lemma 4.2.1. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (4.1.1), then we have the following properties*

a) *(EASY MIN principle) if $f(x)$ is continuous POSITIVE function ($f(x) > 0$ for any $x \in \Omega$), then*

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

b) *(EASY MAX principle) if $f(x)$ is continuous NEGATIVE function ($f(x) > 0$ for any $x \in \Omega$), then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u(x).$$

Proof. We shall prove b) only. If

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

for some $x_0 \in \Omega$, then in the point x_0 we have

$$\nabla u(x_0) = 0, \Delta u(x_0) \leq 0.$$

□

Lemma 4.2.2. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (4.1.1), then we have the following properties*

a) *(WEAK MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in \Omega$), then*

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

b) *(WEAK MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in \Omega$), then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u.$$

Remark 4.2.1. *A function satisfying $L(u)(x) \leq 0$ is called a subsolution. We are thus asserting a subsolution attains its maximum on the boundary of Ω. Similarly, if $L(u)(x) \leq 0$ holds, u is a supersolution and attains its minimum on the boundary of Ω.*

Proof. We shall prove only the assertion a). We shall modify $u(x)$ as follows

$$w_\varepsilon(x) = u(x) - \varepsilon z(x),$$

where

$$z(x) = e^{Rx_1}.$$

Then w_ε tends uniformly to u, so $\min_{\overline{U}} w_\varepsilon(x)$ tends to $\min_{\overline{U}} u(x)$ for any open $U \subseteq \Omega$.

Choose $R > 0$ so large that

$$L(w_\varepsilon) \geq -\varepsilon\Delta z + \varepsilon \sum_{j=1}^{n} V_j(x)\partial_{x_j} z(x) = \varepsilon(R^2 - VR)e^{Rx_1} > 0$$

for any $x \in \Omega$. Then the Easy MIN principle implies

$$\min_{\partial\Omega} w_\varepsilon = \min_{\overline{\Omega}} w_\varepsilon(x).$$

Taking the limit as $\varepsilon > 0$ tends to zero we get

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

□

4.3 Hopf lemma and strong maximum principle

In order to prepare strong maximum principle we shall start with Hopf's Lemma. For this we shall assume the boundary of the domain Ω satisfies the ball condition, i.e.

$$\text{(H1)} \qquad \begin{cases} \text{for any } x_0 \in \partial\Omega \text{ there exists a ball } B \subset \Omega \\ \text{so that } x_0 \in \partial B \end{cases}$$

Lemma 4.3.1. *(Hopf's Lemma). Assume* (H1), $u \in C^2(\Omega)) \cap C^1(\overline{\Omega})$, *and*

$$Lu = f \leq 0 \quad in\, \Omega.$$

If $x_0 \in \partial\Omega$ *is such that*

$$u(x_0) > u(x) \quad \textit{for all } x \in \Omega \tag{4.3.2}$$

then

$$\partial_\nu u(x_0) > 0,$$

where $\nu(x)$ *is the exterior unit normal at* $x \in \partial\Omega$.

Proof. Without loss of generality we can assume the ball in the assumption (H1) is $B(0,r)$, so that $|x_0| = r$ and

$$\nu(x_0) = x_0/r.$$

Then we take

$$w_\varepsilon(x) = u(x) + \varepsilon z(x),$$

where

$$z(x) = e^{-Rx^2} - e^{-Rr^2}$$

Then $z(x_0) = 0$ and

$$L(z)(x) = e^{-Rx^2}\left(-4R^2x^2 + 2nR - \sum_{j=1}^{n} 2V_j x_j R\right) < 0$$

for R big enough and $x \in B(0,r) \setminus B(0,r/2)$. In view of (4.3.2) we can find $\varepsilon > 0$ so that

$$w_\varepsilon(x_0) = u(x_0) > u(x) + \varepsilon z(x) = w_\varepsilon(x) \;\; |x| = r/2.$$

We have also

$$w_\varepsilon(x_0) = u(x_0) \geq u(x) + \varepsilon z(x) = w_\varepsilon(x), \;\; |x| = r.$$

Applyi ng the weak maximum principle for $w_\varepsilon(x) - w_\varepsilon(b)$ and the domain $B(0,r)\setminus \overline{B(0,r/2)}$ we find

$$\max_{\overline{B(0,r)}\setminus B(0,r/2)} w_\varepsilon - w_\varepsilon(b) \leq 0$$

and hence

$$\partial_\nu w_\varepsilon(x_0) \geq 0.$$

In this way we find

$$\partial_\nu u(x_0) \geq -\varepsilon \partial_\nu z(x_0) = 2\varepsilon R r e^{-Rr^2} > 0.$$

□

Lemma 4.3.2. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (4.1.1), then we have the following properties*

a) *(STRONG MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in \Omega$) and f has minimum in internal point $x_0 \in \Omega$, then it is a constant;*

b) *(STRONG MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in \Omega$)and f has maximum in internal point $x_0 \in \Omega$, then it is a constant.*

Proof. We shall prove only b). Set

$$M := \max_{\overline{\Omega}} u$$

and assuming u is not a constant we can decompose Ω as

$$\Omega = C \cup V,$$

where

$$C := \{x \in \Omega \mid u(x) = M\},$$

$$V := \{x \in \Omega \mid u(x) < M\}.$$

This means we can find two points $P \in C$ and $Q \in V$ and connect them with an arc in Ω. Let R be the closest point on the arc to Q such that $u(R) = M$ and $u(x) < M$ for all x on the arc between Q and R. Since R is in Ω, we can find y sufficiently close to R on the arc QR so that

$$y \in V, \; d(y,R) < d(y,\partial\Omega).$$

Then we can define the largest open ball $B(y,r) \subset V$ so that $\partial B(y,r)$ has a point $x_* \in C$. In this way we can apply for $B(y,r)$ the Hopf lemma and deduce $\partial_\nu u(x_*) > 0$ and this is a contradiction with the fact u has a maximum in x_*. The contradiction shows that V is empty. □

4.4 Maximum principle for Laplace operator with Coulomb potential

A natural question is to ask if the linear operator

$$P_\omega = -\Delta - \frac{1}{|x|} + \omega,$$

satisfies the weak maximum principle in the sense that

$$u \in H^2, \ P_\omega(u) = g \geq 0, \Longrightarrow u \geq 0. \tag{4.4.3}$$

The above maximum principle is incomplete, since additional behavior of u and g at infinity has to be imposed, namely, we shall suppose that

$$(1+|x|)^{-M} e^{\sqrt{\omega}|x|} u \in H^2, \quad (1+|x|)^{-M} e^{\sqrt{\omega}|x|} g \in H^2, \tag{4.4.4}$$

for some real number $M > 0$.

Note, that the energy levels of the hydrogen atom are described by the eigenvalues $\omega_k > 0$ of the eigenvalue problem

$$\Delta e_k(x) + \frac{e_k(x)}{|x|} = \omega_k e_k(x), \ \ e_k(x) \in H^2.$$

One has

$$\omega_k = \frac{1}{4(k+1)^2}, \quad k = 0, 1, \ldots$$

and $e_0(x) = ce^{-|x|/2}, c > 0$. The first observation is that all eigenfunctions $e_k(x)$, $k \geq 1$, are expressed in terms of Laguerre polynomials of $|x|$, having exactly k roots. This fact guarantees that the maximum principle is not valid for $\omega = \omega_k$. More precisely, we can show the following.

Lemma 4.4.1. *The weak maximum principle 4.4.3 is valid if an only if*

$$\omega \geq \frac{1}{4}.$$

4.5 Appendix: Maximum principle for subharmonic functions.

4.5.1 Mean value theorem. Harmonic functions

The Gauss Green identity

$$\int_\Omega (\Delta u v - u \Delta v)\, dy = \int_{\partial\Omega} (\partial_N u v - u \partial_N v)\, dS_y,$$

enables one to take $v(y) = 1/|x-y|$, where $x \in \Omega$ and modify the domain Ω as follows

$$\Omega_\delta = \{y : y \in \Omega, |y-x| \geq \delta\}$$

provided $\delta > 0$ is small.

Taking the limit as $\delta \to 0$ we get

Lemma 4.5.1. *(integral representation) If $u \in C^2(\mathbb{R}^n)$ then*

$$u(x) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{\partial_N u}{|x-y|} - u\partial_N\left(\frac{1}{|x-y|}\right)\right) dS_y$$
$$-\frac{1}{4\pi}\int_\Omega \frac{\Delta u}{|x-y|} dy.$$

In the particular case $\Omega = \{|x-x_0| < R\}$ we get

Lemma 4.5.2. *(Gauss mean value theorem) If $u \in C^2$ is a harmonic in $\{|x-x_0| < R\}$, then*

$$u(x_0) = \frac{1}{4\pi R^2}\int_{|x-x_0|=R} u(y)dS_y.$$

Problem 4.5.1. *(Strong Maximum principle) Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of*

$$\Delta u = 0, x \in \Omega. \tag{4.5.5}$$

If

$$u(c) = \max_{\overline{\Omega}} u(x)$$

for some $c \in \Omega$, then $u(x)$ is a constant.

4.5.2 Maximum principle for subharmonic functions

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$. Set

$$M(u)(x,R) = \frac{1}{\mu(|y| \leq R)}\int_{|y|\leq R} u(x+y)dy \tag{4.5.6}$$

and

$$M_S(u)(x,R) = \frac{1}{\mu(\mathbf{S}^{n-1})}\int_{|\omega|=1} u(x+R\omega)d\omega \tag{4.5.7}$$

Since $\mu(|y| \leq R) = \mu(\mathbf{S}^{n-1})R^n/n$, we have the relation

$$\int_0^R M_S(u)(x,r)r^{n-1}dr = \frac{R^n}{n}M(u)(x,R).$$

Definition 4.5.1. *A function* $u \in L^1_{loc}$ *is called subharmonic if* $u(x) \le M(u)(x,R)$ *for any* $R > 0$ *such that* $\{y + x; |y| \le R\} \subset \Omega$ *and for a.e. in* $x \in \Omega$.

Problem 4.5.2. *If* $u \in C^\infty(\Omega)$ *then* u *is subharmonic if and only if* $\Delta u \ge 0$.

Hint. If $\Delta u \ge 0$, then Gauss-Green gives

$$0 \le \int |y| \le R \Delta u(x+y)dy = cR^{n-1}\frac{d}{dR}M_S(u)(x,R),$$

so $M_S(u)(x,R) \ge M_S(u)(x,0) = u(x)$ and

$$M(u)(x,R) = nR^{-n}\int_0^R M_S(u)(x,r)r^{n-1}dr \ge u(x)nR^{-n}\int_0^R r^{n-1}dr = u(x).$$

Problem 4.5.3. *If* $u \in L^1_{loc}(\Omega)$ *then* u *is subharmonic if and only if* $\Delta u \ge 0$.

Problem 4.5.4. *If* $u \in C(\overline{\Omega})$ *is subharmonic, then the condition* $u(c) = \max_{\overline{\Omega}} u(x)$ *for some* $c \in \Omega$ *implies* $u = cost$.

The function

$$M(u)(x) = \sup_{R>0} M(u)(x,R) = \sup_{R>0}\frac{1}{\mu(|y| \le R)}\int_{|y|\le R} u(x+y)dy \tag{4.5.8}$$

is called Hardy - Littlewood MAXIMAL function.

Problem 4.5.5. *Show that there exists a constant* $C > 0$ *so that for any* $u \in C_0^\infty(\mathbf{R}^n)$

$$\|M(u)\|_{L^2} \le C\|u\|_{L^2}.$$

Chapter 5

Fundamental solution of Laplace operator in $\mathbb{R}^n$ and applications

5.1 Laplace equation in $\mathbb{R}^n$

Among the most important of all partial differential equations are undoubtedly Laplace's equation

$$\Delta u = 0 \tag{5.1.1}$$

and Poisson's equation

$$-\Delta u = f. \tag{5.1.2}$$

In both (5.1.1) and (5.1.2) , $x \in \Omega$ and the unknown is $u : \bar{\Omega} \to \mathbb{R}$, $u = u(x)$ where $\Omega \subset \mathbb{R}^n$ is a given open set. In (5.1.2) the function $f : U \to \mathbb{R}$ is also given. Remember that the Laplacian of u is

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}.$$

Definition 5.1.1. *A C^2 function u satisfying* (5.1.1) *is called a harmonic function.*

5.2 Decomposition of the Laplace operatore into radial and angular part

Start with the relation

$$|x|^2 \Delta = L^2 + (n-2)L + \Delta_{S^{n-1}}, \tag{5.2.3}$$

where

$$\Delta_{S^{n-1}} = \sum_{j,k=1}^{n} \Omega_{jk}^2, \Omega_{jk} = x_j\partial_k - x_k\partial_j \tag{5.2.4}$$

and

$$L = r\partial_r = \sum_{j=1}^{n} x_j\partial_j.$$

Per $n = 2$ we have

$$\Delta_{S^1} = x_1\partial_2 - x_2\partial_1.$$

Introduce polar coordinates

$$x_1 = r\cos\phi, x_2 = r\sin\phi.$$

Then

$$\partial_1 = \frac{x_1}{r}\partial_r - \frac{x_2}{x_1^2 + x_2^2}\partial_\phi = \cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi$$

and

$$\partial_2 = \frac{x_2}{r}\partial_r + \frac{x_1}{x_1^2 + x_2^2}\partial_\phi = \sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi.$$

So we get

$$\Omega_{12} = \partial_\phi$$

and we find

$$\Delta_{S^1} = \partial_\phi^2.$$

For $n = 3$ we have

$$x_1 = r\cos\phi\sin\theta, x_2 = r\sin\phi\sin\theta, x_3 = \cos\theta.$$

Then

$$\partial_1 = \frac{x_1}{r}\partial_r - \frac{x_2}{x_1^2 + x_2^2}\partial_\phi - \frac{x_3 x_1}{r^2(x_1^2 + x_2^2)^{1/2}}\partial_\theta =$$

$$= \cos\phi\sin\theta\partial_r - \frac{\sin\phi}{r\sin\theta}\partial_\phi - \frac{\cos\theta}{r}\partial_\theta.$$

$$\partial_2 = \frac{x_2}{r}\partial_r + \frac{x_1}{x_1^2 + x_2^2}\partial_\phi - \frac{x_3 x_2}{r^2(x_1^2 + x_2^2)^{1/2}}\partial_\theta$$

and

$$\partial_3 = \frac{x_3}{r}\partial_r + \frac{(x_1^2 + x_2^2)^{1/2}}{r^2}\partial_\theta$$

Hence,

$$\Omega_{12} = \partial_\phi, \;\; \Omega_{13} = \sin\phi\cot\theta\partial_\phi + \cos\phi\partial_\theta, \;\; \Omega_{13} = -\cos\phi\cot\theta\partial_\phi + \sin\phi\partial_\theta.$$

Taking the sum of squares, we get

$$\Delta_{S^2} = \partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2.$$

We have the following commutator properties

Problem 5.2.1. *If* $[A,B] = AB - BA$ *is the commutator of two operators then we have*

$$[\partial_j, x_k] = \delta jk,$$
$$[\Delta, \partial_j] = 0,$$
$$[\Delta, x_j] = 2\partial_j,$$
$$[\Delta, \Omega_{jk}] = 0,$$
$$[\Delta, L] = 2\Delta.$$

5.2.1 Spherical harmonics

If $Y(\omega)$ is an eigenfunction of $\Delta_{S^{n-1}}$, one can introduce

$$u(x) = |x|^M Y\left(\frac{x}{|x|}\right)$$

and see that $u(x)$ is harmonic, i.e. $\Delta u(x) = 0$ if and only if

$$\Delta_{S^{n-1}} Y = -M(M+n-2)Y.$$

One can take $u(x)$ to be a harmonic homogeneous POLYNOMIAL of order M (M is a non negative integer) and see that $Y(x/|x|) = |x|^{-M}u(x)$ is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $M(M+n-2)$. The application of Liuoville theorem shows that M is only integer and $u(x)$ has to be a homogeneous polynomial.

5.3 Laplace equation in $\mathbb{R}^n$ and its fundamental solution

One good strategy for investigating any partial differential equation is first to identify some explicit solutions and then, provided the PDE is linear, to assemble more complicated solutions out of the specific ones previously noted. Furthermore, in looking for explicit solutions it is often wise to restrict attention to classes of functions with certain symmetry properties. since Laplace's equation is invariant under rotations, it consequently seems advisable to search first for radial solutions, that is, functions of $r = |x|$ Let us therefore attempt to find a solution u of Laplace's equation in $\Omega = \mathbb{R}^n$, having the form

$$u(x) = v(r)$$

where $r = |x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$ and v is to be selected (if possible) so that $\Delta u = 0$ holds. First note for $i = 1, \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \quad (x \neq 0)$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

for $i = 1, \ldots, n$, and so

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r} v' = 0. \tag{5.3.5}$$

If $v' \neq 0$, we deduce

$$\log\left(v'\right)' = \frac{v''}{v'} = \frac{1-n}{r}$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases}$$

where b and c are constants. These considerations motivate the following.

Definition 5.3.1. *The function*

$$E(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)|B|}\frac{1}{|x|^{n-2}} & (n\geq 3) \end{cases} \tag{5.3.6}$$

defined for $x \in \mathbb{R}^n, x \neq 0$, is the fundamental solution of Laplace's equation.

Remark 5.3.1. *Recall that the volume of the ball $B(0,R)$ in $\mathbb{R}^n$ is*

$$|B(0,R)| = \int_0^R \rho^{n-1} d\rho \mu(S^{n-1}) = \frac{R^n \mu(\mathbb{S}^{n-1})}{n}$$

where $\mu(\mathbb{S}^{n-1})$ is the surface area of the unit sphere in $\mathbb{R}^n$.

The reason for the particular choices of the constants in (5.3.6) will be apparent in a moment.

We will sometimes slightly abuse notation and write $\Phi(x) = \Phi(|x|)$ to emphasize that the fundamental solution is radial. Observe also that we have the estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \left|D^2\Phi(x)\right| \leq \frac{C}{|x|^n} \quad (x \neq 0) \tag{5.3.7}$$

for some constant $C > 0$

5.4 Poisson equation

By construction the function $x \mapsto E(x)$ is harmonic for $x \neq 0$. If we shift the origin to a new point y, the PDE (1) is unchanged; and so $x \mapsto E(x-y)$ is also harmonic as a function of $x, x \neq y$. Let us now take $f : \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $x \mapsto E(x-y)f(y)(x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points y. This reasoning might suggest that the convolution

$$\begin{aligned} u(x) &= \textstyle\int_{\mathbb{R}^n} E(x-y)f(y)dy \\ &= \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log(|x-y|)f(y)dy & (n=2) \\ \frac{1}{n(n-2)|B|}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy & (n\geq 3) \end{cases} \end{aligned} \tag{5.4.8}$$

will solve Laplace equation (5.1.1). However, this is wrong: we cannot just compute

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x E(x-y)f(y)dy = 0$$

Indeed, as intimated by estimate (5.3.7) $D^2\Phi(x-y)$ is not summable near the singularity at $y=x$, and so the differentiation under the integral sign above is unjustified (and incorrect). We must proceed more carefully in calculating Δu Let us for simplicity now assume $f \in C_c^2(\mathbb{R}^n)$; that is, f is twice continuously differentiable, with compact support.

Theorem 5.4.1. *(Solving Poisson's equation). Define u by* (5.4.8) . *Then*

(i) $u \in C^2(\mathbb{R}^n)$

and

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$ *We consequently see that* (5.4.8) *provides us with a formula for a solution of Poisson's equation* (5.1.2) *in* $\mathbb{R}^n$.

Proof. We have

$$u(x) = \int_{\mathbb{R}^n} E(x-y)f(y)dy = \int_{\mathbb{R}^n} E(y)f(x-y)dy$$

hence

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} E(y)\left[\frac{f\left(x+he_i-y\right)-f(x-y)}{h}\right]dy$$

where $h \neq 0$ and $e_i = (0,\ldots,1,\ldots,0)$, the 1 in the i^{th} -slot. But

$$\frac{f\left(x+he_i-y\right)-f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$, and thus

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} E(y)\frac{\partial f}{\partial x_i}(x-y)dy \quad (i=1,\ldots,n)$$

Similarly

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbf{K}^n} E(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)dy \quad (i,j=1,\ldots,n) \tag{5.4.9}$$

As the expression on the right hand side of eq.PE10 is continuous in the variable x, we see $u \in C^2(\mathbb{R}^n)$

Since E blows up at 0, we will need for subsequent calculations to isolate this singularity inside a small ball. So fix $\varepsilon > 0$. Then

$$\Delta u(x) = \int_{B(0,z)} E(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^m - B(0,t)} E(y)\Delta_x f(x-y)dy$$

$$=: I_\varepsilon + J_\varepsilon$$

Now

$$|I_\varepsilon| \le C \left\| D^2 f \right\|_{L^\infty(\mathbb{R}^-)} \int_{B(0,\varepsilon)} |E(y)| dy \le \begin{cases} C\varepsilon^2 |\log\epsilon| & (n=2) \\ C\varepsilon^2 & (n \ge 3) \end{cases} \tag{5.4.10}$$

An integration by parts yields

$$\begin{aligned} J_c &= \int_{\mathbb{R}^n - B(0,c)} E(y) \Delta_y f(x-y) dy \\ &= -\int_{\mathbb{R}^- - B(0,z)} DE(y) \cdot D_y f(x-y) dy \\ &\quad + \int_{\partial B(0,\varepsilon)} E(y) \frac{\partial f}{\partial \nu}(x-y) dS(y) \\ &=: K_\varepsilon + L_\varepsilon \end{aligned} \tag{5.4.11}$$

ν denoting the inward pointing unit normal along $\partial B(0,\varepsilon)$. We readily check

$$|L_\varepsilon| \le \|Df\|_{L^\infty(\mathbf{K}^n)} \int_{\partial B(0,\varepsilon)} |E(y)| dS(y) \le \begin{cases} C\varepsilon |\log\varepsilon| & (n=2) \\ C\varepsilon & (n \ge 3) \end{cases} \tag{5.4.12}$$

We continue by integrating by parts once again in the term K_ε, to discover

$$\begin{aligned} K_\varepsilon &= \int_{\mathbb{R}^- - B(0,\varepsilon)} \Delta E(y) f(x-y) dy - \int_{\partial B(0,\varepsilon)} \frac{\partial E}{\partial \nu}(y) f(x-y) dS(y) \\ &= -\int_{\partial B(0,\varepsilon)} \frac{\partial E}{\partial \nu}(y) f(x-y) dS(y) \end{aligned}$$

since E is harmonic away from the origin. Now $DE(y) = \frac{-1}{n|B|} \frac{y}{y'} (y \neq 0)$ on $\partial B(0,\varepsilon)$. since $n|B|\varepsilon^{n-1}$ is the surface area of the sphere $\partial B(0,\varepsilon)$, we have

$$\begin{aligned} K_\varepsilon &= -\frac{1}{n|B|\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y) dS(y) \\ &= -f_{\partial B(z,t)} f(y) dS(y) \to -f(x) \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{5.4.13}$$

(Remember that a slash through an integral denotes an average.) Combining now (5.4.10)-(5.4.13)and letting $\varepsilon \to 0$, we find $-\Delta u(x) = f(x)$, as asserted. □

5.5 Weak solutions of Poisson equation

Typical example of application of the notion of distribution is the definition of a weak solution of the Laplace equation.

Definition 5.5.1. *If u, f are distributions in $\mathbb{R}^n$, then u is a weak solution of the equation*

$$-\Delta u = f, \ \ \Delta = \partial_1^2 + \cdots \partial_n^2, \tag{5.5.14}$$

if for any test function $\varphi(x)$ we have

$$< u, \Delta\varphi >= - < f, \varphi > .$$

5.5.1 Case $n = 3$

One can verify that

$$\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta$$

in the sense of distributions in $\mathbb{R}^3$. Indeed taking any test function φ we apply Gauss - Green formula for the domain $\{|x| \geq \varepsilon\}$ and using the fact that

$$\Delta\left(\frac{1}{|x|}\right) = 0 \quad |x| \neq 0,$$

we find

$$\int_{|x|>\varepsilon} \left(\Delta\left(\frac{1}{|x|}\right)\right) \varphi(x) dx - \int_{|x|>\varepsilon} \left(\frac{1}{|x|}\right) \Delta\varphi(x) dx =$$
$$- \int_{|x|=\varepsilon} \partial_r \left(\frac{1}{|x|}\right) \varphi(x) dS_x + \int_{|x|=\varepsilon} \left(\frac{1}{|x|}\right) \partial_r\varphi(x) dS_x,$$

where here and below

$$\partial_r = \sum_{j=1}^{n} \frac{x_j}{|x|}\partial_j.$$

Taking into account the fact that

$$\partial_r \left(\frac{1}{|x|}\right) = -\frac{1}{|x|^2}$$

and introducing spherical coordinate $x = \varepsilon\omega, |\omega| = 1$, we find

$$\int_{|x|=\varepsilon} \partial_r \left(\frac{1}{|x|}\right) \varphi(x) dS_x = -\int_{|\omega|=1} \varphi(\varepsilon\omega) dS_\omega,$$

$$\int_{|x|=\varepsilon} \left(\frac{1}{|x|}\right) \partial_r\varphi(x) dS_x = \varepsilon \int_{|\omega|=1} \partial_r\varphi(\varepsilon\omega) d\omega$$

so taking the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \partial_r \left(\frac{1}{|x|}\right) \varphi(x) dS_x = -4\pi\varphi(0),$$

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \left(\frac{1}{|x|}\right) \partial_r \varphi(x) dS_x = 0,$$

so we arrive at

$$-\int_{\mathbb{R}^3} \left(\frac{1}{|x|}\right) \Delta\varphi(x) dx = 4\pi\varphi(0)$$

and the identity

$$-\frac{1}{4\pi}\Delta\left(\frac{1}{|x|}\right) = \delta. \tag{5.5.15}$$

5.5.2 Case $n \geq 3$.

The function

$$E(x) \in C^\infty(\mathbb{R}^n \setminus 0)$$

satisfying

$$-\Delta E = \delta$$

in the sense of distributions is called fundamental solutions of the Laplace operator and they enable one to represent the solution of the Poisson equation

$$-\Delta u = f, \;\; f \in C_0^\infty,$$

as follows

$$u(x) = \int_{\mathbb{R}^n} E(x-y) f(y) dy.$$

To verify that

$$u(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

is a weak solution of

$$\Delta u = f,$$

we consider

$$u_\varepsilon(x) = c\int_{|x-y|\geq\varepsilon} \frac{f(y)}{|x-y|^{n-2}} dy = c\int_{|y|\geq\varepsilon} \frac{f(x-y)}{|y|^{n-2}} dy$$

$$= c\int_\varepsilon^\infty \int_{|\omega|=1} \frac{f(x-r\omega)}{r^{n-2}} r^{n-1} dr.$$

One can easily derive that if $f \in C^2$ and has a compact support, then $u_\varepsilon \in C^2$ and

$$\Delta u_\varepsilon(x) = c\int_{|y|\geq\varepsilon} \frac{\Delta_y f(x-y)}{|y|^{n-2}}dy$$

Applying the Gauss - Green formula, we find

$$\Delta u_\varepsilon(x) = c\int_{|y|\geq\varepsilon} f(x-y)\Delta\left(\frac{1}{|y|^{n-2}}\right)dy+$$

$$+c\int_{|y|=\varepsilon}\frac{\partial_N f(x-y)}{|y|^{n-2}}dy - c\int_{|y|=\varepsilon} f(x-y)\partial_N\left(\frac{1}{|y|^{n-2}}\right)dy,$$

where $\partial_N = -\sum_j y_j/|y|\partial_j = -\partial_r$, so we get

$$\Delta u_\varepsilon(x) = c\int_{|\omega|=1}\frac{\partial_r f(x-\varepsilon\omega)}{\varepsilon^{n-2}}\varepsilon^{n-1}d\omega - (n-2)c\int_{|\omega|=1} f(x-\varepsilon\omega)d\omega$$

so taking the limit as $\varepsilon \to 0$, we obtain

$$\Delta u(x) = -(n-2)c\mu(S^{n-1})f(x).$$

So

$$c = -\frac{1}{(n-2)\mu(S^{n-1})}.$$

5.6 Mean-value formulas.

Consider now an open set $\Omega \subset \mathbb{R}^n$ and suppose u is a harmonic function within Ω. We next derive the important mean-value formulas, which declare that $u(x)$ equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$, provided $B(x,r) \subset \Omega$. These implicit formulas involving u generate a remarkable number of consequences, as we will momentarily see.

Theorem 5.6.1. *(Mean-value formulas for Laplace's equation). If $u \in C^2(\Omega)$ is harmonic, then*

(5.6.16)
$$u(x) = ⨍_{\partial B(x,r)} u\,dS = ⨍_{B(x,r)} u\,dy$$

for each ball $B(x,r) \subset \Omega$.

Proof. Set

$$\phi(r) := ⨍_{\partial B(x,r)} u(y)dS(y) = ⨍_{\partial B(0,1)} u(x+rz)dS(z)$$

Then

$$\phi'(r) = ⨍_{\partial B(0,1)} Du(x+rz)\cdot z dS(z)$$

and consequently, using Green's formulas we compute

$$\begin{aligned}\phi'(r) &= ⨍_{\partial B(z,r)} Du(y)\cdot \frac{y-x}{r} dS(y)\\ &= ⨍_{\partial B(x,r)} \frac{\partial u}{\partial v} dS(y)\\ &= \frac{r}{n} ⨍_{B(x,r)} \Delta u(y) dy = 0\end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0} ⨍_{\partial B(x,t)} u(y) dS(y) = u(x)$$

Observe next that our employing polar coordinates, gives

$$\begin{aligned}\int_{B(x,r)} u dy &= \int_0^r \left(\int_{\partial B(x,s)} u dS\right) ds\\ &= u(x)\int_0^r n\alpha(n)s^{n-1} ds = \alpha(n) r^n u(x)\end{aligned}$$

□

Theorem 5.6.2. *THEOREM 3 (Converse to mean-value property). If $u \in C^2(\Omega)$ satisfies*

$$u(x) = ⨍ u dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. Proof. If $\Delta u \neq 0$, there exists some ball $B(x,r) \subset \Omega$ such that, say, $\Delta u > 0$ within $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y) dy > 0$$

a contradiction. □

5.7 Properties of harmonic functions.

We now present a sequence of interesting deductions about harmonic functions, all based upon the mean-value formulas. Assume for the following that $\Omega \subset \mathbb{R}^n$ is open and bounded.

5.7.1 Strong maximum principle, uniqueness.

Theorem 5.7.1. *(Strong maximum principle). Suppose* $u \in C^2(\Omega) \cap C(O)$ *is harmonic within* Ω.

(i) Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(ii) Furthermore, if Ω *is connected and there exists a point* $x_0 \in \Omega$ *such that*

$$u(x_0) = \max_{\overline{\Omega}} u$$

then u is constant within Ω. *Assertion (i) is the marimum principle for Laplace's equation and (ii) is the strong maximum principle. Replacing u by* $-u$, *we recover also similar assertions with "min" replacing "max".*

Proof. Suppose there exists a point $x_0 \in \Omega$ with $u(x_0) = M := \max_{\Omega} u$ Then for $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, the mean-value property asserts

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dy \leq M$$

As equality holds only if $u \equiv M$ within $B(x_0, r)$, we see $u(y) = M$ for all $y \in B(x, r)$. Hence the set $\{x \in \Omega \mid u(x) = M\}$ is both open and relatively closed in Ω, and thus equals Ω if Ω is connected. This proves assertion (ii), from which (i) follows. □

Remark 5.7.1. *The strong maximum principle asserts in particular that if* Ω *is connected and* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfies*

$$\begin{cases} \Delta u = 0 \ in\, \Omega \\ \ \ u = g \ on\, \partial\Omega \end{cases}$$

where $g \geq 0$, *then u is positive everywhere in* Ω *if g is positive somewhere on* $\partial\Omega$.

An important application of the maximum principle is establishing the uniqueness of solutions to certain boundary-value problems for Poisson's equation.

Theorem 5.7.2. *(Uniqueness). Let* $g \in C(\partial\Omega), f \in C(\Omega)$. *Then there exists at most one solution* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *of the boundary-value problem*

$$\begin{cases} -\Delta u = f \ in\, U \\ \quad u = g \ on\, \partial U \end{cases} \tag{5.7.17}$$

Proof. If u and $\tilde{u}$ both satisfy (5.7.17) one can apply apply Theorem 5.7.1 to the harmonic functions $w := \pm(u - \tilde{u})$ □

5.7.2 Regularity

Now we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus harmonic functions are automatically infinitely differentiable. This sort of assertion is called a regularity theorem. The interesting point is that the algebraic structure of Laplace's equation $\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$ leads to the analytic deduction that all the partial derivatives of u exist, even those which do not appear in the PDE.

Theorem 5.7.3. *THEOREM 6 (Smoothness). If u E $C(U)$ satisfies the mean-tualue property (16) for each ball $B(x,r) \subset U$, then*

$$u \in C^\infty(U)$$

Note carefully that u may not be smooth, or even continuous, up to ∂U.

Proof. Proof. Let η be a standard mollifier, as described in \$C.4, and recall that η is a radial function. Set $u^\epsilon := \eta_c * u$ in $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$ As shown in \$C.4, $u^r \in C^\infty(U_e)$ We will prove u is smooth by demonstrating that in fact $u \equiv u^f$ on U_ε Indeed if $x \in U_{e,}$ then Thus $u^\epsilon \equiv u$ in U_ε, and so $u \in C^\infty(U_e)$ for each $\epsilon > 0$. □

5.8 Liouville's Theorem.

Next we see that there are no nontrivial bounded harmonic functions on all of R^n

THEOREM 8 (Liouville's Theorem). Suppose u : R" → R is harmonic and bounded. Then u is constant. Proof. Fix $x_0 \in \mathbb{R}^n, r > 0$, and apply Theorem 7 on $B(x_0, r)$:

$$\begin{aligned} |Du(x_0)| &\leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(z_0,r))} \\ &\leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0 \end{aligned}$$

as $r \to \infty$. Thus $Du \equiv 0$, and so u is constant. THEOREM 9 (Representation formula). Let $f \in C_c^2(\mathbb{R}^n), n \geq 3$. Then any bounded solution of

$$-\Delta u = f \quad \text{in } \mathrm{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^0} \Phi(x-y) f(y) dy + C \quad \left(x \in R^n\right)$$

for some constant C.

5.9 Liouville theorem for harmonic functions in $\mathbb{R}^n$, $n \geq 3$.

The Laplace equation

$$\Delta u = f, \ \ f \in C_0^\infty,$$

has a solution

$$u(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

provided $n \geq 3$ and

$$c = -\frac{1}{(n-2)\mu(S^{n-1})}.$$

We shall use this fundamental solution to derive the Liouville theorem. For simplicity we consider only the case $n = 3$.

Lemma 5.9.1. *(Liouville) If $u \in C^2(\mathbb{R}^n)$ is a harmonic function, i.e. $\Delta u = 0$ in $\mathbb{R}^n$, then the condition*

$$\max_{|x|=R} |u(x)| \leq C$$

implies $u = cost$.

Idea of Proof: Take cut off function $\varphi(s)$ such that $\varphi(s) = 1$ for $|s| \leq 1$ and $\varphi(s) = 0$ for $|s| \geq 2$. Then setting $\varphi_R(x) = \varphi(|x|/R)$ we have

$$\Delta(\varphi_R u) = 2\nabla(u \nabla \varphi_R) - u \Delta \varphi_R,$$

since u is harmonic. Applying the formula for the fundamental solution, we get

$$\varphi_R(x) u(x) = -\frac{1}{4\pi} \int_{R^3} \frac{2\nabla(u(y)\nabla\varphi_R(y)) - u(y)\Delta\varphi_R(y)}{|x-y|} dy =$$

$$= -\frac{1}{4\pi} \int_{R^3} \frac{2(u(y)(x-y)\nabla\varphi_R(y))}{|x-y|^3} dy + \frac{1}{4\pi} \int_{R^3} \frac{u(y)\Delta\varphi_R(y)}{|x-y|} dy.$$

Take $|x| \sim R/2$. Then $|y| \sim R$ on the support of $\nabla\varphi_R(y)$. So one can show that

$$|\nabla u(x)| \leq \frac{C}{|x|}$$

provided $|x| \sim R/2$. Since $\nabla u(x)$ is also harmonic, applying the maximum principle of Lemma 5.9.1 we complete the proof that $\nabla u = 0$.

Problem 5.9.1. *(Liouville) If $u \in C^2(\mathbb{R}^n)$ is a harmonic function, i.e. $\Delta u = 0$ in $\mathbb{R}^n$, then the condition*

$$\max_{|x|=R} |u(x)| \leq C(1+|x|)^M$$

implies $u(x)$ is a polynomial.

Chapter 6

Harmonic functions in domains (ball and semispace) and conformal transform.

First step in this section is to make change of variables in the Laplace operator. Indeed, let us take

$$y \to x = F(y)$$

invertible change of variables in $\mathbb{R}^n$.

Our goal is the compute

$$\Delta_y u(F(y)).$$

For the purpose we start with the representation

$$u(F(y)) = \frac{1}{(2\pi)^n} \int e^{i<F(y).\xi>} \hat{u}(\xi) d\xi.$$

Since

$$\Delta e^{i<F(y).\xi>} = \left(-\sum_k < \partial_k F(y), \xi >< \partial_k F(y), \xi > + i < \Delta F(y), \xi > \right) e^{i<F(y).\xi>},$$

we can set

$$g^{jm}(y) = \sum_k \partial_k F_j(y) \partial_k F_m(y)$$

and obtain

$$\Delta e^{i<F(y).\xi>} = \left(-\sum_{jm} g^{jm}(y) \xi_j \xi_m + i < \Delta F(y), \xi > \right) e^{i<F(y).\xi>}.$$

Using $i\xi_k\hat{u}(\xi) = \widehat{\partial_k u}(\xi)$, we

$$\Delta_y u(F(y)) = \sum_{jm} g^{jm}(y)(\partial_j\partial_m u)(F(y)) + < \Delta F(y), (\nabla u)(F(y)) > .$$

In the particular case

$$F(y) = \frac{y}{|y|^2}$$

one can verify that we have

$$\left(g^{jm}\right)_{j,m=1}^n = \frac{I}{|y|^4}, \ \ \Delta F(y) = \frac{2y}{|y|^4}.$$

Given any $u(x) \in C_0^\infty$ we define

(6.0.1) $$u^*(y) = |y|^{-1}u(y/|y|^2)$$

Then we can assume $n = 3$ and derive

(6.0.2) $$\Delta_y u^*(y) = |y|^{-5}\Delta_x u(y/|y|^2).$$

Idea of Proof: Introduce polar coordinates $R = |x|$, $r = |y|$ and $\omega = x/|x| = y/|y|$.Then the transform is given by

$$R = \frac{1}{r}.$$

We have

$$\Delta_x = \partial_R^2 + \frac{2}{R}\partial_R + \frac{1}{R^2}\Delta_\omega$$

$$\Delta_y = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_\omega$$

Note that

$$R\partial_R = -r\partial_r, \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\frac{f}{r} = \frac{1}{r}\partial_r^2 f,$$

so we obtain

$$\left(\partial_r^2 + \frac{2}{r}\partial_r\right)\frac{f}{r} = R(R^2\partial_R)^2 f = R^5\left(\partial_R^2 + \frac{2}{R}\partial_R\right)f.$$

Problem 6.0.1. *How can be generalized the conformal low (6.0.2) for dimensions $n \geq 3$.?*

Turning to the Dirichlet problem, we study the boundary value problem

$$\Delta u = 0, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

For the purpose we look for

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - h(x,y)\right)$$

so that $h(x,y)$ is harmonic with respect to y and on the boundary $\partial\Omega$ we have

$$h(x,y) = \frac{1}{|x-y|}, y \in \partial\Omega.$$

The Gauss Green identity

$$\int_\Omega (\Delta u v - u\Delta v)\, dy = \int_{\partial\Omega} (\partial_N u v - u \partial_N v)\, dS_y,$$

enables one to take $v(y) = 1/|x-y|$, where $x \in \Omega$ and modify the domain Ω as follows

$$\Omega_\delta = \{y : y \in \Omega, |y-x| \geq \delta\}$$

provided $\delta > 0$ is small.

As in Lemma 4.5.1 taking the limit as $\delta \to 0$ we get

$$u(x) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{\partial_N u}{|x-y|} - u\partial_N\left(\frac{1}{|x-y|}\right)\right) dS_y$$
$$-\frac{1}{4\pi}\int_\Omega \frac{\Delta u}{|x-y|} dy.$$

If we take $v(y) = h(x,y)$ we can use the fact that $h(x,y)$ is more regular than $1/|x-y|$ and find

$$0 = \frac{1}{4\pi}\int_{\partial\Omega}\left(h(x,y)\partial_N u - u\partial_N h\right) dS_y$$
$$-\frac{1}{4\pi}\int_\Omega h(x,y)\Delta u dy.$$

The above relations imply

$$u(x) = \int_{\partial\Omega}\left(G(x,y)\partial_N u - u\partial_N G(x,y)\right) dS_y$$
$$-\int_\Omega G(x,y)\Delta u dy.$$

Since $G(x,y) = 0$ on the boundary and since u is harmonic, we get

Lemma 6.0.1. *(Green function) Let $u \in C^2(\Omega)$ be a harmonic function in the domain Ω, i.e. $\Delta u = 0$ in Ω. If*

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - h(x,y)\right)$$

where $h(x,y)$ is harmonic with respect to y and on the boundary $\partial\Omega$ we have

$$h(x,y) = \frac{1}{|x-y|}, y \in \partial\Omega,$$

then the solution to the problem

$$\Delta u = 0, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

is given by

$$u(x) = -\int_{\partial\Omega} \left(f(y)\partial_N G(x,y)\right) dS_y$$

Example 6.0.1. *Let $\Omega = \{x, |x| < 1\}$. To construct $h(x,y)$ we consider the exterior of the domain $\Omega_{ext} = \{x, |x| > 1\}$ and use the conformal map*

$$y \in \Omega_{ext} \to \frac{y}{|y|^2} \in \Omega.$$

Given any x, $|x| < 1$ the function

$$\frac{1}{|x-y|}$$

is harmonic in Ω_{ext} so following (6.0.1) we see that

$$h(x,y) = \frac{1}{|y||x - y/|y|^2|}$$

is harmonic in Ω. Introduce polar coordinates $y = r\omega$. Then Lemma 6.0.1 implies that

$$u(x) = -\int_{|\omega|=1} \left(f(\omega)\partial_r G(x, r\omega)\right) d\omega$$

Since

$$4\pi G(x, r\omega) = \frac{1}{|x - r\omega|} - \frac{1}{|rx - \omega|}$$

and since

$$\partial_r \frac{1}{|x - r\omega|} = -\frac{< r\omega - x, \omega >}{|x - r\omega|^3}$$

$$\partial_r \frac{1}{|rx-\omega|} = -\frac{< rx-\omega, x >}{|rx-\omega|^3}$$

we obtain taking $r = 1$

$$4\pi\partial_r G(x, r\omega) = -\frac{1-|x|^2}{|x-\omega|^3}.$$

Finally, we can write

$$u(x) = \frac{1-|x|^2}{4\pi} \int_{|\omega|=1} \left(f(\omega) \frac{d\omega}{|x-\omega|^3} \right).$$

and this is the unique solution to the Dirichlet problem

$$\Delta u = 0, x \in \Omega$$
$$\lim_{x\to\infty} u(x) = 0,$$
$$u(x) = f(x), x \in \partial\Omega.$$

To show that this is a solution really, we take into account the relation

$$(2L_x + 1)\left(\frac{1}{|x-y|}\right) = \left(\frac{1-|x|^2}{|x-y|^3}\right), \ \ |y| = 1, \ |x| \neq 1, \ \ L_x = x.\nabla_x$$

as well as the commutator relations

$$[\Delta, L] = 2\Delta.$$

So from

$$\Delta_x \left(\frac{1}{|x-y|}\right) = 0$$

we can derive

$$\Delta_x \left((2L_x+1)\left(\frac{1}{|x-y|}\right)\right) = \Delta_x \left(\frac{1-|x|^2}{|x-y|^3}\right) = 0$$

so

$$u(x) = \frac{1-|x|^2}{4\pi} \int_{|\omega|=1} \left(f(\omega) \frac{d\omega}{|x-\omega|^3} \right)$$

is a harmonic function!!!

Example 6.0.2. *Let* $\Omega = \{x, |x| > 1\}$. *To construct* $h(x, y)$ *we consider the interior of the domain* $\Omega_{int} = \{x, |x| < 1\}$ *and use the conformal map*

$$y \in \Omega_{ext} \to \frac{y}{|y|^2} \in \Omega.$$

Given any x, $|x| > 1$ the function

$$\frac{1}{|x-y|}$$

is harmonic in Ω_{int} so following (6.0.1) we see that

$$h(x,y) = \frac{1}{|y||x - y/|y|^2|}$$

is harmonic with respect to y in Ω. Introduce polar coordinates $y = r\omega$. Then Lemma 6.0.1 implies that

$$u(x) = -\int_{|\omega|=1} \big(f(\omega)\partial_r G(x, r\omega)\big)\, d\omega$$

Since

$$4\pi G(x, r\omega) = \frac{1}{|x - r\omega|} - \frac{1}{|rx - \omega|}$$

and since

$$\partial_r \frac{1}{|x - r\omega|} = -\frac{< r\omega - x, \omega >}{|x - r\omega|^3}$$
$$\partial_r \frac{1}{|rx - \omega|} = -\frac{< rx - \omega, x >}{|rx - \omega|^3}$$

we obtain taking $r = 1$

$$4\pi\partial_r G(x, r\omega) = -\frac{1 - |x|^2}{|x - \omega|^3}.$$

Finally, we can write

$$u(x) = \frac{1 - |x|^2}{4\pi} \int_{|\omega|=1} \left(f(\omega) \frac{d\omega}{|x - \omega|^3} \right).$$

and this is the unique solution to the Dirichlet problem

$$\Delta u = 0, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

Problem 6.0.2. *Let $u(x)$ be a solution to*

$$\Delta u = 0, |x| > 1$$
$$\lim_{x\to\infty} u(x) = 0,$$
$$u(x) = f(x), |x| = 1.$$

Show that

$$|u(x)| \le \frac{C}{1 + |x|} \max_{|x|=1} |f(x)|.$$

Problem 6.0.3. *Construct the Green function for the domains:*
a)$\{x, |x| < R\}$,
b)$\{x, |x| > R\}$,

Answer: a)
$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{|y|=R} \left(f(y) \frac{dS_y}{|x-y|^3} \right).$$

Problem 6.0.4. *Let* $u(x)$ *be a solution to*

$$\Delta u = 0, |x| > R$$
$$u(x) = f(x), |x| = R.$$

Show that
$$|u(x)| \leq \frac{C}{1+|x|} \max_{|x|=R} |f(x)|.$$

Problem 6.0.5. *Construct the Green function for the domain* $\{x \in \mathbb{R}^3, x_3 > 0\}$.

Answer:

$$u(x) = \frac{x_3}{2\pi} \int_{\mathbb{R}^2} \left(f(y') \frac{dy'}{(|x'-y'|^2 + x_3^2)^{3/2}} \right), \;\; x' = (x_1, x_2).$$

Chapter 7

Applications: a priori estimates

7.0.1 Laplace equation in the space $\mathbb{R}^n$.

The equation

$$\Delta u(x) = f(x), x \in \mathbb{R}^n$$

has a unique solution provided

$$\lim_{x\to\infty} u(x) = 0$$

Taking $f(x) \in C(\mathbb{R}^n)$ with compact support one can represent the unique solution as follows (for simplicity we take $n = 3$.)

$$u(x) = -\frac{1}{4\pi}\int_K \frac{f(y)}{|x-y|}dy,$$

where here and below K denotes the support of f.

Problem 7.0.1. *(smoothing property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 7.0.2. *(smoothing property) If $f(x) \in L^\infty(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 7.0.3. *(smoothing property) If $f(x) \in C^k(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^{k+1}(\mathbb{R}^n)$*

Problem 7.0.4. *(convergence) If $f_k(x)$ have a fixed compact support K and tend uniformly in $C^k(K)$, then*

$$u_k(x) = -\frac{1}{4\pi}\int_K \frac{f_k(y)}{|x-y|}dy,$$

converge uniformly in $C^{k+1}(\mathbb{R}^n)$.

Problem 7.0.5. *(smoothing and decay property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$ and*

$$|u(x)| \leq \frac{C\|f\|_{C(K)}}{1+|x|}.$$

Problem 7.0.6. *(decay property for f without compact support) If $f(x) \in C(\mathbb{R}^n)$ has the property*

$$\|(1+|x|)^{3+\varepsilon} f\|_{L^\infty(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3} |(1+|x|)^{3+\varepsilon} f(x)| < \infty,$$

then $u(x) \in C^1(\mathbb{R}^n)$ and

$$|u(x)| \leq \frac{C\|(1+|x|)^{3+\varepsilon} f\|_{L^\infty(\mathbb{R}^3)}}{1+|x|}.$$

Problem 7.0.7. *(Coulomb behaviour property) If $f(x) \in C(\mathbb{R}^n)$ is non-negative, has a compact support and $f(x_0) > 0$ for some point $x_0 \in \mathbb{R}^n$, then there is a positive constant C_0 so that*

$$|u(x)| \geq \frac{C_0}{1+|x|}.$$

Hence u is not square integrable. More precisely, we have

$$\int_{|x|\leq R} |u(x)|^2 dx \geq CR$$

for R sufficiently large.

7.0.2 Laplace equation in bounded domain with Dirichlet boundary condition.

The general problem

$$\Delta u = F, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

can be solved extending F in $\mathbb{R}^n$ and representing u as follows

$$u = u_0 + w,$$

where

$$u_0(x) = -\frac{1}{4\pi}\int_\Omega \frac{F(y)}{|x-y|} dy.$$

Then w solves

$$\Delta w = 0, x \in \Omega$$
$$w(x) = g(x), x \in \partial\Omega.$$

where

$$g = f - u_0.$$

Applying the maximum principle we get

$$\|w\|_{L^\infty(\Omega)} \leq C\|g\|_{C(\partial\Omega)} \leq C\left(\|f\|_{C(\partial\Omega)} + \|F\|_{L^\infty(\Omega)}\right).$$

This estimate guarantees the uniqueness of the solution. The existence is delicate: needs or sub and supersolutions (Peron method) or methods from Functional analysis (Friedrich's extentions). For the concrete domains: interior or exterior ball, half space we have concrete solutions.

7.0.3 Weak and strong solutions in R^n

We shall define the meaning of solution to the Cauchy problem

$$-\Delta u = f, \tag{7.0.1}$$

Definition 7.0.1. *Let $f \in L^2(\mathbb{R}^n)$. A function $u(x) \in L^2(\mathbb{R}^n)$ is a weak solution of the Cauchy problem* (7.0.1)*, if u solves $-\Delta u = f$, in distributional sense in $\mathbb{R}^n$, i.e. for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have*

$$-\int_{\mathbb{R}^n} u(x)\Delta_x\varphi(x)dx = \int_{\mathbb{R}^n} f(x)\varphi(x)dx \tag{7.0.2}$$

Definition 7.0.2. *Let $f \in L^2(\mathbb{R}^n)$. A function*

$$u(x) \in L^2(\mathbb{R}^n)$$

is a strong solution of the Cauchy problem (7.0.1)*, if*

$$u(x) \in H^2(\mathbb{R}^n)$$

and $-\Delta u = f$ in sense of L^2 functions.

Lemma 7.0.1. *(H^1 estimates) For any $\nu > 0$ the exists $C_\nu > 0$ so that if $f \in L^2(\mathbb{R}^n)$ and $u(x) \in L^2(\mathbb{R}^n)$ is a weak solution of the Cauchy problem* (7.0.1)*, then we have the estimates*

$$\|\nabla u\|_{L^2}^2 \leq \|f\|_{L^2}\|u\|_{L^2}. \tag{7.0.3}$$

$$\|\nabla u\|_{L^2} \leq C_\nu\|f\|_{L^2} + \nu\|u\|_{L^2}. \tag{7.0.4}$$

Proof with Fourier transform. Using the Fourier transform we easily deduce

$$u \in L^2, \Delta u \in L^2 \implies \partial_k u \in L^2$$

and we have the estimate

$$\sum_{|\alpha|\leq 1} \|\partial_x^\alpha u\|_{L^2} \lesssim \|\Delta u\|_{L^2} + \|u\|_{L^2}.$$

□

Proof with Yosida approximation. We shall use the Yosida approximation of u defined for $\varepsilon \in (0,1)$ as

$$u_\varepsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix.\xi} (1+\varepsilon|\xi|^2)^{-1} \widehat{u}(\xi) d\xi. \tag{7.0.5}$$

To be more precise this integral is well - defined for $f \in C_0^\infty$ and obeys the relation

$$\widehat{u_\varepsilon}(\xi) = (1+\varepsilon|\xi|^2)^{-1} \widehat{u}(\xi).$$

Moreover, we have the estimates

$$\|u_\varepsilon\|_{L^2} = c\|\widehat{u_\varepsilon}\|_{L^2} = c\|(1+\varepsilon|\xi|^2)^{-1}\widehat{u}\|_{L^2} \leq \|u\|_{L^2},$$

$$\|u_\varepsilon\|_{H^2} = \|(1+\xi^2)\widehat{u_\varepsilon}\|_{L^2} \leq \|(1+\xi^2)(1+\varepsilon|\xi|^2)^{-1}\widehat{u}\|_{L^2} \leq \frac{1}{\varepsilon}\|u\|_{L^2}.$$

These estimates show that the operator $u \to u_\varepsilon$ defined by (7.0.5) can be extended by density argument to an operator

$$u : L^2 \to u_\varepsilon = (1-\varepsilon\Delta)^{-1} u \in H^2 \tag{7.0.6}$$

so that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^2} = 0.$$

If $u \in L^2$ is a weak solution to (7.0.1), then

$$u_\varepsilon = (1-\varepsilon\Delta)^{-1} u$$

is also a weak solution to

$$\Delta u_\varepsilon = f_\varepsilon. \tag{7.0.7}$$

Since for fixed $\varepsilon > 0$

$$u_\varepsilon \in C([0,T]; H^2)$$

due to (8.1.20), we can multiply the equation (7.0.7) by u_ε and after integrating we get

$$\|\nabla u_\varepsilon(t)\|_{L^2}^2 = (f_\varepsilon, u_\varepsilon)_{L^2}.$$

Taking the limit $\varepsilon \to 0$, we find (7.0.3). From this estimate and simple estimate

$$ab \le \frac{1}{4\nu^2} a^2 + \nu^2 b^2$$

we get (7.0.8). □

Lemma 7.0.2. *(H^2 estimates)There exists $C > 0$ so that if $f \in L^2(\mathbb{R}^n)$ and $u(x) \in L^2(\mathbb{R}^n)$ is a weak solution of the Cauchy problem* (7.0.1)*, then we have the estimates*

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha u\|_{L^2} \le C\left(\|f\|_{L^2} + \|u\|_{L^2}\right). \tag{7.0.8}$$

Proof. Using the Fourier transform we easily deduce

$$u \in H^1, \Delta u \in L^2 \implies \partial_j \partial_k u \in L^2$$

and we have the estimate

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha u\|_{L^2} \lesssim \|\Delta u\|_{L^2} + \|u\|_{L^2}.$$

□

A small modification in the above result can be done for the first order perturbation.

We can define the meaning of weak solution to the Cauchy problem

$$-\Delta u = f + \sum_{j=1}^n \partial_{x_j} F_j, \tag{7.0.9}$$

where $F_j(x)$ are L^2 functions.

Lemma 7.0.3. *(H^1 estimates) There exists a constant $C > 0$ so that if $f, F_j \in L^2(\mathbb{R}^n)$ and $u(x) \in L^2(\mathbb{R}^n)$ is a weak solution of the Cauchy problem* (7.0.9)*, then we have the estimates*

$$\|\nabla u\|_{L^2} \le C\left(\|f\|_{L^2} + \sum_{j=1}^n \|F_j\|_{L^2} + \|u\|_{L^2}\right). \tag{7.0.10}$$

Proof. Again we use the Yosida approximation.

If $u \in L^2$ is a weak solution to (7.0.9), then

$$u_\varepsilon = (1-\varepsilon\Delta)^{-1} u$$

is also a weak solution to

$$\Delta u_\varepsilon = f_\varepsilon + \sum_{j=1}^{n} \partial_{x_j} F_{j,\varepsilon}. \tag{7.0.11}$$

Then we can multiply the equation (7.0.11) by u_ε and after integrating we get

$$\|\nabla u_\varepsilon(t)\|^2_{L^2} = (f_\varepsilon, u_\varepsilon)_{L^2} + \sum_{j=1}^{n} (\partial_{x_j} F_{j,\varepsilon}, u_\varepsilon)_{L^2} =$$

$$= (f_\varepsilon, u_\varepsilon)_{L^2} - \sum_{j=1}^{n} (F_{j,\varepsilon}, \partial_{x_j} u_\varepsilon)_{L^2} \le \|f_\varepsilon\|^2_{L^2} + \|u_\varepsilon\|^2_{L^2} + \frac{1}{4\nu^2} \sum_{j=1}^{n} \|F_{j,\varepsilon}\|^2_{L^2} + \nu^2 \|\nabla u_\varepsilon(t)\|^2_{L^2},$$

so taking ν sufficiently small we absorb the term

$$\nu^2 \|\nabla u_\varepsilon(t)\|^2_{L^2}$$

and get

$$\|\nabla u_\varepsilon(t)\|^2_{L^2} \lesssim \|f_\varepsilon\|^2_{L^2} + \|u_\varepsilon\|^2_{L^2} + \sum_{j=1}^{n} \|F_{j,\varepsilon}\|^2_{L^2}$$

Taking the limit $\varepsilon \to 0$, we find (7.0.10). □

7.0.4 Weak and strong solutions in domains

We shall define the meaning of solution to the Cauchy problem

$$-\Delta u = f \tag{7.0.12}$$

in a domain $U \subset \mathbb{R}^n$.

Definition 7.0.3. *Let $f \in L^2(U)$. A function $u(x) \in L^2(U)$ is a weak solution of the Cauchy problem* (7.0.12)*, if u solves $-\Delta u = f$, in distributional sense in U, i.e. for any $\varphi \in C_0^\infty(U)$ we have*

$$-\int_U u(x)\Delta_x \varphi(x)\,dx = \int_U f(x)\varphi(x)\,dx. \tag{7.0.13}$$

Definition 7.0.4. *Let $f \in L^2(U)$. A function*

$$u(x) \in L^2(U)$$

is a strong solution of the Cauchy problem (7.0.12)*, if*

$$u(x) \in H^2(U)$$

and $-\Delta u = f$ in sense of L^2 functions.

Lemma 7.0.4. *(H^1 interior estimates) There exists $C > 0$ so that if $f \in L^2(U)$ and $u(x) \in L^2(U)$ is a weak solution of the Cauchy problem* (7.0.12)*, then for any open BOUNDED set $V \subset U$ we have the estimate*

$$\sum_{|\alpha|=1} \|\partial_x^\alpha u\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{7.0.14}$$

Proof. We shall use cut – of function $\varphi \in C_0^\infty(U)$, such that $\varphi = 1$ on U. The φu is a solution to

$$-\Delta(\varphi u) = \varphi f - 2\nabla\varphi\nabla u - (\Delta\varphi)u = \varphi f - 2\nabla\left(u\nabla\varphi\right) + (\Delta\varphi)u.$$

Using the fact that

$$|\nabla\varphi(x)| + |\Delta\varphi(x)| \lesssim 1,$$

we apply (7.0.10) and find

$$\|\nabla u\|^2_{L^2(V)} \lesssim \|\nabla(\varphi u)\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(U)}\|u\|_{L^2(U)} + \|u\|^2_{L^2(U)} \lesssim (\|f\|_{L^2(U)} + \|u\|_{L^2(U)})^2.$$

□

Lemma 7.0.5. *(H^2 interior estimates) There exists $C > 0$ so that if $f \in L^2(U)$ and $u(x) \in L^2(U)$ is a weak solution of the Cauchy problem* (7.0.12)*, then for any open BOUNDED set $V \subset U$ we have the estimate*

$$\sum_{|\alpha|\leq 2} \|\partial_x^\alpha u\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{7.0.15}$$

Proof. We shall use two cut – of function $\varphi \in C_0^\infty(U)$, such that $\varphi = 1$ on U and $\psi \in C_0^\infty(U)$, such that $\psi = 1$ on the support of φ. Then ψu is a solution to

$$-\Delta(\psi u) = \varphi f - 2\nabla\left(u\nabla\psi\right) + (\Delta\psi)u.$$

Using the fact that

$$|\nabla\psi(x)| + |\Delta\psi(x)| \lesssim 1,$$

we apply (7.0.10) and find

$$\|\nabla(\psi u)\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(U)} + \|u\|^2_{L^2(U)}. \tag{7.0.16}$$

Further the equation for φu is

$$-\Delta(\varphi u) = \varphi f - 2\nabla\left(u\nabla\varphi\right) + (\Delta\varphi)u$$

so the H^2 free estimate of Lemma 7.0.2 gives

$$\sum_{|\alpha|\leq 2} \|\partial_x^\alpha(\varphi u)\|_{L^2(R^n)} \lesssim \|f\|_{L^2(U)} + \|u\|_{L^2(U)} + \|\nabla(\psi u)\|_{L^2(U)}$$

and applying (7.0.16), we complete the proof. □

7.0.5 Some elliptic estimates

We as always assume that $U \subset \mathbb{R}^n$ is a bounded, open set. Suppose also $u \in H_0^1(U)$ is a weak solution of the PDE(1), where L has the divergence form (4)

$$Lu = -\Delta u$$

We begin studying the local behaviour of (weak) solutions of the system of equations

$$\begin{cases} -\partial_\alpha\left(A_{ij}^{\alpha\beta}\partial_\beta u^j\right) = f_i - \partial_\alpha F_i^\alpha \quad i = 1,\dots,m \\ u \in H^1_{\text{loc}}(\Omega;\mathbb{R}^m) \end{cases}$$

with $A_{ij}^{\alpha\beta} \in L^\infty(\Omega), f_i \in L^2_{\text{loc}}(\Omega)$ and $F_i^\alpha \in L^2_{\text{loc}}(\Omega)$. From now on we shall use $|\cdot|$ for the Hilbert-Schmidt norm of matrices and tensors, even though some estimates would still be valid with the (smaller) operator norm.

Theorem 4.1 (Caccioppoli-Leray inequality). If the Borel coefficients $A_{ij}^{\alpha\beta}$ satisfy the Legendre condition $(L)_\lambda$ with $\lambda > 0$ and

$$\sup_{x\in\Omega}\left|A_{ij}^{\alpha\beta}(x)\right| \leq \Lambda < \infty$$

then there exists a positive constant $c = c(\lambda,\Lambda)$ such that for any ball $B_R(x_0)$? Ω and any $k \in \mathbb{R}^m$ it holds

$$c\int_{B_{R/2}(x_0)} |\nabla u|^2 dx \leq R^{-2}\int_{B_R(x_0)} |u(x)-k|^2 dx + R^2\int_{B_R(x_0)} |f(x)|^2 dx + \int_{B_R(x_0)} |F(x)|^2 dx$$

THEOREM 1 (Interior H^2-regularity). Assume (5) $a^{ij} \in C^1(U), b^i, c \in L^\infty(U) \quad (i,j = 1,\dots,n)$ and (6)

$$f \in L^2(U)$$

Suppose furthermore that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \quad \text{in } U$$

Then (7)

$$u \in H^2_{\text{loc}}(U)$$

and for each open $V \subset\subset U$ we have the estimate

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha u\|_{L^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right), \tag{7.0.17}$$

the constant C depending only on V, U, and the coefficients of L.

Remarks. (i) Note carefully that we do not require $u \in H_0^1(U)$; that is, we are not necessarily assuming the boundary condition $u = 0$ on ∂U in the trace sense. (ii) Observe additionally that since $u \in H^2_{\text{loc}}(U)$, we have

$$Lu = f \quad \text{a.e. in } U$$

Thus u actually solves the PDE, at least for a.e. point within U. To see this, note that for each $v \in C_c^\infty(U)$, we have

$$B[u,v] = (f,v)$$

since $u \in H^2_{\text{loc}}(U)$, we can integrate by parts:

$$B[u,v] = (Lu,v)$$

Thus $(Lu - f, v) = 0$ for all $v \in C_c^\infty(U)$, and so $Lu = f$ a.e.

7.0.6 Idea of Peron method

4.8. Metodo di Perron per il problema di Dirichlet su domini generici Ω. Una volta risolto il problema di Dirichlet su $B_R(0)$ si puo' risolvere il problema di Dirichlet su una famiglia molto ampia di domini Ω senza bisogno di costruire la funzione di Green associata (come fatto su $B_R(0)$). Introduciamo le funzioni subarmoniche

Definition 7.0.5. *Una funzione $u : \Omega \to \mathbb{R}$ si dice subarmonica e diremo*

$$u \in \text{SUB}(\Omega)$$

se $u \in C(\Omega)$ (ossia e' continua) ed inoltre

$$\forall x \in \Omega \quad \exists r(x) > 0 \text{ t.c. } u(x) \le \fint_{S_r(x)} u\,d\sigma, \quad \forall r \in (0, r(x))$$

Proposition 7.0.1. *Per ogni funzione* $u \in \mathrm{SUB}(\Omega)$ *si ha* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

Proof. Segue dallo stesso argomento usato per provare il principio del massimo per funzioni armoniche (usando il teorema della media). □

Proposition 7.0.2. *Siano* $u, v \in \mathrm{SUB}(\Omega)$ *allora* $\max\{u, v\} \in \mathrm{SUB}(\Omega)$

Proof. E ben noto che $\max\{u, v\}$ e'continua se u e v sono continue. Abbiamo inoltre per ipotesi

$$u(x) \leq \fint_{S_r(x)} u d\sigma \leq \fint_{S_r(x)} \max\{u, v\} d\sigma$$

e $v(x) \leq \fint_{S_r(x)} u d\sigma \leq \fint_{S_r(x)} \max\{u, v\} d\sigma$ quindi

$$\max\{u, v\}(x) \leq \fint_{S_r(x)} \max\{u, v\} d\sigma$$

□

Proposition 7.0.3. *Se u e' armonica su* Ω *allora* $u \in \mathrm{SUB}(\Omega)$.

Proof. Abbiamo provato che per ogni funzione armonica vale l'identita' della media. □

Proposition 7.0.4. *Sia* u_n *una successione di funzioni armoniche su* B_R *che siano monotone crescenti ed uniformemente limitate. Allora la funzione* $u(x)$, *definita come il limite puntuale di* $u_n(x)$ *e' armonica, ossia* $u \in C^2(\Omega)$ *e* $\Delta u = 0$.

Proof. Basta provare che $u(x)$e$'$ continua e $u(x) = \fint_{S_R(x)} u d\sigma$. A tal fine osserviamo che per il principio della media (sia su B_R che su S_R) si ha

$$u_n(x) = \fint_{S_R(x)} u_n(y) d\sigma \text{ e } u_n(x) = \fint_{B_R(x)} u_n(y) dy$$

Passando al limite abbiamo che $u(x) = \fint_{B_R(x)} u(y) dy$ e da questa rappresentazione e facile dedurre che $u(x)$ e' continua (basta applicare il teorema di Lebesgue alla successione di funzioni $u(y)\chi_{B_R}(x_n)$ dove $x_n \overset{n\to\infty}{\to} x$). Passando sempre al limite (nell'integrale di superficie) si ha $u(x) = \fint_{S_R(x)} u(y) d\sigma$ □

Proposition 7.0.5. *Proposizione* 4.10. *Sia* $B_R(x_0) \subset \Omega$ *e* $u \in \mathrm{SUB}(\Omega)$. *Sia inoltre* v *tale che* $\Delta v = 0$ *su* $B_R(x_0)$ *e* $v = u$ *su* $\partial B_R(x_0)$. *Allora si ha* $w \in \mathrm{SUB}(\Omega)$ *dove*

$$w(x) = \begin{cases} v(x), & x \in B_R(x_0) \\ u(x), & x \in \Omega \setminus B_R(x_0) \end{cases}$$

Proof. Ovviamente w e' continua. Per provare che e' subarmonica consideriamo tre casi. Se $x \in \Omega \backslash B_R(x_0)$ allora essendo u subarmonica si ha

$$w(x) = u(x) \leq \fint_{S_r(x)} u d\sigma = \fint_{S_r(x)} w d\sigma$$

per r abbastanza piccolo e per $x \in \Omega \backslash B_R(x_0)$.

Se $x \in B_R(x_0)$ invece essendo v armonica su $B_R(x_0)$ concludiamo

$$w(x) = v(x) = \fint_{S_r(x)} v d\sigma = \fint_{S_r(x)} w d\sigma$$

per r abbastanza piccolo.

Resta il caso in cui $x \in \partial B_R(x_0)$. Allora osserviamo che $u - v \in \mathrm{SUB}(B_R(x_0))$ ed inoltre

$u - v = 0$ su $S_R(x_0)$. Quindi per la Proposizione 7.0.1 si ha $u - v \leq 0$ su $B_R(x_0)$. Da cio' deduciamo

$$\begin{aligned} \fint_{S_r(x)} w d\sigma &= \fint_{S_r(x) \cap B_R(x_0)} v d\sigma + \fint_{S_r(x) \cap B_R^c(x_0)} u d\sigma \\ &\geq \fint_{S_r(x) \cap B_R(x_0)} u d\sigma + \fint_{S_r(x) \cap B_R^s(x_0)} u d\sigma \geq u(x) = w(x) \end{aligned}$$

dove abbiamo usato il fatto che $u(x) = w(x)$ su $\partial B_R(x_0)$. □

Possiamo ora introdurre la candidata soluzione al problema (24)

$$\begin{cases} \Delta u = 0, & y \in \Omega \\ u(y) = g(y), & y \in \partial\Omega \end{cases} \tag{7.0.18}$$

dove $g(y) \in \mathscr{C}(\partial\Omega)$. Nel seguito assumeremo $\bar{\Omega}$ compatto. Definiamo (25)

$$v(x) = \sup\{w(x) \mid w \in \mathrm{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega}), w(y) \leq g(y) \quad \forall y \in \partial\Omega\}. \tag{7.0.19}$$

Osserviamo che l'insieme di funzioni $w \in \mathscr{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ tale che $w(y) \leq g(y) \quad \forall y \in \partial\Omega$ e' non vuoto. Infatti basta osservare che la funzione costante $w(x) = \inf_{\partial\Omega} g$ soddisfa questa proprieta'. Osserviamo anche che necessariamente $v(x) < \infty$. A tal fine osserviamo che per la Proposizione 4.6 necessariamente $w(x) \leq \sup_{y \in \partial\Omega} g(y)$ per ogni $w \in \mathscr{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ tale che $w(y) \leq g(y) \quad \forall y \in \partial\Omega$. Come primo passo provi- amo che $\Delta v = 0$ in Ω. Il secondo passo consistera' nell'individuare delle condizioni su Ω che garantiscano $v \in \mathscr{C}(\bar{\Omega}), v(y) = g(y)$ per $y \in \partial\Omega$.

Theorem 7.0.1. *Teorema 4.11. La funzione $v(x)$ definita in* (7.0.19) *é armonica in Ω.*

Proof. sia $x_0 \in \Omega$ e sia $B_R(x_0) \subset \Omega$. Selezioniamo quindi $w_{k,x_0}(x)$ tali che $w_{k,x_0} \in \mathcal{SUB}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ e $w_{k,x_0}(y) \leq g(y) \quad \forall y \in \partial\Omega$ ed inoltre $w_{k,x_0}(x_0) \stackrel{k\to\infty}{\to} v(x_0)$. Allora os- serviamo che siccome $v(x)$ e'definito come sup possiamo assumere le funzioni $w_n(x)$ crescenti (se non lo fossero basta lavorare con

$$\sup_{i=1,\ldots,n} \left\{ w_{1,x_0}(x), \ldots, w_{n,x_0}(x) \right\}.$$

Inoltre possiamo anche assumere che $w_{k,x_0}(x)$ sia armonica su $B_R(x_0)$. Infatti se cosi' non fosse basterebbe considerare

$$\tilde{w}_{k,x_0}(x) = \begin{cases} w_{k,x_0}(x), & x \in \Omega \backslash B_R(x_0) \\ v_k(x), & x \in B_R(x_0) \end{cases}$$

dove $\Delta v_k = 0$ e $v_k(y) = w_{k,x_0}(y)$ per $y \in \partial B_R(x_0)$. Allora dalle proprieta' delle funzioni subarmoniche si vede che $\tilde{w}_{k,x_0}(x) \in \mathcal{SUB}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, inoltre $\tilde{w}_{k,x_0}(y) \leq g(y) \quad \forall y \in \partial\Omega$ ed anche $\tilde{w}_{k,x_0}(x) \geq w_{k,x_0}(x)$. Questa disuguaglianza e ovvia in $\Omega \backslash B_R(0)$ ed e' anche vera su $B_R(x_0)$ poiche' $-\bar{w}_{k,x_0}(x) + w_{k,x_0}(x) = -v_k(x) + w_{k,x_0}(x)$ e' subarmonica su $B_R(x_0)$ ed e' nulla su $\partial B_R(x_0)$. Quindi per Proposizione 4.6 si ha $\tilde{w}_{k,x_0}(x) \geq w_{k,x_0}(x)$. Riassumendo abbiamo $\tilde{w}_{k,x_0}(x)$ armoniche su $\overline{B_R}(x_0)$ ed inoltre $\tilde{w}_{k,x_0}(x)$ crescenti rispetto a k. Come conseguenza del Teorema della media e del Teorema di passaggio al limite sotto il segno di integrale si puo' provare che il limite puntuale di

funzioni armoniche, crescenti e uniformemente limitate e' armonica. Pertanto abbiamo che detta $\bar{w}_{x_0}(x) = \lim_{k\to\infty} \tilde{w}_{k,x_0}(x)$ si ha che $\bar{w}_{x_0}(x)$ e' armonica su $B_R(x_0)$ ed inoltre $\bar{w}_{x_0}(x_0) = v(x_0)$ Dico che necessariamente (26)

$$v(x) = \bar{w}_{x_0}(x), \quad \forall x \in B_R(x_0)$$

e questo ci permetterebbe di conclude che $v(x)$ e' armonica su $B_R(x_0)$ e quindi su Ω data l'arbitrarieta' della palla $B_R(x_0)$. Se (26) non fosse vera avremmo $v(y) > \bar{w}_{x_0}(y)$ per qualche $y \in B_R(x_0)$ e ragionando come sopra potremmo considerare una successione $w_{k,y}$ tali che $w_{k,y}(y) \stackrel{k\to\infty}{\to} v(y)$. A meno di considerare $\max\left\{w_{k,y}, \tilde{w}_{k,x_0}\right\}$ possiamo assumere che $w_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$. Inoltre ragionando come sopra possiamo costruire una ulteriore nuova successione

$$\tilde{w}_{k,y}(x) = \begin{cases} w_{k,y}(x), & x \in \Omega \backslash B_R(x_0) \\ u_k(x), & x \in B_R(x_0) \end{cases}$$

dove $\Delta u_k = 0$ su $B_R(x_0)$ e $u_k = w_{k,y}$ su $\partial B_R(x_0)$ Usando il principio del massimo ed il fatto che $w_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ per ogni $x \in \Omega$ e quindi anche per $x \in \partial B_R(x_0)$ avremo che $\tilde{w}_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ per ogni $x \in \Omega$ Definendo quindi $\bar{w}_y(x) = \lim_{k\to\infty} \tilde{w}_{k,y}(x)$ avremo che $\bar{w}_y$ e' armonica su $B_R(x_0)$ ed inoltre (27)

$$v(x_0) = \bar{w}_y(x_0) \geq \bar{w}_{x_0}(x_0) = v(x_0)$$

La disuguaglianza in (27) segue dal fatto che $\tilde{w}_{k,y}(x) \geq \tilde{w}_{k,x_0}(x)$, per ogni $x \in \Omega$ invece la prima uguaglianza in (27) segue da

$$v(x_0) = \lim_{k\to\infty} \tilde{w}_{k,x_0}(x_0) \leq \lim_{k\to\infty} \tilde{w}_{k,y}(x_0) = v(x_0)$$

dove nell'ultima uguaglianza abbiamo usato che (data la definizione di $v(x_0)$ come sup) $v(x_0) \geq \lim_{k\to\infty} \tilde{w}_{k,y}(x_0) \geq \lim_{k\to\infty} \tilde{w}_{k,x_0}(x_0) = v(x_0)$. Osserviamo inoltre che sic- come $\bar{w}_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ su $B_R(x_0)$ e siccome entrambe sono armoniche e monotone crescenti e' facile dedurre che i loro limiti puntuali saranno armonici su $B_R(x_0)$ ed inoltre $\bar{w}_y(x) \geq \bar{w}_{x_0}(x)$ su $B_R(x_0)$. Siccome da (27) segue che $\bar{w}_y(x_0) = \bar{w}_{x_0}(x_0)$ ne de- duciamo per il principio del massimo che $\bar{w}_y = \vec{w}_{x_0}$ su $B_R(x_0)$ e quindi $v(y) = \bar{w}_y(y) = \bar{w}_{x_0}(y)$, e quindi data l'arbitrarieta' di y abbiamo

Resta solo da capire se $v \in \mathscr{C}(\bar{\Omega})$ e se $v(y) = g(y)$ su $\partial\Omega$. Senza ipotesi ulteriori su Ω questa proprieta' e' falsa ma e' vera per una ampia classe di aperti come vedremo di seguito.

7.0.7 Eigenvalues of Laplace equation in bounded domain with Dirichlet boundary condition.

The general eigenvalue problem

$$\Delta u = \lambda u, x \in \Omega$$
$$u(x) = 0, x \in \partial\Omega.$$

for the case of bounded domain Ω can have only negative eigenvalues. Indeed, multiplying by u and integrating into Ω we find

$$-\int_\Omega |\nabla u|^2 = \lambda \int_\Omega |u|^2$$

so if $\lambda \geq 0$ we get $u = cost = 0$. The fact that λ is real follows from this relation too.

Any two eigenfunctions $(j = 1,2)$

$$\Delta u_j = \lambda_j u_j, x \in \Omega$$
$$u_j(x) = 0, x \in \partial\Omega.$$

we have the orthogonality relation

$$\int_\Omega u_1(x) u_2(x) dx = 0.$$

The set of all eigenfunctions is complete, i.e. if $f(x)$ is orthogonal to all eigenfunctions, then $f = 0$. This property needs more details from functional analysis and we shall stop here this argument.

7.0.8 Application: Nonlinear problem for Laplace equation with Dirichlet data

Let $\Omega = \{|x| < R\}$ be a bounded domain with smooth boundary $\partial\Omega = \{|x| = R\}$. Consider the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned} \tag{7.0.20}$$

where $F(u)$ is a C^1 function, such that

$$F(u) = O(|u|^p)$$

near $u = 0$.

Lemma 7.0.6. *(small data solutions) Let*

$$p > 1.$$

There exists $\varepsilon > 0$ such that for any $f(x) \in C(\partial\Omega)$ with

$$\max_{\partial\Omega} |f(x)| \le \varepsilon,$$

there exists a unique solution $u(x) \in C(\overline{\Omega}) \cap C^1(\Omega)$ to the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Idea of the proof. Let u_k be the sequence defined as follows $u_0 = 0$, and

$$\begin{aligned} \Delta u_{k+1} &= F(u_k), x \in \Omega \\ u_{k+1}(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Set

$$X_k = \sup_{|x| \le R} |u_k(x)|.$$

The the apriori estimates imply

$$X_k \le C\varepsilon + C X_{k+1}^p.$$

Show that this inequality implies that there exists a constant $C_0 > C$ so that

$$X_k \le C_0 \varepsilon.$$

Apply the contraction principle for the sequence u_k showing that

$$Y_k = \max_{|x|\leq R} |u_k(x) - u_{k+1}(x)|$$

satisfies

$$Y_k \leq q Y_{k-1}$$

with some $q \in (0,1)$. So $u_k(x)$ tends uniformly to a function $u(x)$ Using the Poisson formula one can show that $\partial_j u_k(x)$ tends uniformly to a function $\partial_j u(x)$ for $x \in K$, where K is any compact set in Ω.

7.0.9 Laplace equation in exterior domain with Dirichlet boundary condition.

Let $\Omega \subset \mathbb{R}^3$ be the exterior of a compact K with smooth boundary $\partial\Omega$. The general problem

$$\begin{aligned} \Delta u &= F, x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

has to be considered together with a condition that guarantees the uniqueness of the solution, i.e.

$$u(x) = o(1)$$

as $x \to \infty$. As before the problem can be solved extending F in $\mathbb{R}^3$ and representing u as follows

$$u = u_0 + w,$$

where

$$u_0(x) = -\frac{1}{4\pi} \int_\Omega \frac{F(y)}{|x-y|} dy.$$

Then w solves

$$\begin{aligned} \Delta w &= 0, x \in \Omega \\ w(x) &= g(x), x \in \partial\Omega. \end{aligned}$$

where

$$g = f - u_0.$$

Applying the estimates of the previous sections, we get

$$\|(1+|x|)w\|_{L^\infty(\Omega)} \leq C\|g\|_{C(\partial\Omega)} \leq C\left(\|f\|_{C(\partial\Omega)} + \|(1+|x|)u_0\|_{L^\infty(\Omega)}\right) \leq$$

$$\leq C\left(\|f\|_{C(\partial\Omega)} + \|(1+|x|)^{3+\varepsilon} u_0\|_{L^\infty(\Omega)}\right).$$

Problem 7.0.8. *Generalize the above argument for $n \geq 3$.*

This estimate guarantees the uniqueness of the solution. The existence is delicate: needs or sub and supersolutions (Peron method) or methods from Functional analysis (Friedrich's extentions). For the concrete domains: interior or exterior ball, half space we have concrete solutions.

7.0.10 Exterior nonlinear problem for Laplace equation with Dirichlet data

Let $\Omega = \{|x| > R\}$ be an exterior domain with smooth boundary $\partial\Omega = \{|x| = R\}$. Consider the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned} \tag{7.0.21}$$

where $F(u)$ is a C^1 function, such that

$$F(u) = O(|u|^p)$$

near $u = 0$.

Lemma 7.0.7. *(small data solutions) Let*

$$p > 3.$$

There exists $\varepsilon > 0$ such that for any $f(x) \in C(\partial\Omega)$ with

$$\max_{\partial\Omega} |f(x)| \leq \varepsilon,$$

there exists a unique solution $u(x) \in C(\overline{\Omega}) \cap C^1(\Omega)$ to the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Idea of the proof. Let u_k be the sequence defined as follows $u_0 = 0$, and

$$\begin{aligned} \Delta u_{k+1} &= F(u_k), x \in \Omega \\ u_{k+1}(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Set

$$X_k = \sup_{|x| \geq R} |(1 + |x|) u_k(x)|.$$

The the apriori estimates and the assumption $p > 3$ imply

$$X_k \leq C\varepsilon + CX_{k+1}^p.$$

Show that this inequality implies that there exists a constant $C_0 > C$ so that

$$X_k \leq C_0\varepsilon.$$

Apply the contraction principle for the sequence u_k showing that

$$Y_k = \max_{|x|\leq R} |u_k(x) - u_{k+1}(x)|$$

satisfies

$$Y_k \leq qY_{k-1}$$

with some $q \in (0,1)$. So $u_k(x)$ tends uniformly to a function $u(x)$ Using the Poisson formula one can show that $\partial_j u_k(x)$ tends uniformly to a function $\partial_j u(x)$ for $x \in K$, where K is any compact set in Ω. Hence $F(u_k)$ tends uniformly in $C^1(K)$ and we conclude that $u \in C^2(\Omega)$.

Problem 7.0.9. *Generalize the above argument for $n \geq 3$.*

7.0.11 Bessel functions and Stoke's phenomena

The standard Bessel functions $J_\nu(z)$ are solutions to the differential equation

$$z^2 \frac{d^2}{dz^2} w(z) + z\frac{d}{dz} w(z) + (z^2 - \nu^2)w(z) = 0 \tag{7.0.22}$$

More precisely, we have the definition of $J_\nu(z)$ given by

$$J_\nu(z) = z^\nu \left(\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m m!\Gamma(m+1+\nu)} \right), \tag{7.0.23}$$

where $z^\nu = e^{\nu \log z}$ with $\log z = \log|z| + i \arg z$ being the branch of the logarithm defined via the choice of $\arg z$. For example the principle branch is with $\arg z \in (-\pi,\pi)$. Therefore, $J_\nu(z)$ is well defined and analytic for $|\arg z| < \pi$ and has singularity at $z = 0$ when ν is not an integer. The equation (7.0.22) has two linearly independent solutions $J_\nu(z)$ and $J_{-\nu}(z)$.

Near the origin we have the asymptotic expansion

$$J_\nu(z) = z^\nu \left(1 + O(|z|^2)\right), \ \ z \to 0. \tag{7.0.24}$$

The fact that a solution of an ODE, near an irregular singularity, in different sectors of the complex plane in general shows different asymptotic behavior was observed and studied by Stokes [?] and is, therefore, named Stokes' phenomenon.

More precisely, the asymptotic expansion for the $J_\nu(x)$ can be found from section 7.2 in [?]

$$J_\nu(z) = c_1 \left(\frac{2}{\pi}\right)^{1/2} e^{-(\log|z| + \mathrm{i}\arg z)/2} e^{i(z-\pi\nu/2-\pi/4)} \left(1 + O\left(|z|^{-1}\right)\right) + \\ + c_2 \left(\frac{2}{\pi}\right)^{1/2} e^{-(\log|z| + \mathrm{i}\arg z)/2} e^{-i(z-\pi\nu/2-\pi/4)} \left(1 + O\left(|z|^{-1}\right)\right), \tag{7.0.25}$$

where the constants c_1, c_2 depend on the choice of the sector in the complex plane.

Indeed, for any integer p we have

$$c_1 = c_2 = \frac{1}{2} e^{\mathrm{i}p(2\nu+1)\pi} \text{ if } \arg z \in ((2p-1)\pi, (2p+1)\pi) \tag{7.0.26}$$

and

$$c_1 = \frac{1}{2} e^{\mathrm{i}(p+1)(2\nu+1)\pi}, c_2 = \frac{1}{2} e^{\mathrm{i}p(2\nu+1)\pi} \text{ if } \arg z \in (2p\pi, (2p+2)\pi) \tag{7.0.27}$$

due to the Stokes phenomenon (see [?] or section 7.2 in [?]).

The proof of these asymptotic expansions can be deduced via the relation

$$J_\nu(z e^{\mathrm{i}m\pi}) = e^{\mathrm{i}m\pi\nu} J_\nu(z) \tag{7.0.28}$$

valid for any integer m. The Neumann function is defined as a linear combination of $J_\pm$

$$Y_\nu(z) = \frac{1}{\sin(\nu\pi)} \left(\cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)\right). \tag{7.0.29}$$

The Hankel functions are given by

$$H_\nu^{(1)}(z) = J_\nu(z) + \mathrm{i} Y_\nu(z), \\ H_\nu^{(2)}(z) = J_\nu(z) - \mathrm{i} Y_\nu(z). \tag{7.0.30}$$

Their integral representation (see 7.3.6 (28) in [?]) provided $\arg z \in (0, \pi)$

$$\pi H_\nu^{(1)}(z) = -\mathrm{i} e^{-\mathrm{i}\nu\pi/2} \int_{-\infty}^{\infty} e^{\mathrm{i}z\cosh t} e^{-\nu t} dt = \\ = -\mathrm{i} e^{-\mathrm{i}\nu\pi/2} \int_0^{\infty} e^{\mathrm{i}z(s+1/s)/2} s^{-\nu-1} ds \tag{7.0.31}$$

Some partial values of $\nu = \pm 1/2$ give the following result

$$H_{-1/2}(w) = i H_{1/2}(w)$$

$$H_{1/2}(w) = -i\left(\frac{2}{\pi w}\right)^{1/2} e^{iw}.$$

In particular we shall need for $w = iy$ with $y > 0$ the following

$$H_{1/2}(iy) = -ie^{-i\pi/4}\left(\frac{2}{\pi y}\right)^{1/2} e^{-y}. \tag{7.0.32}$$

$$H_{-1/2}(iy) = e^{-i\pi/4}\left(\frac{2}{\pi y}\right)^{1/2} e^{-y}. \tag{7.0.33}$$

The modified Bessel functions are combinations of $J_\nu(iz), J_{-\nu}(\mathrm{i}z)$ and therefore they are solutions to the differential equation

$$z^2\frac{d^2}{dz^2}w(z) + z\frac{d}{dz}w(z) - (z^2+\nu^2)w(z) = 0. \tag{7.0.34}$$

The modified Bessel function $I_\nu(z)$ is defined by

$$I_\nu(z) = e^{-\mathrm{i}\nu\pi/2}J_\nu(\mathrm{i}z). \tag{7.0.35}$$

For $\nu = 0$ we have the expansion near the origin

$$I_0(z) = 1 + O(|z|^2). \tag{7.0.36}$$

The relation (7.0.28) implies

$$I_\nu(ze^{\mathrm{i}m\pi}) = e^{\mathrm{i}m\pi\nu}I_\nu(z) \tag{7.0.37}$$

valid for any integer m.

Then two linearly independent solutions to (7.0.34) are $I_\nu(z)$ and $I_{-\nu}(z)$, for $\nu \neq 0$.

The asymptotics expansion of $I_\nu(z)$ for $|z| \to \infty$ depends where is $\mathrm{Arg} z$, therefore we have to be careful for Stokes phenomena. If $\arg z \in (-\pi/2, 3\pi/2)$, then we have the asymptotics

$$\begin{aligned} I_\nu(z) = {} & \frac{e^z}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}\left(1+O(|z|^{-1})\right)+ \\ & +\frac{e^{-z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right). \end{aligned} \tag{7.0.38}$$

From this we take $\arg z \in (-\pi/2+m\pi, 3\pi/2+m\pi)$ we get

$$\begin{aligned} I_\nu(z) = {} & \frac{e^{(-1)^m z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}m(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right)+ \\ & +\frac{e^{-(-1)^m z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}(m+1)(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right). \end{aligned} \tag{7.0.39}$$

In this way we conclude

(7.0.40) $$\arg z \in (-\pi/2 + m\pi, 3\pi/2 + m\pi) \implies I_\nu(z) =$$
$$= \frac{e^{-\mathrm{i}\arg z)/2}}{\sqrt{2\pi|z|}} \left[e^{(-1)^m z} e^{\mathrm{i}m(\nu+1/2)\pi} + e^{-(-1)^m z} e^{\mathrm{i}(m+1)(\nu+1/2)\pi} \right] \left(1 + O(|z|^{-1})\right).$$

If $\arg z \in (-3\pi/2 + 2p\pi, \pi/2 + 2p\pi)$, p integer, then we take $m = 2p-1$ and then we get

(7.0.41) $$I_\nu(z) = \frac{e^{-\mathrm{i}\arg z)/2}}{\sqrt{2\pi|z|}} \left[e^{-z} e^{\mathrm{i}(2p-1)(\nu+1/2)\pi} + e^{z} e^{\mathrm{i}2p(\nu+1/2)\pi} \right] \left(1 + O(|z|^{-1})\right).$$

The modified Bessel function $K_\nu(z)$ is defined by

(7.0.42) $$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)} \left(I_{-\nu}(z) - I_\nu(z)\right) =$$
$$= \frac{\pi}{2\sin(\nu\pi)} \left(e^{\mathrm{i}\nu\pi/2} J_{-\nu}(\mathrm{i}z) - e^{-\mathrm{i}\nu\pi/2} J_\nu(\mathrm{i}z)\right).$$

Obviously, we have

(7.0.43) $$K_\nu(z) = K_{-\nu}(z).$$

We have the following relation between $K_\nu(z)$ and Hankel functions (see 10.27.8 in [**?**]) with $-\pi \le \arg z \le \pi/2$

(7.0.44) $$K_\nu(z) = \frac{\pi \mathrm{i}}{2} e^{\nu \mathrm{i}\pi/2} H_\nu^{(1)}(\mathrm{i}z).$$

The differential equation satisfied by $K_\nu(z)$ is again (7.0.34).

Similarly to (7.0.49) , we have the relations

(7.0.45) $$K_\nu(z e^{\mathrm{i}m\pi}) = e^{-\mathrm{i}m\pi\nu} K_\nu(z) - i\pi \frac{\sin(m\nu\pi)}{\sin(\nu\pi)} I_\nu(z).$$

valid for any integer m.

Near the origin we have the expansion

(7.0.46) $$K_0(z) = -I_0(z) \log\left(\frac{z}{2}\right) + O(1) = -\log(z) + O(1), \quad z \to 0.$$

We have the following asymptotic expansion valid if $|z| \to \infty$ and $\arg z \in (-3\pi/2, 3\pi/2)$ (see relation (20), section 7.23 in [**?**])

(7.0.47) $$K_\nu(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-(\log|z| + \mathrm{i}\arg z)/2} e^{-z} \left(1 + O(|z|^{-1})\right).$$

For $\arg w \in (\pi/2, 5\pi/2)$ we can take $w = e^{i2\pi} z$ so that

$$\arg w \in (\pi/2, 5\pi/2) \Longrightarrow \operatorname{Arg} z \in (-3\pi/2, 3\pi/2)$$

use (7.0.49), so we can write

$$K_\nu(w) = K_\nu(e^{2i\pi} z) = \frac{\pi}{2\sin(\nu\pi)} \left(I_{-\nu}(e^{2i\pi} z) - I_\nu(e^{2i\pi} z) \right) =$$

$$= \frac{\pi}{2\sin(\nu\pi)} \left(e^{-2i\pi\nu} I_{-\nu}(z) - e^{2i\pi\nu} I_\nu(z) \right)$$

and using the asymptotic expansions (7.0.41) for z we arrive at

$$K_\nu(w) = \sqrt{\frac{\pi}{2}} \frac{e^{-i(arg w)/2}}{\sqrt{|w|}} \left[2i\cos(\nu\pi) e^{w} + e^{-w} \right] \left(1 + O(|w|^{-1}) \right) \tag{7.0.48}$$

$$I_\nu(ze^{im\pi}) = e^{im\pi\nu} I_\nu(z) \tag{7.0.49}$$

For $\nu = 1/4$ we get

$$\begin{aligned} K_{1/4}(z) &= \frac{\pi}{\sqrt{2}} (I_{-1/4}(z) - I_{1/4}(z)) = \\ &\frac{\pi}{\sqrt{2}} \left(e^{i\pi/8} J_{-1/4}(iz) - e^{-i\pi/8} J_{1/4}(iz) \right). \end{aligned} \tag{7.0.50}$$

The modified Bessel function $K_\nu(z)$ has the integral representation (see Eq. (16) in section 7.3.4 in [2])

$$\Gamma(\nu + 1/2) K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \int_0^\infty e^{-t} t^{\nu - 1/2} \left(1 + \frac{t}{2z} \right)^{\nu - 1/2} dt \tag{7.0.51}$$

provided $\operatorname{Re}(\nu) > -1/2$, $|\arg(z)| < \pi$. If $z > 0$, the a change of variable implies

$$\Gamma(\nu + 1/2) K_\nu(z) = \sqrt{\pi} \frac{e^{-z}}{2^\nu} z^\nu \int_0^\infty e^{-tz} t^{\nu - 1/2} (2 + t)^{\nu - 1/2} dt. \tag{7.0.52}$$

In equivalent way, we can write

$$\sqrt{\pi} e^{-z} \int_0^\infty e^{-tz} t^{\nu - 1/2} (2 + t)^{\nu - 1/2} dt = z^{-\nu} 2^\nu \Gamma(\nu + 1/2) K_\nu(z). \tag{7.0.53}$$

7.0.12 Foundamental solution of the Helmholtz equation

The Helmholtz equation

$$(\lambda - \Delta)G = \delta$$

with $\lambda > -$ has a fundamental solution

$$G_{\lambda,n}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} \frac{d\xi}{\lambda + |\xi|^2} = (2\pi)^{-n} \mathrm{Re}\left(\int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} \frac{d\xi}{\lambda + |\xi|^2} \right).$$

Here and below $\lambda > 0$. Setting

$$G_n(x) = G_{1,n}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} \frac{d\xi}{1 + |\xi|^2},$$

we see that

$$G_{\lambda,n}(x) = \lambda^{(n-2)/2} G_n(\sqrt{\lambda}|x|). \tag{7.0.54}$$

The main result of the section is the following

Lemma 7.0.8. *We have the relation*

$$G_{\lambda,n}(x) = (2\pi)^{-n/2} \lambda^{(n-2)/4} \frac{K_{(n-2)/2}(\sqrt{\lambda}|x|)}{|x|^{(n-2)/2}}. \tag{7.0.55}$$

Proof. **Case:** $n = 1$

In this case, the right side of (7.0.55) is given by

$$(2\pi)^{-1/2} |x|^{1/2} K_{-1/2}(\sqrt{\lambda}|x|)$$

Since

$$K_{1/2}(|x|) = K_{-1/2}(|x|) = \sqrt{\frac{\pi}{2}} \frac{e^{-|x|}}{\sqrt{|x|}},$$

we see that (7.0.55) becomes

$$G_{\lambda,1}(x) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x|} \tag{7.0.56}$$

We have the relation

$$G_1(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-\mathrm{i}x\xi} \frac{d\xi}{1 + \xi^2} =$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos(x\xi) \frac{d\xi}{1 + \xi^2} = (2\pi)^{-1} \int_{\infty}^{\infty} e^{\mathrm{i}x\xi} \frac{d\xi}{1 + \xi^2}.$$

The function $K_1(x)$ is even and hence it is sufficient to consider the case $x > 0$. A simple application of the Cauchy theorem implies that for any $x > 0$ we have the identities

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}xz}}{1+z^2}\, dz = \frac{\mathrm{i}^{-1}}{2} e^{-x} \tag{7.0.57}$$

and more generally for any $\lambda > 0$ we have

$$\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}xz}}{\lambda+z^2}\, dz = \int_{-\infty}^{\infty} \frac{\cos(xz)}{\lambda+z^2}\, dz = \frac{\pi}{\sqrt{\lambda}} e^{-\sqrt{\lambda}|x|} \tag{7.0.58}$$

so

$$G_1(x) = \frac{e^{-|x|}}{2} \tag{7.0.59}$$

and hence we have (7.0.56).

Case: $n = 2$**.**

We assume $x = (0, |x|)$ and then

$$G_2(x) = (2\pi)^{-2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{e^{-i|x|\xi_2}\, d\xi_2}{1+\xi_1^2+\xi_2^2} \right) d\xi_1.$$

So applying (7.0.58) with $\lambda = 1 + \xi_1^2$, we find

$$G_2(x) = \int_0^{\infty} \frac{1}{2\pi\sqrt{1+\xi_1^2}} e^{-\sqrt{1+\xi_1^2}|x|} d\xi_1$$

Now we make change of variables $\xi_1 \in (0,\infty) \to t \in (0,\infty)$ defined by

$$1+\xi_1^2 = (1+t)^2, \implies \xi_1 = \sqrt{t(t+2)},\ \ \xi_1 d\xi_1 = (t+1)dt$$

and get

$$G_2(x) = (2\pi)^{-1} e^{-|x|} \int_0^{\infty} e^{-t|x|} t^{-1/2} (2+t)^{-1/2} dt$$

Using (7.0.53), we find

$$G_2(x) = \frac{1}{2\pi^{3/2}} \Gamma(1/2) K_0(|x|).$$

Using the identity

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

we find

$$G_2(x) = (2\pi)^{-1} K_0(|x|). \tag{7.0.60}$$

Further, (7.0.54) implies

$$G_{\lambda,2}(x) = (2\pi)^{-1} K_0(\sqrt{\lambda}|x|). \tag{7.0.61}$$

Case: $n = 3$**.**
We have the relations

$$G_3(x) = (2\pi)^{-2} \int_0^\infty \left(\int_0^\pi e^{-i|x|\rho\cos\theta} \sin\theta d\theta \right) \frac{\rho^2 d\rho}{1+\rho^2} =$$

$$= \frac{1}{(2\pi)^2 \mathrm{i}|x|} \int_0^\infty \left(e^{i|x|\rho} - e^{-i|x|\rho} \right) \frac{\rho d\rho}{1+\rho^2} = \frac{1}{2\pi^2|x|} \int_0^\infty \sin(|x|\rho) \frac{\rho d\rho}{1+\rho^2}.$$

A simple application of the Cauchy theorem implies that for any $R > 0$ we have the identities

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{e^{\mathrm{i}Rz}}{1+z^2}\, z^k dz = \frac{\mathrm{i}^{k-1}}{2} e^{-R}. \tag{7.0.62}$$

Hence, we have the relations

$$\pi \mathrm{i}^k e^{-R} = \int_{-\infty}^\infty \frac{e^{\mathrm{i}Rz}}{1+z^2}\, z^k dz =$$

$$= \int_0^\infty \frac{e^{\mathrm{i}Rz} + (-1)^{k+2} e^{-\mathrm{i}Rz}}{1+z^2}\, z^k dz.$$

Now we can use the fact that

$$e^{\mathrm{i}Rz} + (-1)^{k+2} e^{-\mathrm{i}Rz} = \begin{cases} 2\cos(Rz), & \text{if } k \text{ is even, } z > 0, R > 0; \\ 2\mathrm{i}\sin(Rz), & \text{if } k \text{ is odd, } z > 0, R > 0. \end{cases}$$

Thus for k even we have

$$\int_0^\infty \frac{2\cos(Rz)}{1+z^2}\, z^k dz = \pi \mathrm{i}^k e^{-R}$$

and we deduce

$$\int_0^\infty \frac{\cos(R\rho)}{1+\rho^2}\, \rho^k dz = \frac{\pi}{2} (-1)^{k/2} e^{-R}. \tag{7.0.63}$$

For k odd we have

$$\int_0^\infty \frac{\sin(R\rho)}{1+\rho^2}\, \rho^k dz = \frac{\pi}{2} (-1)^{(k-1)/2} e^{-R}. \tag{7.0.64}$$

Applying now (7.0.64) with $k=1$, we find

$$G_3(x) = \frac{1}{4\pi|x|}\, e^{-|x|}. \tag{7.0.65}$$

Then (7.0.54) implies

$$G_{\lambda,3}(x) = \frac{1}{4\pi|x|}\, e^{-\sqrt{\lambda}|x|}. \tag{7.0.66}$$

General case $n \geq 3$

Since $G_n(x)$ is a radial function, we can take $x=(0,\cdots,0,x_n)=(0,\cdots,0,|x|)$. We have to compute

$$G_{1,n}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi}\frac{d\xi}{1+|\xi|^2} =$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\left(\int_{-\infty}^{\infty} e^{-\mathrm{i}|x|\xi_n}\frac{d\xi_n}{1+|\xi'|^2+\xi_n^2}\right)d\xi'.$$

Applying the identity (7.0.58), we find

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}|x|\xi_n}\frac{d\xi_n}{1+|\xi'|^2+\xi_n^2} = \frac{\pi}{\sqrt{1+|\xi'|^2}}e^{-\sqrt{1+|\xi'|^2}|x|}$$

and introducing polar coordinates $\rho=|\xi'|$ in R^{n-1} we get

$$G_{1,n}(x) = (2\pi)^{-n}\mu(S^{n-2})\int_0^{\infty}\frac{\pi}{\sqrt{1+|\rho|^2}}e^{-\sqrt{1+|\rho|^2}|x|}\rho^{n-2}d\rho$$

Now we make change of variables $\rho\in(0,\infty)\to t\in(0,\infty)$ defined by

$$1+\rho^2 = (1+t)^2, \implies \rho=\sqrt{t(t+2)}, \;\; \rho d\rho = (t+1)dt$$

and get

$$G_{1,n}(x) = (2\pi)^{-n}\mu(S^{n-2})\pi e^{-|x|}\int_0^{\infty} e^{-t|x|}t^{(n-2)/2-1/2}(2+t)^{(n-2)/2-1/2}dt$$

Using (7.0.53), we find

$$G_{1,n}(x) = (2\pi)^{-n}\mu(S^{n-2})\sqrt{\pi}2^{\nu}\Gamma(\nu+1/2)\frac{K_\nu(|x|)}{|x|^\nu}, \nu=\frac{n-2}{2} \tag{7.0.67}$$

Comparing the relation (7.0.65) with (7.0.67) and using the relations

$$K_{1/2}(|x|) = \sqrt{\frac{\pi}{2}}\frac{e^{-|x|}}{\sqrt{|x|}}, \;\; \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \;\; \mu(\mathbb{S}^1)=2\pi,$$

we see that (7.0.67) with $n = 3$ coincides with (7.0.65).

Since $\mu(\mathbb{S}^{n-2}) = (2\pi)^{(n-1)/2}/\Gamma((n-1)/2)$, we find

$$G_n(x) = G_{1,n}(x) = (2\pi)^{-n/2}\frac{K_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}. \tag{7.0.68}$$

Further, (7.0.54) implies (7.0.55).

□

Now we turn to the case of Helmholtz equation

$$(-z^2 - \Delta)\mathfrak{G} = \delta$$

with $z = i\sqrt{\lambda}$. Its fundamental solution is

$$\mathfrak{G}(x) = \mathfrak{G}_{z,n}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi}\frac{d\xi}{-z^2+|\xi|^2} = G_{-z^2,n}(|x|).$$

First we consider the case $n = 1$. Then for any z with $\mathrm{Im} z > 0$ we have

$$\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}xw}}{-z^2+w^2}\,dw = \frac{\mathrm{i}\pi}{z}e^{\mathrm{i}z|x|} \tag{7.0.69}$$

and therefore

$$\mathfrak{G}_{z,1}(x) = \frac{\mathrm{i}}{2z}e^{\mathrm{i}z|x|} \tag{7.0.70}$$

From Lemma 7.0.8 we have the relation

$$G_{\lambda,n}(x) = (2\pi)^{-n/2}\lambda^{(n-2)/4}\frac{K_{(n-2)/2}(\sqrt{\lambda}|x|)}{|x|^{(n-2)/2}}$$

so

$$\mathfrak{G}_{z,n}(x) = G_{-z^2,n}(|x|) = (2\pi)^{-n/2}|z|^{(n-2)/2}\frac{K_{(n-2)/2}(-iz|x|)}{|x|^{(n-2)/2}}.$$

Now we use the relation (7.0.44) and find

$$K_{(n-2)/2}(-iz|x|) = \frac{\pi\mathrm{i}}{2}e^{(n-2)\mathrm{i}\pi/4}H^{(1)}_{(n-2)/2}(z|x|)$$

so

$$\mathfrak{G}_{z,n}(x) = (2\pi)^{-n/2}\frac{\pi\mathrm{i}}{2}|z|^{(n-2)/2}e^{(n-2)\mathrm{i}\pi/4}\frac{H^{(1)}_{(n-2)/2}(z|x|)}{|x|^{(n-2)/2}}. \tag{7.0.71}$$

For $n = 1$ we use

$$\mathfrak{G}_{z,1}(x) = (2\pi)^{-1/2}\frac{\pi\mathrm{i}}{2}|z|^{-1/2}e^{-\mathrm{i}\pi/4}|x|^{1/2}H^{(1)}_{-1/2}(z|x|)$$

and (7.0.33) and find

$$\mathfrak{G}_{z,1}(x) = (2\pi)^{-1/2}\frac{\pi \mathrm{i}}{2}|z|^{-1/2}e^{-\mathrm{i}\pi/4}|x|^{1/2}e^{-i\pi/4}\left(\frac{2}{\pi|z||x|}\right)^{1/2} e^{\mathrm{i}z|x|} =$$

$$= (2\pi)^{-1/2}\frac{\pi \mathrm{i}}{2}\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{\mathrm{i}z|x|}}{z} = \frac{\mathrm{i}}{2z}e^{\mathrm{i}z|x|}$$

and this relation is compatible with (7.0.70). For $n = 2$ we have

$$\mathfrak{G}_{z,2}(x) = (2\pi)^{-1}\frac{\pi \mathrm{i}}{2}H_0^{(1)}(z|x|) = \frac{i}{4}H_0^{(1)}(z|x|).$$

This result is compatible with (5.15) Chapter I.5 in [1].

7.0.13 Helmholz equation in the space $\mathbb{R}^3$.

The equation

$$\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3$$

is called Helmholz equation Taking $f(x) \in C(\mathbb{R}^3)$ with compact support one can represent the unique solution as follows

$$u(x) = -\frac{1}{4\pi}\int_K \frac{e^{\pm i\lambda|x-y|}f(y)}{|x-y|}dy,$$

where here and below K denotes the support of f.

One can verify that

$$(\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x-y|}}{|x|}\right) = -4\pi\delta$$

in the sense of distributions in $\mathbb{R}^3$. Indeed taking any test function φ we apply Gauss - Green formula for the domain $\{|x| \geq \varepsilon\}$ and using the fact that

$$(\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) = 0 \quad |x| \neq 0,$$

we find

$$\int_{|x|>\varepsilon}\left((\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\right)\varphi(x)dx - \int_{|x|>\varepsilon}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)(\Delta + \lambda^2)\varphi(x)dx =$$

$$-\int_{|x|=\varepsilon}\partial_r\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\varphi(x)dS_x + \int_{|x|=\varepsilon}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\partial_r\varphi(x)dS_x,$$

where here and below

$$\partial_r = \sum_{j=1}^{n} \frac{x_j}{|x|}\partial_j.$$

Taking into account the fact that

$$\partial_r\left(\frac{1}{|x|}\right) = -\frac{1}{|x|^2}$$

and introducing spherical coordinate $x = \varepsilon\omega, |\omega| = 1$, we find

$$\int_{|x|=\varepsilon} \partial_r\left(\frac{1}{|x|}\right)\varphi(x)dS_x = -\int_{|\omega|=1}\varphi(\varepsilon\omega)dS_\omega,$$

$$\int_{|x|=\varepsilon}\left(\frac{1}{|x|}\right)\partial_r\varphi(x)dS_x = \varepsilon\int_{|\omega|=1}\partial_r\varphi(\varepsilon\omega)d\omega$$

so taking the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\partial_r\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\varphi(x)dS_x = -4\pi\varphi(0),$$

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\partial_r\varphi(x)dS_x = 0,$$

so we arrive at

$$-\int_{\mathbb{R}^3}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)(\Delta+\lambda^2)\varphi(x)dx = 4\pi\varphi(0)$$

and the identity

$$-\frac{1}{4\pi}(\Delta+\lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) = \delta.$$

The function

$$E_\pm(x) \in C^\infty(\mathbb{R}^3\setminus 0)$$

satisfying

$$(\Delta+\lambda^2)E_\pm = \delta$$

in the sense of distributions is called fundamental solutions of the Helmholz operator and they enable one to represent the solution of the Laplace equation

$$\Delta u = f,\ \ f \in C_0^\infty,$$

as follows

$$u_\pm(x) = \int_{\mathbb{R}^n} E_\pm(x-y)f(y)dy.$$

The uniqueness of the solution is guaranteed by the radiation condition

$$u(x) = \frac{e^{\pm i\lambda|x|}}{|x|}a\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|^2}\right)$$

at infinity.

Problem 7.0.10. *(smoothing property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 7.0.11. *(smoothing property) If $f(x) \in L^\infty(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 7.0.12. *(smoothing property) If $f(x) \in C^k(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^{k+1}(\mathbb{R}^n)$*

Problem 7.0.13. *(smoothing and decay property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$ and*

$$|u(x)| \leq \frac{C\|f\|_{C(K)}}{1+|x|}.$$

Problem 7.0.14. *(representation) If $f(x) \in C(\mathbb{R}^3)$ satisfies the estimate*

$$|f(x)| \leq ce^{-|x|}$$

and $u(x) \in C^2(\mathbb{R}^3)$ is a bounded function that is a solution to

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

then

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\lambda|x-y|}f(y)}{|x-y|}dy.$$

Hint: Apply the max principle of Problem ?? and derive the uniqueness.

Problem 7.0.15. *(a priori estimate) If $f(x) \in C(\mathbb{R}^3)$ satisfies the estimate*

$$|f(x)| \leq ce^{-A|x|}$$

and $u(x) \in C^2(\mathbb{R}^3)$ is a bounded function that is a solution to

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

with $0 < \lambda < A$ then

$$|u(x)| \leq \frac{C\, e^{-\lambda|x|}}{|x|}.$$

Problem 7.0.16. *(some integrals for radial functions)*

$$\int_{\S^2} F(|x+r\omega|)d\omega = \frac{c}{|x|r}\int_{||x|-r|}^{|x|+r} F(\lambda)\lambda d\lambda$$

Problem 7.0.17. *(representation for radial solutions) If* $f(x) = f(|x|) \in C(\mathbb{R}^3)$ *satisfies the estimate*

$$|f(x)| \le ce^{-|x|}$$

and $u(x) = u(|x|) \in C^2(\mathbb{R}^3)$ *is a bounded radial function that is a solution to*

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

then

$$u(x) = -\frac{c}{|x|}\int_0^\infty e^{-r}\int_{||x|-r|}^{|x|+r} f(\lambda)\lambda d\lambda dr =$$

$$= \int_0^{|x|} e^{-|x|}\sinh(\lambda) f(\lambda)\lambda d\lambda + \int_{|x|}^\infty \sinh(|x|)e^{-\lambda} f(\lambda)\lambda d\lambda.$$

Chapter 8

Heat equations

8.1 Heat equation in $\mathbb{R}^n$.

8.1.1 Fundamental solution

The heat equation has the form

$$\partial_t u = \Delta u, t \in \mathbb{R}, x \in \mathbb{R}^n.$$

The corresponding Cauchy problem has initial data

$$u(0, x) = f(x).$$

Applying a Fourier transform, we find

$$\widehat{u}(t, \xi) = e^{-t|\xi|^2} \widehat{f}(\xi).$$

So the inverse Fourier transform gives

$$u(t, x) = c \int_{\mathbb{R}^n} E(t, x - y) f(y) dy,$$

where $E(t, x)$ is the Fourier transform of

$$e^{-|\xi|^2 t}$$

We start with the Fourier transform of the function

$$\mathrm{e}^{-\xi^2} \in S(\mathbf{R}^n),$$

i.e. our goal is to compute

$$\hat{f}(x) = \int_{\mathbf{R}^n} \mathrm{e}^{-i\xi x - \xi^2} d\xi.$$

We shall consider only the case $n = 1$, since the case $n \geq 2$ can be treated in a similar manner.

Using the identity

$$-ix\xi - \xi^2 = -(ix/2+\xi)^2 - x^2/4,$$

we get

$$\hat{f}(x) = \mathrm{e}^{-x^2/4} \int \mathrm{e}^{-(\xi+ix/2)^2} d\xi.$$

Let $x > 0$ for determinacy. Since the function

$$z = \xi + ix/2 \in \mathbf{C} \to \mathrm{e}^{-z^2}$$

is an entire function and for z in the strip $\{z; \mathrm{Im} z \in [0, x/2]\}$ we have the estimate

$$|\mathrm{e}^{-z^2}| \leq C(\xi)\mathrm{e}^{x^2/4},$$

we can change the path of integration $\mathrm{Im} z = x/2$ in

$$\hat{f}(x) = \mathrm{e}^{-x^2/4} \int_{\mathrm{Im} z = x/2} \mathrm{e}^{-z^2} dz.$$

into $\mathrm{Im} z = 0$ so

$$\hat{f}(x) = \mathrm{e}^{-x^2/4} \int_{\mathrm{Im} z = 0} \mathrm{e}^{-z^2} dz = \mathrm{e}^{-x^2/4} \int_{\mathbf{R}} \mathrm{e}^{-v^2} dv.$$

To determine the constant

$$c = \int_{\mathbf{R}} \mathrm{e}^{-v^2} dv,$$

we use the fact that

$$c^2 = \int_{\mathbf{R}} \mathrm{e}^{-x_1^2} dx_1 \int_{\mathbf{R}} \mathrm{e}^{-x_2^2} dx_2.$$

Introducing polar coordinates in $\mathbf{R}^2$

$$x = (x_1, x_2) = \rho(\cos\varphi, \sin\varphi),$$

where $\rho > 0, \varphi \in [0, 2\pi)$, we get

$$c^2 = 2\pi \int_0^\infty \mathrm{e}^{-\rho^2} \rho d\rho = \pi.$$

So $c = \sqrt{\pi}$ and

$$\widehat{\mathrm{e}^{-x^2}}(\xi) = \sqrt{\pi}\mathrm{e}^{-\xi^2/4}$$

provided the space dimension is $n = 1$.

For $n-$dimensional case the Fourier transform $F(f)(x)$ of

$$f(\xi) = \mathrm{e}^{-R\xi^2}, \xi \in \mathbf{R}^n$$

is

$$F(\mathrm{e}^{-R^2\xi^2})(x) = R^{-n/2}\pi^{n/2}\mathrm{e}^{-x^2/(4R)}$$

for any space dimension $n \geq 1$ and any $R > 0$.

Hence, with

$$E(t,x) = c\frac{e^{-x^2/4t}}{t^{n/2}}, \ c = (4\pi)^{-n/2}, \tag{8.1.1}$$

we have

$$u(t,x) = E(t) * f(x) = c\int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4t}}{t^{n/2}} f(y)dy.$$

8.1.2 Smoothing properties of the convolution $E(t) * f$

Using the simple property

$$E(t,x) = c\frac{e^{-|x|^2/4t}}{t^{n/2}} \in L^r(\mathbb{R}^n_x)$$

we see that

$$\|E(t,\cdot)\|^r_{L^r(\mathbb{R}^n)} = c^r t^{-nr/2}\int_{\mathbb{R}^n} e^{-r|x|^2/4t}dx =$$

$$= c^r(4/r)^{n/2}t^{-nr/2+n/2}\int_{\mathbb{R}^n} e^{-|z|^2}dz = c^r(4/r)^{n/2}t^{-nr/2+n/2}\pi^{n/2} \sim t^{-nr/2+n/2},$$

and hence

$$\|E(t,\cdot)\|^r_{L^r(\mathbb{R}^n)} = (4\pi)^{-n/2(1-1/r)}r^{-n/2}t^{-n/2(1-1/r)}. \tag{8.1.2}$$

Using the Young inequality we find

$$\|E(t) * f\|_{L^p(R^n)} \leq C(p,q)t^{-n/2(1/q-1/p)}\|f\|_{L^q(\mathbb{R}^n)} \tag{8.1.3}$$

where

$$1 \leq q \leq p \leq \infty$$

and

$$C(p,q) = (4\pi)^{-n/2(1/q-1/p)}(1+1/p-1/q)^{-n/2}.$$

This implies

$$\|E(t) * f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \tag{8.1.4}$$

Lemma 8.1.1. *The operator*

$$f \in C_0^\infty(\mathbb{R}^n) \to E(t) * f$$

can be extended to an operator on L^p such that

a) $E(t) * f \in L^q(\mathbb{R}^n), q \geq p$ *with norm bounded by*

$$C(p,q)t^{-n/2(1/q-1/p)}, \;\; C(p,q) = (4\pi)^{-n/2(1/q-1/p)}(1+1/p-1/q)^{-n/2};$$

b) $E(t) * f \in C([0,T]; L^p(\mathbb{R}^n));$

c) *if $p = 2$, then $E(t) * f \in C((0,T); H^k(\mathbb{R}^n)$ for any $k > 0$ integer;*

d) *if $p = 2$, then $E(t) * f \in C^\ell((0,T); H^k(R^n)$ for any $\ell, k > 0$ integers.*

Proof. The property a) follows from (8.1.4). b) follows from (8.1.4) with $p = \infty$ and the Lebesgue convergence theorem. Indeed, we have

$$E(t+h) * f(x) = c\int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4(t+h)}}{(t+h)^{n/2}} f(y)dy = c\int_{\mathbb{R}^n} e^{-|z|^2/4} f(x-\sqrt{t+h}z)dz$$

and

$$E(t+h) * f(x) - E(t) * f(x) = c\int_{\mathbb{R}^n} e^{-|z|^2/4}\left(f(x-\sqrt{t+h}z) - f(x\sqrt{t}z)\right)dz.$$

So

$$\|E(t+h) * f(x) - E(t) * f(x)\|_{L^p} \leq c\int_{\mathbb{R}^n} e^{-|z|^2/4}\left\|f(x-\sqrt{t+h}z) - f(x\sqrt{t}z)\right\|_{L^p_x} dz.$$

To show the smallness of this L^p norm, when h is small and $t > 0$ is fixed it is sufficient to show the smallness of

$$\int_{|z|\leq R} e^{-|z|^2/4}\left\|f(x-\sqrt{t+h}z) - f(x\sqrt{t}z\right\|_{L^p_x} dz$$

but this follows from

$$\lim_{h\to 0}\left\|f(x-\sqrt{t+h}z) - f(x\sqrt{t}z\right\|_{L^p_x} = 0$$

for z bounded, $t > 0$ fixed. We turn to the proof of c). First, we shall show that

$$u(t,x) = E(t) * f(x) \in C((0,T); H^k(\mathbb{R}^n)) \tag{8.1.5}$$

Indeed, for any fixed $t > 0$ we have $\partial_x^\alpha E(t,x)$ is continuous in $t > 0$ and

$$|\partial_x^\alpha E(t,x)| \le c(t)(1+|x|)^{|\alpha|} e^{-x^2/(4t)}. \tag{8.1.6}$$

Hence its L^1 norm is bounded by some $c_1(t) > 0$ and by Young inequality we get (8.1.5).

Finally, we prove d). Using the relation

$$\partial_t^\ell E = \Delta \partial_t^{\ell-1} E$$

and (8.1.6), we deduce by induction in ℓ that

$$u(t,x) = E(t) * f(x) \in C^\ell((0,T); H^k(\mathbb{R}^n)). \tag{8.1.7}$$

□

8.1.3 Weak and strong solution for ODE

Consider the Cauchy problem for ODE

$$y'(t) = g(t), \quad y(0) = y_0 \in \mathbb{R}. \tag{8.1.8}$$

Definition 8.1.1. *Let $g(t) \in C(0,T)$. A function $y(t) \in C([0,T))$ is a weak solution of the Cauchy problem* (8.1.8)*, if*

$$\lim_{t \searrow 0} |y(t) - y_0| = 0 \tag{8.1.9}$$

and y solves $\partial_t y = g$, in distributional sense in $(0,T)$ i.e. for any $\varphi \in C_0^\infty((0,T))$ we have

$$-\int_0^T y(t)\partial_t\varphi(t)\,dt = \int_0^T g(t)\varphi(t)\,dt. \tag{8.1.10}$$

Definition 8.1.2. *Let $g(t) \in C(0,T)$. A function $y(t) \in C([0,T)$ is a strong solution of the Cauchy problem* (8.1.8)*, if,* (8.1.9) *is satisfied ,*

$$y(t) \in C^1(0,T))$$

and $y'(t) = g(t)$ for any $t \in (0,T)$.

Lemma 8.1.2. *If $g \in C(0,T)$ and $y(t) \in C([0,T))$ is a weak solution of* (8.1.8)*, then it is a strong solution.*

Proof. Assume $y(t) \in C([0,T))$ is a weak solution of (8.1.8). Take approximation of unity, i.e. take a non - negative function $\varphi \in C_0^\infty(R)$ with support in $(-1,1)$ so that $\int_{\mathbb{R}} \varphi = 1$ and for fixed $t_0 \in (0,T)$ consider the function

$$\varphi_\varepsilon(s) = \varepsilon^{-1}\varphi\left(\frac{s-t_0}{\varepsilon}\right).$$

Then for any continuous function $h(t)$ we can write

$$\int_0^T h(t)\varphi_\varepsilon(t)dt = \int_{-1}^1 \varphi(z)h(t_0+\varepsilon z)dz \to h(t_0) \tag{8.1.11}$$

as $\varepsilon \to 0$. Our goal is to show that y is differentiable in t_0. Therefore, using (8.1.11), we need the limit as $h \to 0$ of

$$\frac{y(t_0+h)-y(t_0)}{h} = \lim_{\varepsilon\to 0}\frac{1}{h}\int_0^T (y(t+h)-y(t))\varphi_\varepsilon(t)dt =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{h}\int_0^T y(t)\left(\varphi_\varepsilon(t-h)-\varphi_\varepsilon(t)\right)dt$$

We have further

$$\int_0^T y(t)\left(\varphi_\varepsilon(t-h)-\varphi_\varepsilon(t)\right) = -\int_0^h\int_0^T y(t)\partial_t\varphi_\varepsilon(t-\tau)dtd\tau.$$

Applying the identity (8.1.10), we find

$$-\int_0^T y(t)\partial_t\varphi_\varepsilon(t-\tau)dt = \int_0^T g(t)\partial_t\varphi_\varepsilon(t-\tau)dt \to g(t_0+\tau)$$

as $\varepsilon \to 0$ due to (8.1.11). Taking the limit as $\varepsilon \to 0$, we deduce

$$\frac{y(t_0+h)-y(t_0)}{h} = \frac{1}{h}\int_0^h g(t_0+\tau)d\tau.$$

Since

$$\frac{1}{h}\int_0^h g(t_0+\tau)d\tau = g(t_0)+o(h)$$

due to the mean value theorem, we arrive at

$$\frac{y(t_0+h)-y(t_0)}{h} = g(t_0)+o(h)$$

Hence $y'(t)$ exists in $(0,T)$ and coincides with the continuous function $g(t)$. $\square$

Further , we can consider the case, when $y(t)$ is a function with image in Banach space $\mathscr{B}$.

Given an interval $I \subseteq \mathbb{R}$ we define the space of continuous functions

$$C(I;\mathscr{B}) = \{f(t) : I \to \mathscr{B};\ \lim_{s\in I, s\to t} \|f(s) - f(t)\|_{\mathscr{B}} = 0,\ \forall t \in I\}.$$

The Frechet derivative of a function $f : I \to \mathscr{B}$ is defined for interior points $t \in I$ as the standard limit

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = f'(t)$$

in $\mathscr{B}$. Then assuming I is open interval, we define

$$C^1(I;\mathscr{B}) = \{f(t) \in C(I;\mathscr{B}); f'(t) \text{exists for any } t \in I \text{ and belongs to } C(I;\mathscr{B})\}.$$

Then we can define for any $\ell \geq 2$ the space $C^\ell(I;\mathscr{B})$ inductively in ℓ as follows

$$C^\ell(I;\mathscr{B}) = \{f(t) \in C^{\ell-1}(I;\mathscr{B}); f'(t) \in C^{\ell-1}(I;\mathscr{B})\}.$$

Consider the Cauchy problem for ODE in the Banach space $\mathscr{B}$

$$y'(t) = g(t),\ \ y(0) = y_0 \in \mathscr{B}. \tag{8.1.12}$$

Definition 8.1.3. *Let $g(t) \in C((0,T);\mathscr{B})$. A function $y(t) \in C([0,T);\mathscr{B})$ is a weak solution of the Cauchy problem* (8.1.12)*, if*

$$\lim_{t\searrow 0} \|y(t) - y_0\|_{\mathscr{B}} = 0 \tag{8.1.13}$$

and y solves $\partial_t y = g$, in distributional sense in $(0,T)$ i.e. for any $\varphi \in C_0^\infty((0,T))$ we have

$$-\int_0^T y(t)\partial_t\varphi(t)dt = \int_0^T g(t)\varphi(t)dt. \tag{8.1.14}$$

where the last identity is in sense that in the left and right side we have of elements in $\mathscr{B}$.

Definition 8.1.4. *Let $g(t) \in C((0,T);\mathscr{B})$. A function $y(t) \in C([0,T);\mathscr{B})$ is a strong solution of the Cauchy problem* (8.1.12)*, if,* (8.1.13) *is satisfied ,*

$$y(t) \in C^1((0,T);\mathscr{B})$$

and $y'(t) = g(t)$ for any $t \in (0,T)$. Again the last identity is in sense that in the left and right side we have of elements in $\mathscr{B}$.

We can prove the following statement in the same way as we have established Lemma 8.1.2.

Lemma 8.1.3. *If $g \in C(0,T)$ and $y(t) \in C([0,T))$ is a weak solution of* (8.1.12)*, then it is a strong solution.*

8.1.4 Weak and strong solutions in Sobolev spaces

We shall define the meaning of solution to the Cauchy problem

$$\begin{aligned} \partial_t u &= \Delta u, \\ u(0,x) &= f(x). \end{aligned} \tag{8.1.15}$$

Definition 8.1.5. *Let $f \in L^2(\mathbb{R}^n)$. A function $u(t,x) \in C([0,T]; L^2(\mathbb{R}^n))$ is a weak solution of the Cauchy problem* (8.1.15)*, if*

$$\lim_{t \searrow 0} \|u(t,\cdot) - f\|_{L^2(\mathbb{R}^n)} = 0 \tag{8.1.16}$$

and u solves $\partial_t u = \Delta u$, in distributional sense in $(0,T)\times\mathbb{R}^n$, i.e. for any $\varphi \in C_0^\infty((0,T)\times\mathbb{R}^n)$ we have

$$-\int_0^T \int_{\mathbb{R}^n} u(t,x)\partial_t\varphi(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^n} u(t,x)\Delta_x\varphi(t,x)\,dx\,dt. \tag{8.1.17}$$

Definition 8.1.6. *Let $f \in L^2(\mathbb{R}^n)$. A function*

$$u(t,x) \in C([0,T]; L^2(\mathbb{R}^n))$$

is a strong solution of the Cauchy problem (8.1.15)*, if* (8.1.16) *is satisfied ,*

$$u(t,x) \in C((0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n))$$

and $\partial_t u = \Delta u$ in sense of L^2 functions.

Lemma 8.1.4. *If $f \in L^2(\mathbb{R}^n)$ and $u(t,x) \in C([0,T]; L^2(\mathbb{R}^n))$ is a weak solution of the Cauchy problem* (8.1.15)*, then we have the estimate*

$$\|u(t)\|_{L^2} \le \|f\|_{L^2} \tag{8.1.18}$$

for any $t \in [0,T)$.

Proof. We shall use the Yosida approximation of f defined for $\varepsilon \in (0,1)$ as

$$f_\varepsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix.\xi}(1+\varepsilon|\xi|^2)^{-1}\widehat{f}(\xi)\,d\xi. \tag{8.1.19}$$

To be more precise this integral is well - defined for $f \in C_0^\infty$ and obeys the relation

$$\widehat{f_\varepsilon}(\xi) = (1+\varepsilon|\xi|^2)^{-1}\widehat{f}(\xi).$$

Moreover, we have the estimates

$$\|f_\varepsilon\|_{L^2} = c\|\widehat{f_\varepsilon}\|_{L^2} = c\|(1+\varepsilon|\xi|^2)^{-1}\widehat{f}\|_{L^2} \le \|f\|_{L^2},$$

$$\|f_\varepsilon\|_{H^2} = \|(1+\xi^2)\widehat{f_\varepsilon}\|_{L^2} \le \|(1+\xi^2)(1+\varepsilon|\xi|^2)^{-1}\widehat{f}\|_{L^2} \le \frac{1}{\varepsilon}\|f\|_{L^2}$$

These estimates show that the operator $f \to f_\varepsilon$ defined by (8.1.19) can be extended by density argument to an operator

$$f : L^2 \to f_\varepsilon = (1-\varepsilon\Delta)^{-1} f \in H^2 \tag{8.1.20}$$

so that

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^2} = 0.$$

If $u \in C([0,T];L^2)$ is a weak solution to (8.1.15), then

$$u_\varepsilon(t) = (1-\varepsilon\Delta)^{-1} u(t)$$

is also a weak solution to

$$\begin{aligned} \partial_t u_\varepsilon &= \Delta u_\varepsilon, \\ u_\varepsilon(0,x) &= f_\varepsilon(x). \end{aligned} \tag{8.1.21}$$

Since for fixed $\varepsilon > 0$

$$u_\varepsilon \in C([0,T);H^2)$$

due to (8.1.20), we can interpret the right side of the equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon, \tag{8.1.22}$$

as element in L^2 and then this equation has the meaning of weak solution of ODE of type (8.1.12). Applying Lemma 8.1.3 we see that

$$u_\varepsilon \in C^1((0,T);L^2)$$

and therefore we can multiply the equation (8.1.22) by u_ε and after integrating we get

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|_{L^2}^2 + \|\nabla u_\varepsilon(t)\|_{L^2}^2 = 0.$$

This estimate implies

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{L^2}^2 \le 0, \forall t \in (0,T).$$

So

$$\|u_\varepsilon(t)\|_{L^2} \le \|f_\varepsilon\|_{L^2}.$$

Taking the limit $\varepsilon \to 0$, we deduce (8.1.18). □

8.1.5 Weak solutions are strong, uniqueness in $\mathbb{R}^n$

We study the Cauchy problem

$$\partial_t u(t,x) = \Delta_x u(t,x),\ x \in \mathbb{R}^n,\ 0 < t < T.$$
$$u(0,x) = f(x).$$

Lemma 8.1.5. *(uniqueness) If*

$$u \in C([0,T); L^2(\mathbb{R}^n)$$

is a weak solution (in the sense of Definition 8.1.5) of

$$\partial_t u(t,x) = \Delta_x u(t,x), x \in \mathbb{R}^n,\ 0 < t < T,$$
$$u(0,x) = f(x),$$

then the condition $f(x) = 0$ implies $u(t,x) = 0, 0 < t < T$.

Proof. Follows from Lemma 8.1.4. □

Now we can show that weak solution in sense of Definition 8.1.5 is also a strong one (in the sense of Definition 8.1.6).

Lemma 8.1.6. *(weak implies strong) If*

$$u \in C([0,T); L^2(\mathbb{R}^n)$$

is a weak solution (in the sense of Definition 8.1.5) of

$$\begin{aligned} &\partial_t u(t,x) = \Delta_x u(t,x), x \in \mathbb{R}^n,\ 0 < t < T, \\ &u(0,x) = f(x), \end{aligned} \tag{8.1.23}$$

then u is also strong one

Proof. Let $U(t,x) = E(t) * f(x)$ be the solution to the equation constructed via fundamental solution. We use Lemma 8.1.1 and see that U is a strong solution to (8.1.6). The uniqueness from Lemma 8.1.5 gives $u = U$ and hence we have strong solution u. □

8.1.6 Decay estimates for heat semigroup

Problem 8.1.1. *Using the representation of the fundamental solution for*

$$\partial_t u = \Delta u,\ \ u(0,x) = f \in C_0^\infty(\mathbb{R}^n)$$

verify the estimates

$$\|u(t,\cdot)\|_{L^\infty} \le C\|f\|_{L^\infty}$$

and

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C}{t^{n/2}}\|f\|_{L^1}.$$

8.1.7 Appendix: Generalizations of Fourier transform of the Gauss density

It is easy to extend the previous result

saying that the Fourier transform $F(f)(\xi)$ of

$$f(x) = \mathrm{e}^{-R^2x^2}, x \in \mathbf{R}^n$$

is

$$F(\mathrm{e}^{-R^2x^2})(\xi) = R^{-n/2}\pi^{n/2}\mathrm{e}^{-\xi^2/(4R)}$$

to the case, when $R \in \mathbf{C}$ and $\mathrm{Re}R > 0$.

In fact, we can use the fact that $z^{1/2}$ is a well - defined analytic function for $\mathrm{Re}z > 0$.

Using the formula

$$F(\mathrm{e}^{-R^2x^2})(\xi) = R^{-n/2}\pi^{n/2}\mathrm{e}^{-\xi^2/(4R)}$$

with $\mathrm{Re}R > 0$, we set

$$R = i + \varepsilon$$

and taking the limit as $\varepsilon \to 0$, we can establish the following property: the Fourier transform of the distribution

$$\mathrm{e}^{ix^2} \in S'(\mathbf{R}^n)$$

is

$$\mathrm{e}^{-i\pi n/4}\pi^{n/2}\mathrm{e}^{-i\xi^2/4}.$$

Using a diagonalization for any symmetric matrix Q we obtain on the basis of the last argument the following.

Lemma 8.1.7. *If Q is a symmetric $n \times n$ matrix with determinant*

$$\mathrm{det}Q \neq 0,$$

then the Fourier transform of the distribution

$$\mathrm{e}^{i(Qx,x)} \in S'(\mathbf{R}^n),$$

where $(.,.)$ is the scalar product in $\mathbf{R}^n$, is

$$\mathrm{e}^{-i\pi \mathrm{sgnQ}/4}\pi^{n/2}|\mathrm{det}Q|^{-1/2}\mathrm{e}^{-i(Q^{-1}\xi,\xi)/4}.$$

Here $\mathrm{sgn}Q$ is the signature of the symmetric matrix Q.

Problem 8.1.2. *(dualita' della luce, ottica geometrica) Find the Fourier transform of*

$$e^{iRx^2}.$$

8.2 Maximum principle for the heat equaiton

8.2.1 1 - D Maximum principle for the heat equation

Consider 1-D heat equation

$$\partial_x^2 u(t,x) - \partial_t u(t,x) = f(t,x), a < x < b,\ 0 < t < T.$$

Here and below the rectangular $E = \{a < x < b,\ 0 < t < T\}$ has sides

$$S_a = \{(a,t),\ 0 \le t \le T\}, S_b = \{(b,t),\ 0 \le t \le T\},\ S_0 = \{(x,0),\ a \le x \le b\}.$$

Given any $f \in C[\overline{E}]$ we look for solutions to (3.1.1) $u \in C[\overline{E}] \cap C^2(E)$.

Lemma 8.2.1. *(EASY MAX principle) If $u \in C[\overline{E}] \cap C^2(E)$ is a solution of*

$$\partial_x^2 u(t,x) - \partial_t u(t,x) = f(t,x)$$

$f(x)$ is any bounded POSITIVE function then

$$\max_{S_a,S_b,S_0} u(t,x) = \max_E u(t,x).$$

Proof: If

$$u(c) = \max_E u(t,x)$$

for some $c \in E$, then in the point c we have

$$\partial_x u(c) = \partial_t u(c) = 0, \partial_{xx}^2 u(c) \le 0.$$

Similar argument works if $c = (x_0, T), a < x_0 < b$.

Lemma 8.2.2. *(weak MAX principle) If $u \in C[\overline{E}] \cap C^2(E)$ is a solution of*

$$\partial_x^2 u(t,x) - \partial_t u(t,x) = f(t,x)$$

$f(x)$ is any bounded NON-NEGATIVE function then

$$\max_{S_a,S_b,S_0} u(t,x) = \max_E u(t,x).$$

Idea of Proof: If

$$u(c) = \max_E u(t,x)$$

for some $c = (x_0, t_0) \in E$, then we shall modify $u(t,x)$ as follows

$$w_\varepsilon(t,x) = u(t,x) + \varepsilon z(x),$$

where

$$z(x) = (x - x_0)^2.$$

Then the Easy MAX principle can be applied.

Complete the proof.

Lemma 8.2.3. *(STRONG MAX principle) If* $u \in C[\overline{E}] \cap C^2(E)$ *is a solution of the heat equation*

$$\partial_x^2 u(t,x) - \partial_t u(t,x) = f(t,x),$$

$f(t,x)$ *is any bounded NON-NEGATIVE function and if*

$$u(c) = \max_E u(t,x)$$

for some $c \in E$*, then* $u(x)$ *is a constant.*

Problem 8.2.1. *Generalize the above maximum principle for n dimentional case.*

Chapter 9

Main hyperbolic equations of math physics

9.0.1 Wave and K-G equation

Our first step in this chapter is to formulate some of the most important hyperbolic equations in mathematical physics.

In these lectures we shall focus our attention mainly to the wave and Klein-Gordon equations as the basic examples of hyperbolic equations in mathematical physics.

The wave equation is an important problem in continuum mechanics. A derivation of this equation in the model of vibrating string can be found in [61] Chapter 2.

The same equation plays a crucial role in relativistic quantum mechanics, since it is connected with a model of a massless relativistic field $u = u(t,x)$, where t is the time variable and

$$x = (x_1,...,x_n) \in \mathbf{R}^n$$

are the space variables. The wave equation satisfied by the field u has the form

$$(-\partial_t^2 + \Delta)u = F, \tag{9.0.1}$$

where

$$\Delta = \partial_{x_1}^2 + ... + \partial_{x_n}^2$$

is the Laplace operator and $F = F(t,x)$ is a given known function. Usually, the operator

$$\square = -\partial_t^2 + \Delta$$

is called D'Alembert operator.

For the case, when a scalar relativistic field has a mass, the corresponding equation is called Klein-Gordon equation and this equation has the form

$$(-\partial_t^2 + \Delta - M^2)u = F, \tag{9.0.2}$$

where $M > 0$ is the mass of the field.

In general we can consider the wave equation as a partial case of Klein-Gordon equation with mass zero.

The first important physical law for these equation is the conservation of energy, when the external force F is identically zero.

Indeed, let us assume the solution is smooth and for any fixed t has a compact support. Then multiplying (9.0.2) by $\partial_t u$ we see that the energy

$$\begin{aligned} E(t) = \frac{1}{2}\int |\partial_t u(t,x)|^2 + \\ + |\nabla_x u(t,x)|^2 + M|u(t,x)|^2 dx \end{aligned} \tag{9.0.3}$$

is a constant independent of the time variable t.

We shall rewrite the Klein-Gordon equation (and therefore the wave equation) as abstract evolution equation of the form

$$\partial_t v = Av, \tag{9.0.4}$$

where A is a skew-selfadjoint operator in a suitable Hilbert space H. We refer to [33] for complete information about this reduction.

For simplicity we shall consider here only the case of positive mass M.

It is clear that we can define the operator $M^2 - \Delta$ on the space of smooth compactly supported functions in $\mathbf{R}^n$. Then this operator is a symmetric with respect to the scalar product

$$(f,g)_{L^2} = \int_{\mathbf{R}^n} f(x)\overline{g(x)}dx.$$

Setting

$$v = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ \Delta - M^2 & 0 \end{pmatrix},$$

we see that the nonlinear Klein-Gordon equation (9.0.2) takes the form (9.0.4). The form of the energy in (9.0.3) suggests us to consider the Hilbert space $H = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. For any couple $v = (v_1, v_2) \in H$ the corresponding norm is defined by

$$\|v\|_H^2 = \int |\nabla v_1|^2 + M^2|v_1|^2 + |v_2|^2 dx \tag{9.0.5}$$

Denote by $(,)_H$ is the corresponding Hilbert norm.

A dense domain for the operator A can be choosen as

$$D(A) = C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n). \tag{9.0.6}$$

Then we see that A can be extended to a skew-selfadjoint operator with a dense domain

$$D(A) = H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n).$$

Applying the Stone theorem we see that A is a generator of an unitary group $U(t)$ acting in the Hilbert space H.

The fact that A is a generator of the group $U(t)$ means that

$$\lim_{t\to 0} \frac{U(t)f - f}{t} = A(f)$$

for $f \in D(A)$. The fact that A is skew-selfadjoint assures that $U(t)$ is a unitary operator

$$\|U(t)f\|_H = \|f\|_H \tag{9.0.7}$$

The abstract Cauchy problem associated with the generator A can be written in the form

$$\partial_t v = Av,$$
$$v(0) = f.$$

The unique solution of this linear Cauchy problem can be represented as $v = U(t)f$.

Turning back to our original formulation of the Klein-Gordon equation we can state the corresponding Cauchy problem as follows

$$(-\partial_t^2 + \Delta - M^2)u = 0,$$
$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x).$$

Here $f = (f_0, f_1) \in H$.

For the case of nontrivial external force F we have the Cauchy problem

$$(-\partial_t^2 + \Delta - M^2)u = F,$$
$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x).$$

The energy conservation law for the linear wave equation is represented in (9.0.7) so the norm in the Hilbert space H has an interpretation as energy.

9.0.2 Fundamental solution for 1 D wave equation

Consider the wave equation

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, \quad t, x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = 0, \; \partial_t u(0,x) = \varphi(x).$$

Introducing coordinates

$$\xi = x - t, \;\; \eta = t + x,$$

one can see that the wave equation becomes

$$\partial_\xi \partial_\eta u = 0$$

and we can be looked for solutions of type

$$u(\xi,\eta) = f(\xi) + g(\eta).$$

or

$$u(t,x) = f(x-t) + g(t+x):$$

Taking into account the initial data we find.

Lemma 9.0.1. *(D'Alambert formula) For any $\varphi(x) \in C^1$ there exists $u \in C^2$ so that*

$$u(t,x) = \frac{1}{2}\int_{x-t}^{x+t} \varphi(y)dy$$

is solution to

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, \quad t, x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = 0, \; \partial_t u(0,x) = \varphi(x).$$

Problem 9.0.1. *(weak Huygens' principle, finite propagation speed) Show that the condition*

$$supp\, \varphi \subseteq \{|y| \leq R\}$$

implies

$$supp\; u(t,x) \subseteq \{|x| \leq t + R\}$$

for $t \geq 0$.

Problem 9.0.2. *(strong Huygens' principle) Show that the condition*

$$\varphi(y) = 0, \ \forall y, \ |y| \le R,$$

implies

$$u(t,x) = 0, \forall t \in [0,R], \ \forall \ |x| \le R - t.$$

The general inhomogeneous case is treated in the next.

Lemma 9.0.2. *(D'Alambert formula) For any $\varphi(x) \in C^1, \psi(x) \in C$ and $F \in C$ there exists $u \in C^2$ so that*

$$u(t,x) = \frac{\psi(x+t)+\psi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \varphi(y)dy +$$

$$+\frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+t-\tau} F(\tau,y)dyd\tau.$$

is solution to

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = F, \ \ t \ge 0, x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = \psi(x), \ \partial_t u(0,x) = \varphi(x).$$

One can consider also the case of half line $x > 0$ with Dirichlet data on the boundary

$$u(t,0) = 0$$

Lemma 9.0.3. *(D'Alambert formula on half line, exterior of the light cone) For any $\varphi(x) \in C^1, \psi(x) \in C$ and $F \in C$ satisfying the condition*

$$\varphi(0) = 0, \psi(0) = 0, F(t,0) = 0$$

there exists $u \in C^2$ so that for $x > t \ge 0$

$$u(t,x) = \frac{\psi(x+t)+\psi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \varphi(y)dy +$$

$$+\frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+t-\tau} F(\tau,y)dyd\tau.$$

is solution to

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = F, \ \ t \ge 0, x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = \psi(x), \ \partial_t u(0,x) = \varphi(x).$$

Lemma 9.0.4. *(D'Alambert formula on half line, interior of the light cone) For any $\varphi(x) \in C^1, \psi(x) \in C$ and $F \in C$ satisfying the condition*

$$\varphi(0) = 0, \psi(0) = 0, F(t,0) = 0$$

there exists $u \in C^2$ so that for $x \in (0, +\infty), t \geq 0$

$$u(t,x) = \frac{\psi(x+t) - sgn(t-x)\psi(|t-x|)}{2} + \frac{1}{2}\int_{|t-x|}^{x+t} \varphi(y)dy +$$

$$+\frac{1}{2}\int_0^t \int_{|x-(t-\tau)|}^{x+t-\tau} F(\tau, y) dy d\tau$$

is solution to

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = F, \quad t \geq 0, x \in \mathbb{R},$$

subject to the initial conditions

$$u(0,x) = \psi(x), \ \partial_t u(0,x) = \varphi(x).$$

9.0.3 Fundamental solution for 3 D wave equation

Consider the wave equation

$$\partial_t^2 u(t,x) - \Delta u(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^3,$$

subject to the initial conditions

$$u(0,x) = 0, \ \partial_t u(0,x) = \varphi(x).$$

Consider first the case of radially symmetric data

$$\varphi(x) = \varphi(|x|).$$

Then $u(t,x) = u(t,|x|)$ and introducing polar coordinates

$$r = |x|, \quad u(t,r) = \frac{v(t,r}{r}, \quad \varphi(r) = \frac{\chi}{r}$$

we obtain the following 1D problem

$$\partial_t^2 v - \partial_r^2 v = 0,$$

$$v(0,r) = 0, \ \partial_t v(0,r) = \chi(x).$$

Taking Dirichlet boundary condition on $r = 0$ we apply D'Alambert formula for 1D case and find

$$v(t,r) = \frac{1}{2}\int_{|r-t|}^{r+t} \chi(s)ds$$

so

$$u(t,r) = \frac{1}{2r}\int_{|r-t|}^{r+t} \varphi(s)sds.$$

Taking the limit $r \to 0$ we obtain

$$u(t,0) = \varphi(t)t. \tag{9.0.1}$$

For the general case when $\varphi(x)$ is not radially symmetric, we take arbitrary $x_0 \in \mathbb{R}^3$ and consider the mean of u

$$M_{x_0}(u)(r) = \frac{1}{4\pi}\int_{|\omega|=1} u(t, x_0 + r\omega)d\omega.$$

Then

$$\overline{u}(t,r) = M_{x_0}(u)(r),\ \overline{\varphi}(r) = M_{x_0}(\varphi)(r)$$

satisfy the radial wave equation so applying (9.0.1) we get

$$u(t,x_0) = \overline{u}(t,0) = t\overline{\varphi}(t) = \frac{t}{4\pi}\int_{|\omega|=1} \varphi(x_0 + t\omega)d\omega.$$

Lemma 9.0.5. *(Poisson formula in $\mathbb{R}^3$) For any $\varphi(x) \in C^2$, there exists $u \in C^2$ so that*

$$u(t,x) = \frac{t}{4\pi}\int_{|\omega|=1} \varphi(x + t\omega)d\omega$$

is solution to

$$\partial_t^2 u(t,x) - \Delta u(t,x) = 0,\ \ t \geq 0, x \in \mathbb{R}^3,$$

subject to the initial conditions

$$u(0,x) = 0,\ \partial_t u(0,x) = \varphi(x).$$

Problem 9.0.3. *(weak Huygens' principle, finite propagation speed) Show that the condition*

$$supp\ \varphi \subseteq \{|y| \leq R\}$$

implies

$$supp\ u(t,x) \subseteq \{|x| \leq t + R\}$$

for $t \geq 0$.

Problem 9.0.4. *(strong Huygens' principle) Show that the condition*

$$\varphi(y) = 0, \ \forall y, \ |y| \le R,$$

implies

$$u(t,x) = 0, \forall t \in [0,R], \ \forall \ |x| \le R - t.$$

For the general inhomogeneous case we have

Lemma 9.0.6. *(Poisson formula in $\mathbb{R}^3$) For any $\varphi(x) \in C^2, \psi \in C^3$ and $F \in C^2$ there exists $u \in C^2$ so that*

$$u(t,x) = \partial_t \left(\frac{t}{4\pi} \int_{|\omega|=1} \psi(x+t\omega) d\omega \right) + \frac{t}{4\pi} \int_{|\omega|=1} \varphi(x+t\omega) d\omega$$

$$+ \int_0^t \frac{t-\tau}{4\pi} \int_{|\omega|=1} F(\tau, x+(t-\tau)\omega) d\omega d\tau$$

is solution to

$$\partial_t^2 u(t,x) - \Delta u(t,x) = 0, \ \ t \ge 0, x \in \mathbb{R}^3,$$

subject to the initial conditions

$$u(0,x) = 0, \ \partial_t u(0,x) = \varphi(x).$$

9.0.4 Fundamental solution for 2 D K-G and wave equation

Consider the wave equation

$$\partial_t^2 u(t,x) - \Delta u(t,x) + M^2 u = 0, \ \ t \in \mathbb{R}, x \in \mathbb{R}^2,$$

subject to the initial conditions

$$u(0,x) = 0, \ \partial_t u(0,x) = \varphi(x).$$

The case $M > 0$ corresponds to the case of K-G equation, while $M = 0$ is the wave equation in 2D. Take

$$X = (X', X_3), \ \ X' = x, \ \ U(t, X', X_3) = e^{iMX_3} u(t, X') = e^{iMX_3} u(t,x)$$

and

$$\Phi(X', X_3) = e^{iMX_3} \varphi(x).$$

Then

$$U(t,X) = \frac{t}{4\pi} \int_{|\Omega|=1} \Phi(X + t\Omega) d\Omega.$$

Introduce local coordinates

$$\Omega' = y \in \{|y| \le 1\},\ \Omega_3 = \sqrt{1-|y|^2}.$$

and using

$$d\Omega = \frac{dy}{\sqrt{1-|y|^2}}$$

we get

$$u(t,x) = \frac{t}{4\pi}\int_{|y|\le 1} \cos(Mt\sqrt{1-|y|^2})\varphi(x+ty)\frac{dy}{\sqrt{1-|y|^2}}.$$

The change of variable $z = ty$ yields

$$u(t,x) = \frac{1}{4\pi}\int_{|z|\le 1} \cos(M\sqrt{t^2-|z|^2})\varphi(x+z)\frac{dz}{\sqrt{t^2-|z|^2}}.$$

9.0.5 Fundamental solution of the homogeneous wave equation via Fourier transform

Our purpose in this section is to construct a solution of the problem

$$\begin{gathered}\Box E = 0,\\ E(0,x) = 0\ ,\ \partial_t E(0,x) = \delta(x),\end{gathered} \tag{9.0.1}$$

where

$$\Box = -\partial_t^2 + \Delta.$$

Once the solution $E = E(t,x)$ is found, one can represent the solution of

$$\begin{gathered}\Box u = 0,\\ u(0,x) = 0\ ,\ \partial_t u(0,x) = f(x)\end{gathered} \tag{9.0.2}$$

by

$$u = E(t,.) * f. \tag{9.0.3}$$

Since

$$\hat{u}(t,\xi) = \hat{E}(t,\xi)\hat{f}(\xi),$$

comparing the representation of u with the Fourier transform, we see that

$$\hat{E}(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|}. \tag{9.0.4}$$

To construct fundamental solution we define s_+^{-z} for any complex number z with $\mathrm{Re}z < 1$ by

$$s_+^{-z} = \begin{cases} s^{-z} & \text{if } s > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9.0.5}$$

It is clear that this is a classical function in $L^1_{loc}(\mathbf{R})$ for $\mathrm{Re}z < 1$. Note that

$$\frac{d}{ds} s_+^{-z} = -z\, s_+^{-z-1}, \quad \text{for Re } z < 0. \tag{9.0.6}$$

The above relation enables one to extend the definition of s_+^{-z} for $1 \le \mathrm{Re}z < 2$. Namely, we define (for $1 \le \mathrm{Re}z < 2$)

$$s_+^{-z} = \frac{1}{(-z+1)} \frac{d}{ds}\left(s_+^{-z+1}\right), \tag{9.0.7}$$

where the derivative in the right side is taken in the sense of distributions. Moreover for $k \le \mathrm{Re}z < k+1$ we define s_+^{-z} by the relation

$$\begin{aligned} s_+^{-z} &= \frac{1}{(-z+1)\dots(-z+k)} \left(\frac{d}{ds}\right)^k \left(s_+^{-z+k}\right) \\ &= \frac{\Gamma(-z+1)}{\Gamma(-z+k+1)} \left(\frac{d}{ds}\right)^k s_+^{-z+k}, \end{aligned} \tag{9.0.8}$$

where the derivatives are taken in distribution sense. Take

$$E_z(s) = \frac{c_n}{\Gamma(1-z)} s_+^{-z} \tag{9.0.9}$$

for $z \ne \{1,2,3,\dots\}$. Here the constant $c_n > 0$ will be chosen later on. We can rewrite (9.0.8) as

$$E_z(s) = \frac{d^k}{ds^k} E_{z-k}(s). \tag{9.0.10}$$

It is not difficult to establish the relation

$$\lim_{z \to k} E_z(s) = c_n \delta^{(k-1)}(s) \tag{9.0.11}$$

for any integer $k \ge 1$. Here the limit is taken in distribution sense. In fact, the relation (see (10.1.4)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

implies that

$$\lim_{z \to k} \Gamma(1-z)(z-k) = \frac{(-1)^k}{\Gamma(k)}. \tag{9.0.12}$$

On the other hand, for any $\varphi \in C_0^\infty(\mathbf{R})$ and $z = k+\varepsilon$ we see that the quantity

$$\varepsilon(s_+^{-z},\varphi) = \frac{\varepsilon}{(-z+1)...(-z+k)}(\frac{d^k}{dz^k}(s_+^{-\varepsilon}),\varphi) =$$
$$= \frac{(-1)^k\varepsilon}{(-z+1)...(-z+k)}\int_0^\infty s^{-\varepsilon}\frac{d^k}{ds^k}\varphi(s)ds$$

tends to

$$\frac{1}{\Gamma(k)}\int_0^\infty \frac{d^k}{ds^k}\varphi(s)ds = \frac{-1}{\Gamma(k)}\frac{d^{k-1}}{ds^{k-1}}\varphi(0)$$

as ε tends to 0 and from (9.0.12) we obtain the needed relation (9.0.11).

Further, we consider the following family of distributions depending on $z \in \mathbf{C}$

$$E_z(t,x) = \frac{c_n}{\Gamma(1-z)}(t^2-x^2)_+^{-z}. \tag{9.0.13}$$

As before, this is a classical function for $\mathrm{Re} z < 1$. Using the relation

$$-\Box E_z(t,x) = 4(\frac{n-1}{2}-z)E_{z+1}, \tag{9.0.14}$$

one can extend the definition of $E_z(t,x)$ for $\mathrm{Re} z \neq k+(n-1)/2$, $k = 0,1,2,...$ as a distribution in $D'(\mathbf{R}^{n+1})$. Our next step is to compute the partial Fourier transform of the distribution $D'(\mathbf{R}^n)$. First, we start with the case $\mathrm{Re} z < 1$, when $E_z(t,x)$ is a classical function.

Lemma 9.0.7. *For* $\mathrm{Re}\, z < 1$ *we have*

$$\int_{\mathbf{R}^n} \mathrm{e}^{-ix\xi}(t^2-x^2)_+^{-z}\frac{dx}{\Gamma(1-z)} =$$
$$= \frac{(2\pi)^{n/2}}{2^z}|t|^{-z+n/2}|\xi|^{z-n/2}J_{-z+n/2}(|t\xi|). \tag{9.0.15}$$

Proof. Since the scalar product $x.\xi$ is invariant under the action of the group $SO(n)$ of rotations, we see that the left side of the needed identity is a spherical function in ξ. For this we lose no generality assuming $\xi = (|\xi|,0,...,0)$. Then the integral in the left side of (9.0.15) takes the form

$$I = \int_{\mathbf{R}} \mathrm{e}^{-ix_1|\xi|}\int_{\mathbf{R}^{n-1}}(t^2-x_1^2-|x'|^2)_+^{-z}dx'\frac{dx_1}{\Gamma(1-z)}. \tag{9.0.16}$$

For $n \geq 3$ one can use polar coordinates $r = |x'|$, $\omega' = x'/r \in \mathbf{S}^{n-2}$. So we have

$$\int_{\mathbf{R}^{n-1}} (t^2 - x_1^2 - |x'|^2)_+^{-z} dx' =$$

$$= \mu(\mathbf{S}^{n-2})(t^2 - x_1^2)_+^{-z+(n-1)/2} \int_0^1 (1-r^2)^{-z} r^{n-2} dr.$$

Choosing $\mu(\mathbf{S}^0) = 2$, we see that this identity holds also for $n = 2$. The relations (10.1.6) and (10.1.7) enable one to compute explicitly the integral

$$\int_0^1 (1-r^2)^{-z} r^{n-2} dr = \frac{1}{2} \int_0^1 (1-\rho)^{-z} \rho^{(n-3)/2} dr =$$

$$= \frac{1}{2} B(1-z, \frac{n-1}{2}) = \frac{1}{2} \frac{\Gamma(1-z)\Gamma((n-1)/2)}{\Gamma((n+1)/2 - z)}.$$

So the integral in (9.0.16) becomes

$$I = \frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{2\Gamma((n+1)/2-z)} \int_{\mathbf{R}} e^{-ix_1|\xi|} (t^2 - x_1^2)_+^{-z+(n-1)/2} dx_1 =$$

$$= \frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{2\Gamma((n+1)/2-z)} t^{-2z+n} \int_{-1}^1 \cos(tx_1|\xi|)(1-x_1^2)^{-z+(n-1)/2} dx_1 =$$

$$\frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{\Gamma((n+1)/2-z)} t^{-2z+n} \int_0^1 \cos(tx_1|\xi|)(1-x_1^2)^{-z+(n-1)/2} dx_1.$$

Combining the formula (10.1.9) for the surface of the unit sphere with the Poisson integral representation (10.2.7) of the Bessel function, we get

$$I = (2\pi)^{n/2} 2^{-z} t^{-z+n/2} |\xi|^{z-n/2} J_{-z+n/2}(t|\xi|).$$

This completes the proof of the Lemma.

Remark. Note that the action (E_z, φ) of the distribution $E_z(t,x)$ on any test function $\varphi(t,x) \in C_0^\infty(\mathbf{R}^{n+1})$ is analytic function for $z \neq k + (n-1)/2$ for $k = 0, 1, 2, \ldots$ in view of the recurrence relation (9.0.14).

Since the right side of (9.0.15) is analytic function of z for $|t| \neq 0, |\xi| \neq 0$, we see that (9.0.15) is valid in the sense of distributions for $z \neq k + (n-1)/2$ and $k = 0, 1, 2, \ldots$

Given any function $\varphi(t,x) \in C_0^\infty(\mathbf{R}^{n+1})$ we denote by

$$\tilde{\varphi}(t,\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} \varphi(t,x) dx$$

its inverse partial Fourier transform and then the action of E_z on φ satisfies

(9.0.17) $$\frac{(2\pi)^{-n/2}}{c_n} (E_z, \varphi) =$$

$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} \frac{1}{2^z} |t|^{-z+n/2} |\xi|^{z-n/2} J_{-z+n/2}(|t\xi|) \tilde{\varphi}(t,\xi) d\xi dt.$$

Since

$$J_{1/2}(s) = \sqrt{\frac{2}{s\pi}}\sin s,$$

we have

$$\frac{2^{-1}\pi^{(-n+1)/2}}{c_n}(E_{(n-1)/2},\varphi) =$$

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n}\frac{\sin|t||\xi|}{|\xi|}\tilde{\varphi}(t,\xi)d\xi dt. \tag{9.0.18}$$

Then we choose c_n so that

$$\frac{2^{-1}\pi^{(-n+1)/2}}{c_n} = 1,$$

i.e.

$$c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

Thus we get

$$(E_{(n-1)/2},\varphi) =$$

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n}\frac{\sin|t||\xi|}{|\xi|}\tilde{\varphi}(t,\xi)d\xi dt. \tag{9.0.19}$$

Comparing with (9.0.4), we see that $E(t,x) = E_{(n-1)/2}(t,x)$ is a solution of our problem (9.0.1) and this is the fundamental solution of the initial problem (9.0.1) for the wave equation.

The recurrence relation (9.0.14) shows that the fundamental solution $E_{(n-1)/2}$ can be expressed by the aid of $E_1(t,x)$, $E_0(t,x)$ or $E_{1/2}(t,x)$. For this our next step will be an explicit representation formula for these solutions. In fact, we have

$$(E_{1/2}(t,.),f) = c\int_{|y|<t}(t^2-|y|^2)^{-1/2}f(y)dy =$$

$$= ct^{n-1}\int_0^1\int_{\mathbf{S}^{n-1}}f(tr\omega)(1-r^2)^{-1/2}r^{n-1}d\omega dr \tag{9.0.20}$$

for $f \in C_0^\infty(\mathbf{R}^n)$. Here

$$c = \frac{c_n}{\Gamma(1/2)} = \frac{1}{2\pi^{n/2}}.$$

Further, we have

$$(E_0(t,.),f) = c\int_{|y|<t}f(y)dy =$$

$$= ct^n\int_0^1\int_{\mathbf{S}^{n-1}}f(tr\omega)r^{n-1}d\omega dr \tag{9.0.21}$$

with

$$c = c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

Taking advantage of (9.0.11), we find

$$(E_1(t,.), f) = \frac{c}{t}\int_{|y|=t} f(y)\,dS_y =$$

$$= c t^{n-2} \int_{\mathbf{S}^{n-1}} f(t\omega)\,d\omega, \tag{9.0.22}$$

where

$$c = c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

From recurrence relation (9.0.14) we see that

$$E_{(n-1)/2} = C_1 \Box^{(n-3)/2} E_1 = C_2 \Box^{(n-1)/2} E_0 \tag{9.0.23}$$

for $n \geq 3$ odd. Here

$$C_1 = \frac{(-1)^{(n-3)/2}}{2^{n-3}\Gamma((n-1)/2)}\ ,\ C_2 = \frac{(-1)^{(n-1)/2}}{2^{n-1}\Gamma((n+1)/2)}.$$

For $n \geq 2$ even we have

$$E_{(n-1)/2} = C \Box^{(n-2)/2} E_{1/2} \tag{9.0.24}$$

with

$$C = \frac{(-1)^{(n-2)/2}}{2^{n-2}\Gamma(n/2)}.$$

So we conclude that the expression $\Box^k E_z(t,x)$ for some particular values of k, z will appear in the representation formula for the fundamental solution of (9.0.1). For this we shall establish the following representation of term of this type.

Lemma 9.0.8. *Let $z \in \mathbf{C}$ satisfy*

$$z \neq k + (n-1)/2\ ,\ k = 0, 1, 2, \ldots$$

and let $l \geq 1$ be an integer such that

$$z + l \neq k + (n-1)/2\ ,\ k = 0, 1, 2, \ldots$$

Then we have the relation (in sense of distributions)

$$\Box^l E_z(t,x) = \sum_{|\alpha| \leq l} \partial^\alpha_{t,x}(c_{l,\alpha}(t,x) E_z(t,x)), \tag{9.0.25}$$

where $c_{l,\alpha}(t,x)$ are smooth functions in $\mathbf{R}^{n+1}\setminus 0$ and satisfy for any multiindex β the estimate

$$|\partial^\beta_{t,x} c_{l,\alpha}(t,x) \leq C_{\alpha,\beta,l}(|t|+|x|)^{-|\beta|-2l+|\alpha|} \tag{9.0.26}$$

Proof. It is sufficient to establish (9.0.25) for $l=1$. Our starting point is the representation

$$E_z(t,x) = \varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)E_z(t,x) + \left[1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right]E_z(t,x),$$

where $\varphi(s)\in C_0^\infty(\mathbf{R})$ is a cut-off function, such that $\varphi(s)=1$ for $|s|\leq 1/4$ and $\varphi(s)=0$ for $|s|\geq 1/2$. Note that we have the estimate

$$\left|\partial^\beta_{t,x}\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right| \leq C_\beta(|t|+|x|)^{-|\beta|}. \tag{9.0.27}$$

Then it is sufficient to establish that

$$\begin{aligned}\Box\left(\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)E_z(t,x)\right) &= \\ &= \sum_{|\alpha|\leq 1}\partial^\alpha_{t,x}(d_\alpha(t,x)E_z(t,x))\end{aligned} \tag{9.0.28}$$

and

$$\begin{aligned}\Box\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right) &= \\ \sum_{|\alpha|\leq 1}\partial^\alpha_{t,x}(e_\alpha(t,x)E_z(t,x)),\end{aligned} \tag{9.0.29}$$

where the coefficients $d_\alpha(t,x), e_\alpha(t,x)$ satisfy (9.0.26) with $l=1$.

The relation $\Box E_z = c(z)(t^2-|x|^2)^{-1}E_z$ was established in (9.0.14). Then the Leibnitz rule implies that

$$\Box(\varphi E_z) = -E_z\Box\varphi + 2\sum_{\mu=0}^n \partial_\mu(E_z\partial^\mu\varphi) + \frac{\varphi c(z)E_z}{t^2-|x|^2}. \tag{9.0.30}$$

Note that we have

$$|x| \leq \frac{|t|}{\sqrt{3}} \leq \frac{2t}{3} \tag{9.0.31}$$

on the support of $\varphi(|x|/\sqrt{t^2+|x|^2})$. Combining this fact, the representation formula (9.0.30) and the estimate (9.0.27) we arrive at (9.0.28).

To verify (9.0.29) we represent the operator $\Box$ in polar coordinates

$$-\Box = \partial_t^2 - \partial_r^2 - \frac{n-1}{r}\,\partial_r - \frac{1}{r^2}\,\Delta_{\mathbf{S}^{n-1}}.$$

Since

$$\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)$$

is invariant under any rotation, we have

$$-\Box\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right) = \left(\partial_t^2-\partial_r^2-\frac{n-1}{r}\partial_r\right)\left[\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right].$$

Since

$$\partial_t E_z(t,x) = -\frac{2zt}{t^2-|x|^2}\,E_z(t,x),$$

$$\partial_{x_j} E_z(t,x) = \frac{2zx_j}{t^2-|x|^2}\,E_z(t,x),$$

we have the relation

$$\left(\partial_t^2-\partial_r^2\right)E_z(t,r) = -2z(\partial_t-\partial_r)\left(\frac{1}{t+r}E_z(t,r)\right) = \frac{4z^2}{t^2-|x|^2}E_z. \tag{9.0.32}$$

Now we are in position to apply the following variant of Leibniz rule

$$\Box\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z\right) = = E_z\Box\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right) - 2\sum_{\mu=0}^{n}\partial_\mu\left(E_z\partial^\mu\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right) + + \frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)c(z)E_z}{t^2-|x|^2}. \tag{9.0.33}$$

The relation (9.0.32) implies that

$$-2z\frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z}{t^2-|x|^2} = (\partial_t-\partial_r)\left(\frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z}{t+|x|}\right)+$$

$$+\frac{E_z}{t+|x|}(\partial_t-\partial_r)\left(\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right).$$

Using the fact that on the support of

$$1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)$$

we have

$$r=|x|\geq\frac{t}{\sqrt{15}},$$

while on the support of

$$\varphi'\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)$$

the weights $|x|$ and t are equivalent, we see that (9.0.29) is valid.

From (9.0.28) and (9.0.29) we get the desired representation (9.0.25) and the lemma is proved.

Now we can obtain an explicit formula representing the solution of (9.0.2). From (9.0.22) and

$$\left(\frac{d^k}{(dt)^k}\right)\int_{\mathbf{S}^{n-1}}f(x+t\omega)d\omega=\sum_{|\alpha|=k}\int_{\mathbf{S}^{n-1}}\omega^\alpha\partial^\alpha f(x+t\omega)d\omega$$

we can compute the time derivative of

$$E_1(t,.)*f(x).$$

For $n\geq 3$ odd we combine the representation formula (9.0.23) together with (9.0.22) and applying the above lemma, we get

$$u(t,x)=E_{(n-1)/2}(t,.)*f(x)=$$

$$=\sum_{l=0}^{(n-3)/2}\sum_{|\alpha|=l}\frac{C}{t^{n-2-l}}\int_{|x-y|=t}c_{l,\alpha}\left(\frac{x-y}{|x-y|}\right)\partial_y^\alpha f(y)dS_y,\ t>0, \tag{9.0.34}$$

where $c_{l,\alpha}(\omega)$ are smooth functions on $\mathbf{S}^{n-1}$. Moreover, from

$$E_{(n-1)/2} = c\Box^{(n-1)/2} E_0$$

we have

$$u(t,x) = E_{(n-1)/2}(t,.) * f(x) = \tag{9.0.35}$$

$$= \sum_{l=0}^{(n-1)/2} \sum_{|\alpha|=l} \frac{C}{t^{n-1-l}} \int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f(y) dy, \ t > 0,$$

where $c_{l,\alpha}(y)$ are smooth functions in the unit ball. For $n \geq 2$ even we can use the relation (9.0.24) and in this way we get

$$u(t,x) = E_{(n-1)/2}(t,.) * f(x) = \tag{9.0.36}$$

$$= \sum_{l=0}^{(n-2)/2} \sum_{|\alpha|=l} \frac{C}{t^{n-3/2-l}} \int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f(y) \frac{dy}{\sqrt{t-|x-y|}}, \ t > 0,$$

where $c_{l,\alpha}(y)$ are smooth functions in the unit ball.

Now we can turn to the representation of the solution the the Cauchy problem

$$\Box u = 0,$$

$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x). \tag{9.0.37}$$

It is clear that

$$u(t,x) = \frac{d}{dt} E_{(n-1)/2}(t,.) * f_0(x) + E_{(n-1)/2}(t,.) * f_1(x).$$

For $n = 1$ we have the D'Alembert formula

$$u(t,x) = \tag{9.0.38}$$

$$= \frac{f_0(t+x) + f_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} f_1(y) dy, \ \ t > 0.$$

For $n = 2$ we have the Poisson formula

$$u(t,x) = \tag{9.0.39}$$

$$= \partial_t \left(\frac{1}{2\pi} \int_{|x-y|<t} \frac{f_0(y)}{\sqrt{t^2 - |x-y|^2}} dy \right) +$$

$$+ \frac{1}{2\pi} \int_{|x-y|<t} \frac{f_1(y)}{\sqrt{t^2 - |x-y|^2}} dy, \ t > 0,$$

For $n=3$ we have the Kirchhoff formula

(9.0.40)
$$u(t,x)= \\ = \partial_t\left(\frac{1}{4\pi t}\int_{|x-y|=t} f_0(y)dS_y\right)+ \\ +\frac{1}{4\pi t}\int_{|x-y|=t} f_1(y)dS_y,\ t>0,$$

For $n\geq 3$ odd the solution of (9.0.37) takes the form

(9.0.41)
$$u(t,x)= \\ = \sum_{l=0}^{(n-1)/2}\sum_{|\alpha|=l}\frac{1}{t^{n-1-l}}\int_{|x-y|=t} c_{l,\alpha}\left(\frac{x-y}{|x-y|}\right)\partial_y^\alpha f_0(y)dS_y+ \\ +\sum_{l=0}^{(n-3)/2}\sum_{|\alpha|=l}\frac{1}{t^{n-2-l}}\int_{|x-y|=t} d_{l,\alpha}\left(\frac{x-y}{|x-y|}\right)\partial_y^\alpha f_1(y)dS_y,\ t>0,$$

where $c_{l,\alpha}(\omega),\ d_{l,\alpha}(\omega)$ are smooth functions on $\mathbf{S}^{n-1}$. Moreover, for $n\geq 1$ odd from (9.0.35) we get

(9.0.42)
$$u(t,x)= \\ = \sum_{l=0}^{(n+1)/2}\sum_{|\alpha|=l}\frac{1}{t^{n-l}}\int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f_0(y)dy+ \\ +\sum_{l=0}^{(n-1)/2}\sum_{|\alpha|=l}\frac{1}{t^{n-1-l}}\int_{|x-y|<t} d_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f_1(y)dy,\ t>0,$$

where $c_{l,\alpha}(y),\ d_{l,\alpha}(y)$ are smooth functions on the unit ball. Finally, for $n\geq 2$ even from (9.0.36) we deduce

(9.0.43)
$$u(t,x)= \\ = \sum_{l=0}^{n/2}\sum_{|\alpha|=l}\frac{1}{t^{n-1/2-l}}\int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f_0(y)\frac{dy}{\sqrt{t-|x-y|}}+ \\ +\sum_{l=0}^{(n-2)/2}\sum_{|\alpha|=l}\frac{1}{t^{n-3/2-l}}\int_{|x-y|<t} d_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f_1(y)\frac{dy}{\sqrt{t-|x-y|}},$$

where $c_{l,\alpha}(y),\ d_{l,\alpha}(y)$ are smooth functions on the unit ball.

9.0.6 Fundamental solution of the inhomogeneous wave equation via Fourier transform

The solution of the inhomogeneous wave equation

$$\Box u = F,$$
$$u(0,x) = \partial_t u(0,x) = 0 \tag{9.0.1}$$

can be found as follows

$$u(t,x) = \int_0^t E_{(n-1)/2}(t-s,.) * F(s,.)(x)ds. \tag{9.0.2}$$

Since we shall use the recurrence relation (9.0.14), our first step is to find sufficient conditions so that all boundary terms after integration by parts with respect to time variable in (9.0.2) are identically zero.

The terms at $s=0$ can be neglected if we impose the assumption

$$\mathrm{supp}_s F(s,y) \subset (0,\infty), \tag{9.0.3}$$

because the function $F(s,y)$ together with all space-time derivatives of arbitrary order are zero at $s=0$. To be sure that all terms at $s=t$ vanish we need the following.

Lemma 9.0.9. *If* $\mathrm{Re}z < n/2$, *then*

$$\lim_{t\to 0_+} E_z(t,x) = 0 \tag{9.0.4}$$

in the sense of distributions in $\mathbf{R}^n$.

If $\mathrm{Re}z < (n-1)/2$, *then we also have*

$$\lim_{t\to 0_+} \partial_t E_z(t,x) = 0. \tag{9.0.5}$$

Proof. Given any $\varphi \in C_0^\infty(\mathbf{R}^n)$ we have

$$(E_z(t,.),\varphi) = c2^{-z}t^{-z+n/2}\int_{\mathbf{R}^n} \mathrm{e}^{ix\xi}|\xi|^{z-n/2}J_{-z+n/2}(t|\xi|)\hat{\varphi}(\xi)d\xi \tag{9.0.6}$$

for any $t>0$ according to (9.0.15). In fact, for $\mathrm{Re}z<1$ this follows from (9.0.15). Then using the fact that both sides of (9.0.6) are analytic functions for $z \neq k + (n-1)/2$, $k=0,1,2,\ldots$, we establish (9.0.6).

Then the series expansion (10.2.2) together with the asymptotic expansion for the Bessel function (see [2]) show that for $\mathrm{Re}\nu > -1/2$ we have

(9.0.7) $$|J_\nu(s)| \le C_\nu s^{\mathrm{Re}\,\nu}\,,\ s > 0.$$

Thus we get

$$|(E_z(t,.),\varphi)| \le$$
$$c2^{|\mathrm{Re}z|}t^{n-2\mathrm{Re}z}\int_{\mathbf{R}^n}|\hat{\varphi}(\xi)|d\xi$$

and we see that the property (9.0.4) is fulfilled.

To establish the property (9.0.4) for the time derivative of $E_z(t,x)$ we use the recurrence relation (10.2.3) for the Bessel functions and find

$$\partial_t(t^\nu J_\nu(t|\xi|)) = t^\nu|\xi|J_{\nu-1}(t|\xi|)$$

so we have the following variant of (9.0.6)

$$\partial_t(E_z(t,.),\varphi) =$$
$$c(z)t^{-z+n/2}\int_{\mathbf{R}^n}\mathrm{e}^{ix\xi}|\xi|^{z-n/2+1}J_{-1-z+n/2}(t|\xi|)\hat{\varphi}(\xi)d\xi.$$

Applying the estimate (9.0.7) with $\nu = -1-z-n/2$, we see that $\mathrm{Re}\,\nu > -1/2$ so

$$|\partial_t(E_z(t,.),\varphi)| \le$$
$$c(z)t^{n-2\mathrm{Re}z-1}\int_{\mathbf{R}^n}|\hat{\varphi}(\xi)|d\xi$$

and the assumption $\mathrm{Re}z < (n-1)/2$ implies that (9.0.5) is true.

This proves the Lemma.

After this preparation we can obtain the following representation formula.

Proposition 9.0.1. *If $n \ge 3$ is odd and the inclusion (9.0.3) is fulfilled, then the solution of (9.0.1) is*

$$u(t,x) =$$
$$c_n\int_0^t\frac{1}{t-s}\int_{|x-y|=t-s}\Box^{(n-3)/2}F(s,y)dS_y ds.$$

Proof. It is sufficient to apply the recurrence relation

$$E_{(n-1)/2} = c_n\Box^{(n-3)/2}E_1$$

from the previous section in combination with Lemma 9.0.9 and the representation formula (9.0.22) of the distribution E_1.

In the same way we arrive at

Proposition 9.0.2. *If $n \geq 2$ is even and the inclusion (9.0.3) is fulfilled, then the solution of (9.0.1) is*

$$u(t,x) =$$
$$c_n \int_0^t \int_{|x-y|<t-s} \frac{1}{\sqrt{(t-s)^2 - |x-y|^2}} \Box^{(n-2)/2} F(s,y)\,dy\,ds.$$

9.0.7 Energy and conformal energy estimates for homogeneous wave equation in $\mathbb{R}^n$, $n \geq 2$.

Let $(t, x_1, \cdots, x_n)$ be coordinates in $\mathbb{R}^{1+n}$ and

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -E \end{pmatrix}$$

be the Minkowski metric in $\mathbb{R}^{1+n}$. Using the convention for summing over repeating Greek indices we have

$$\Box \equiv \partial_t^2 - \Delta_x = \partial_\mu \partial^\mu,$$

where

$$\partial_0 = \partial_t, \partial_j = \partial_{x_j},$$

and

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu.$$

Let $u(t,x)$ be a C^2 solution to the wave equation

$$\partial_\mu \partial^\mu u = 0.$$

Introducing the quantity

$$T^{\mu\nu} = \partial^\mu u \partial^\nu u - \frac{L\eta^{\mu\nu}}{2},$$

one can verify.

Problem 9.0.5.

$$\partial_\mu T^{\mu\nu} = 0. \tag{9.0.8}$$

One can easily verify the following formulas

$$T^{00} = T_{00} = \frac{|\nabla_{t,x} u(t,x)|^2}{2}, T^{j0} = -T^0_j = -2\partial_j u \partial_t u,$$

where we adopt the rule for Latin indices varying from 1 to n. Note that we use the metric to raise and lower the indices so

$$T^\nu_\mu = \eta_{\mu\kappa} T^{\kappa\nu}$$

for example.

Given any $R, T > 0$, with $R > T$ and integrating in domain

$$\{(t,x); 0 < t < T, |x| < R - t\}$$

one can derive

Problem 9.0.6.

$$E(T, R-T) \leq E(0,R). \tag{9.0.9}$$

where $E(t,A) \equiv \int_{|x|<A} \frac{|\nabla_{t,x} u(t,x)|^2}{2} dx$.

Let K_ν be a killing vector field,i.e.

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0.$$

Then

Problem 9.0.7.

$$\partial_\mu (K_\nu T^{\mu\nu}) = 0. \tag{9.0.10}$$

More general situation is the case of conformally killing vector fields, i.e.

$$\partial_\mu K_\nu + \partial_\nu K_\mu = \rho \eta_{\mu\nu}.$$

Problem 9.0.8. *We have the identities*

$$\partial_\mu (K_\nu T^{\mu\nu}) + \frac{\rho(n-1)}{2} L = 0 \tag{9.0.11}$$

and

$$\partial_\mu \left(K_\nu T^{\mu\nu} + \frac{\rho(n-1)}{2} u \partial^\mu u - \frac{(\partial^\mu \rho)(n-1)}{4} |u|^2 \right) + \tag{9.0.12}$$

$$+ \frac{(\partial_\mu \partial^\mu \rho)(n-1)}{4} |u|^2 = 0.$$

Introduce space-time coordinates

$$X^\mu,\ X^0 = t,\ X^j = x_j, X_\mu = \eta_{\mu\nu} X^\mu,\ X_0 = t,\ X_j = -x_j.$$

Typical non trivial example of non - constant conformally killing vector field is

$$K_\nu = X_\nu X^0 - \frac{\eta_{\nu 0}}{2}(X_\mu X^\mu). \tag{9.0.13}$$

Note that we have

$$K_0 = K^0 = \frac{t^2 + |x|^2}{2},\ K_j = -K^j = -t x_j.$$

Problem 9.0.9. *Verify the relation*

$$\partial_\mu K_\nu + \partial_\nu K_\mu = \rho \eta_{\mu\nu}$$

with $\rho = 2t$.

Problem 9.0.10. *If the conformally killing filed is defined by (9.0.13) , then we have the identities*

$$\partial_\mu \left(e^\mu + t(n-1) u \partial^\mu u - \eta^{\mu 0} \frac{n-1}{2} |u|^2 \right) = 0 \tag{9.0.14}$$

where

$$e^\mu = K_\nu T^{\mu\nu}.$$

We have in particular

$$e^0 = \frac{(t^2 + |x|^2)}{4} |\partial_t u|^2 + \frac{(t^2 + |x|^2)}{4} |\nabla_x u|^2 + t r \partial_t u \partial_r u.$$

Introduce the generators

$$L_0 = X_\mu \partial^\mu,\ L_{\mu\nu} = X_\mu \partial_\nu - X_\nu \partial_m u.$$

We have the relations

$$L_0 u = t \partial_t u + r \partial_r u, L_{0j} u = t \partial_{x_j} u + x_j \partial_t u.$$

Problem 9.0.11. *If the conformally killing filed is defined by (9.0.13) , then we have the identity*

$$e^0 = \frac{1}{2} \left(|L_0 u|^2 + \sum_{\mu<\nu} |L_{\mu\nu} u|^2 \right).$$

Problem 9.0.12. *(delicate and DIFFICULT estimate) If the conformally killing filed is defined by (9.0.13) and* $n \geq 3$ *, then there are two positive constants* $C_1 < C_2$ *so that*

$$C_1 \leq \frac{\int e^0(t,x) + t(n-1)u\partial_t u - \frac{n-1}{2}|u(t,x)|^2 dx}{\int e^0(t,x)dx + \int |u(t,x)|^2 dx} \leq C_2,$$

provided $u \in C_0^\infty$ *is not identically* 0.

Help: Hardy inequality. Integrating by parts twice in the integral

$$\int_0^\infty |\partial_r u(r)|^2 r^{n-1} dr$$

one can show

Problem 9.0.13. *(Hardy inequality) If* $n \geq 3$ *then we have the estimate*

$$\int_0^\infty \frac{|u(r)|^2}{r^2} r^{n-1} dr \leq C \int_0^\infty |\partial_r u(t,r)|^2 r^{n-1} dr.$$

Find the best constant C*?*

Problem 9.0.14. *Discuss the local energy decay*

$$\int_{|x|<t/2} |\nabla_{t,x} u(t,x)|^2 dx \leq \frac{C}{t^2}$$

for any solution to the wave equation

$$\Box u = 0$$

with initial data

$$u(0,x) = f(x) \in S(\mathbb{R}^n), \partial_t u(0,x) = g(x) \in S(\mathbb{R}^n).$$

9.0.8 Local existence of solution to the Cauchy problem for nonlinear wave equation in $\mathbb{R}^3$.

Consider the Cauchy problem

$$u_{tt} - \Delta u = u^2, \ t \in [0,T], \ x \in \mathbb{R}^3,$$

having initial data

$$u(0,x) = f(x) \in C_0^\infty, u_t(0,x) = g(x) \in C_0^\infty.$$

Set

$$\|f\|_{L^2} = \left(\int_{\mathbb{R}^3} |f(x)|^2 dx\right)^{1/2}.$$

Problem 9.0.15. *For the inhomogeneous wave equation*

$$u_{tt} - \Delta u = F, \ t \in [0,T], \ x \in \mathbb{R}^3,$$

show the energy estimate

$$\|\nabla_{t,x} u(t,.)\|_{L^2} \le \|\nabla_{0,x} u(t,.)\|_{L^2} + C\int_0^t \|F(\tau,.)\|_{L^2} d\tau.$$

Problem 9.0.16. *(Strauss Lemma) If $f(x) = f(|x|)$ is a radial function and*

$$\|f\|_{L^2} + \|\nabla_x f\|_{L^2} < \infty,$$

then

$$|f(x)| \le \frac{C}{|x|}\left(\|f\|_{L^2} + \|\nabla_x f\|_{L^2}\right).$$

9.0.9 Some other hyperbolic problems of mathematical physics

Another important hyperbolic problem is the Dirac system

$$i\gamma_\mu \partial_\mu \psi = 0. \tag{9.0.1}$$

Here $\psi(t,x)$ is a function defined in the Minkowski space $\mathbf{R}^{1+3}$ with values in $\mathbf{C}^4$. Usually, ψ is called a spinor. Moreover, γ_μ are the Dirac matrices defined as follows

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, k = 1,2,3.$$

The Pauli matrices σ_k are determined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The initial data are determined by

$$\psi(0,x) = f(x)$$

The Dirac matrices satisfy the relations

$$\gamma^\mu \gamma^\ni + \gamma^\ni \gamma^\mu = -2\eta^{\mu\ni} \tag{9.0.2}$$

A simple reduction of the Dirac equation to the wave equation can be done by applying the operator $i\gamma_\mu\partial_\mu$ to the Dirac equation in (9.0.1). We use the relations (9.0.2) and find

$$\partial_\mu \partial^\mu \psi = 0.$$

From (9.0.1) we have

$$(\partial_t + \alpha^j \partial_j)\psi = 0. \tag{9.0.3}$$

Here $\alpha_j = \gamma_0\gamma_j$ are selfadjoint matrices. Making the Fourier transform in x we get

$$(\partial_t + i\alpha(\xi))\hat{\psi} = 0$$
$$\hat{\psi}(0,\xi) = \hat{f}(\xi).$$

Here $\alpha(\xi) = \sum \alpha_j \xi_j$ is a selfadjoint matrix. Then the solution of the Cauchy problem for the linear Dirac equation has the form

$$\psi(t,x) = \sum_{\pm} (2\pi)^{-3} \int e^{ix\xi \pm |\xi| t} \pi_{\pm}(\xi) \hat{f}(\xi) d\xi, \tag{9.0.4}$$

where π_+ (respectively π_-) is the positive (respectively negative) eigenspace of the matrix $\alpha(\xi)$.

The Maxwell equations in vacuum have the form

$$\begin{aligned} \partial_t E &= \mathrm{rot} H, \\ \partial_t H &= -\mathrm{rot} E, \\ \mathrm{div} E &= \mathrm{div} H = 0, \end{aligned} \tag{9.0.5}$$

where E (resp. H) is the electric (resp. magnetic) field. Recall that $E(t,x), H(t,x)$ are vector-valued functions from Minkowski space in $\mathbf{R}^3$.

To pose correctly the Cauchy problem for the Maxwell equations we take the initial conditions

$$E(0,x) = e(x) \ , \ H(0,x) = h(x). \tag{9.0.6}$$

Then the equations $\mathrm{div}E = \mathrm{div}H = 0$ in (9.0.5) show that the initial data have to satisfy the constraint conditions

$$\mathrm{div} e = \mathrm{div} h = 0. \tag{9.0.7}$$

Taking the evolution part

$$\begin{aligned} \partial_t E &= \mathrm{rot} H, \\ \partial_t H &= -\mathrm{rot} E, \end{aligned} \tag{9.0.8}$$

of the Maxwell equations, we see that we can solve the Cauchy problem for (9.0.8) with initial data (9.0.6) satisfying the constraint conditions (9.0.7). Then taking the div operator in the equations (9.0.8), we see that

$$\partial_t \mathrm{div} E = \partial_t \mathrm{div} H = 0 \tag{9.0.9}$$

so the constraint conditions (9.0.7) assure the elliptic part div$E$ = div$H$ = 0 in Maxwell equations (9.0.5). Setting $\psi = (E,H)$, we see that the equations (9.0.8) can be written in the form (9.0.3) of a Dirac system so we can use the representation (9.0.4) to solve this system by the aid of the Fourier transform.

Again a simple reduction to the wave equation can be done. In fact taking the time derivative in the first equation in (9.0.5) and using the relation rotrot$E = -\Delta E$ provided div$E = 0$, we get

$$(\partial_t^2 - \Delta)E = 0.$$

In a similar way one can see that H also satisfies the wave equation.

In order to write the system (9.0.5) in relativistic form a natural procedure is to construct the following matrix (called usually electromagnetic tensor).

$$F_{km} = \varepsilon_{kml} H_l \ , \ F_{0k} = -F_{k0} = E_k, \tag{9.0.10}$$

where $\varepsilon_{kml} = 1$ if (kml) is an even permutation of (123), $\varepsilon_{kml} = -1$ if (kml) is an odd permutation of (123) and $\varepsilon_{kml} = 0$ otherwise. Moreover, in (9.0.10) we use the summation convention for repeated Latin indices varying from 1 to 3. It is clear that $F_{\mu\ni}$ is skew-symmetric, i.e. $F_{\mu\ni} = -F_{\ni\mu}$.

By the aid of the metric $\{\eta^{\alpha\beta}\} = diag\{-1,1,...,1\}$ one can freely raise the indices

$$F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\ni}F_{\mu\ni}.$$

The corresponding dual tensor $\tilde{F}_{\mu\ni}$ can be defined as follows

$$\tilde{F}_{\mu\ni} = \varepsilon_{\mu\ni\alpha\beta}F^{\alpha\beta}, \tag{9.0.11}$$

where $\varepsilon_{\mu\ni\alpha\beta} = 1$ if $(\mu \ni \alpha\beta)$ is an even permutation of (0123), $\varepsilon_{\mu\ni\alpha\beta} = -1$ if $(\mu \ni \alpha\beta)$ is an odd permutation of (0123) and $\varepsilon_{\mu\ni\alpha\beta} = 0$ otherwise. Moreover, in (9.0.11) we use the summation convention for repeated Greek indices varying from 0 to 3.

Then the Maxwell equations (9.0.5) take the simple form

$$\partial_\mu F^{\mu\ni} = 0 \ , \ \partial_\mu \tilde{F}^{\mu\ni} = 0. \tag{9.0.12}$$

One can show that if

$$F^{\mu\ni}(t,x)$$

are smooth functions satisfying (9.0.12), then there exist functions $A_\mu(t,x)$, such that

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

Remark. The vector $A_\mu(t,x)$ is called electromagnetic potential. This potential is not unique. Namely, we can take $\tilde{A}_\mu(t,x) = A_\mu(t,x) - \partial_\mu\varphi(t,x)$, where $\varphi(t,x)$ is arbitrary.

Problem 9.0.17. *(Maxwell equations in the form of Dirac equations)*
Let E, H satisfy the Maxwell equations (9.0.5) in vacuum. Consider the vector

$$\chi = \begin{pmatrix} 0 \\ E_1 - iH_1 \\ E_2 - iH_2 \\ E_3 - iH_3 \end{pmatrix}.$$

Find three selfadjoint (4×4) *matrices $\alpha_1, \alpha_2, \alpha_3$ so that χ satisfies the Dirac equation (9.0.3) and the relation*

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}$$

9.0.10 Examples of nonlinear hyperbolic equations

One of the simplest nonlinear hyperbolic equation is the equation of a scalar self-interacting field, that is

$$(-\partial_t^2 + \Delta)u - M^2 u = u^3. \tag{9.0.1}$$

In order to prove the existence of global in time solution even in the case of large initial data we shall use the conservation law of the energy. Indeed, multiplying (9.0.1) by $\partial_t u$ we see that the energy

$$E(t) = \frac{1}{2}\int |\partial_t u(t,x)|^2 dx +$$
$$+\frac{1}{2}\int \left(|\nabla_x u(t,x)|^2 + M|u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 \right) dx \tag{9.0.2}$$

is a constant. As usual the initial data are given by

$$u(0,x) = f_0(x), \partial_t u(0,x) = f_1(x) \tag{9.0.3}$$

To establish the existence of global solution we shall make two steps.

First step. We shall rewrite (9.0.1) in abstract evolution equation of the form

$$\partial_t v = Av + K(v), \tag{9.0.4}$$

where A is a selfadjoint operator in a suitable Hilbert space H and $K(v)$ is an operator in this Hilbert space.

Second step. We shall prove for (9.0.4) a suitable continuation principle. Combining the existence of local solution with this principle we shall establish the existence of global solution.

For simplicity we shall consider here only the case of positive mass M. Setting

$$v = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta - M & 0 \end{pmatrix},$$

$$K(v) = \begin{pmatrix} 0 \\ -v_1^3 \end{pmatrix}, \tag{9.0.5}$$

we see that the nonlinear wave equation (9.0.1) takes the form (9.0.4). The form of the energy in (9.0.2) suggests us to consider the Hilbert space $H = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. For any couple $v = (v_1, v_2) \in H$ the corresponding norm is defined by

$$\|v\|_H^2 = \int |\nabla v_1|^2 + M|v_1|^2 + |v_2|^2 dx \tag{9.0.6}$$

Denote by $(,)_H$ is the corresponding Hilbert norm.

The operator A with dense domain

$$D(A) = H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n). \tag{9.0.7}$$

is skew – selfadjoint.

Turning to the continuation principle we consider the abstract nonlinear evolution problem (9.0.4) assuming A is skew-selfadjoint and K is Lipschitz continuos, i.e. for any ball $B(R)$ of radius R in H there exists a constant $C = C(R)$, such that

$$\|K(u) - K(v)\|_H \le C\|u - v\|_H. \tag{9.0.8}$$

The nonlinear problem (9.0.4) can be written in integral form in the same manner as it is done for ordinary differential equations.

$$v(t) = U(t)f + \int_0^t U(t-s)K(v(s))ds. \tag{9.0.9}$$

Now we are in position to state the continuation principle

Theorem 9.0.1. *Under the above assumptions there exists a maximum interval $[0, \bar{t})$ of existence of a unique solution $v(t) \in C([0, \bar{t}); H)$ of the integral equation (9.0.9). Then either $\bar{t} = \infty$, or else $\|v(t)\|_H \to \infty$ as $t \to \bar{t}$.*

Proof.

For any $t_0 \ge 0$ consider the following local problem

$$v(t) = U(t - t_0)f + \int_{t_0}^t U(t-s)K(v(s))ds. \tag{9.0.10}$$

To establish the existence of solution in the interval $[t_0, t_0+\varepsilon]$ we define inductively the sequence $v_k(t)$ in the Banach space $B = C([t_0, t_0+\varepsilon]; H)$ as follows $v_0(t) = f$,

$$v_{k+1}(t) = U(t-t_0)f + \int_{t_0}^{t} U(t-s)K(v_k(s))ds \tag{9.0.11}$$

Using the fact that $U(t)$ is a unitary operator and K is Lipschitz continuous, we obtain the estimate

$$\| v_{k+1} - v_k \|_B \le C\varepsilon \| v_k - v_{k-1} \|_B \tag{9.0.12}$$

with some constant C independent of ε. When f varies in a ball of radius R in H the constant C in (9.0.12) depends on R, but is independent of f.

The contraction mapping principle shows that a unique solution exists, when $C(R)\varepsilon < 1$.

This means that for $\varepsilon < 1/C(r)$ the life span ε of the solution depends only on R, but it is independent of the concrete choice of f in

$$\{h \in H : |h|_H \le R\}.$$

To finish the proof let us consider the maximal interval $[0, \bar{t})$ of existence of solution with finite $\bar{t}$ and $\| v(t) \|_H \le C$ for $t \in [0, \bar{t})$. Then taking $R = 2C$ and applying the above argument based on the contraction mapping principle we see that one can find $\varepsilon >$ depending only on R so that the local problem (9.0.10) with initial data at t_0 very close to $\bar{t}$ (more precisely our choice is determined by $\bar{t} - \varepsilon < t_0 < \bar{t}$), can be solved in the interval $[t_0, t_0+\varepsilon]$. Since $t_0 + \varepsilon > \bar{t}$ this contradicts the fact that $[0, \bar{t})$ is the maximal interval of existence of solution and completes the proof.

We shall prove that the semi linear problem (9.0.1) for the wave equation has a global solution in case of space dimension $n = 3$. To do this it remains to show that the nonlinear operator K defined in (9.0.5) is Lipschitz continuous. The definition of the norm in H and the Hölder inequality imply that

$$\|K(u) - K(v)\|_H = \| u_1^3 - v_1^3 \|_{L^2} \le C \| u_1 - v_1 \|_{L^6} (\| u_1 \|_{L^6} + \| v_1 \|_{L^6})^2$$

for any two couples $u = (u_1, u_2)^t, v = (v_1, v_2)^t$ in H. Applying the Sobolev inequality

$$\| f \|_{L^6(\mathbf{R}^3)} \le C \| f \|_{H^1(\mathbf{R}^3)}$$

and the definition of the Hilbert space H we arrive at

$$\|K(u) - K(v)\|_H \le C \| u - v \|_H (\| u \|_H + \| v \|_H)^2.$$

Thus K is a Lipschitz operator and the continuation principle assures the existence and the uniqueness of the solution.

Chapter 10

Appendix 0: Some facts about the Euler gamma function, Bessel function and the Legendre function

10.1 Appendix: Some facts about the Euler gamma function and the Legendre function

The Euler function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{10.1.1}$$

for $\mathrm{Re} z > 0$. Using the recurrent relation

$$\Gamma(z+1) = z\Gamma(z), \tag{10.1.2}$$

one can extend the definition of $\Gamma(z)$ for $z \in \mathbf{C}, z \neq 0, 1, 2, \dots$ Since $\Gamma(1) = 1$, we get for any integer $n, n \geq 1$ the relation $\Gamma(n+1) = n!$. Some other relations for the function Γ are given below ([2])

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)}, \tag{10.1.3}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{10.1.4}$$

$$\Gamma(z+1/2)\Gamma(z-1/2) = \frac{\pi}{\cos(\pi z)}. \tag{10.1.5}$$

The function $\Gamma(z)$ is closely related to the function

(10.1.6) $$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

defined for $\mathrm{Re}\,x>0$, $\mathrm{Re}\,y>0$. Namely, we have

(10.1.7) $$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Another integral representation is the following.

(10.1.8) $$B(x,y)=2\int_0^{\pi/2}(\sin t)^{2x-1}(\cos t)^{2y-1}\,dt.$$

It is not difficult to compute the surface of the unit sphere $\mathbf{S}^{n-1}$. In fact for $n=2$ we have $\mu(\mathbf{S}^1)=2\pi$. For $n\geq 3$ we can introduce polar coordinates

$$(\cos\varphi,\ \omega'\sin\varphi)\ ,\omega'\in\mathbf{S}^{n-2}.$$

Then

$$\mu(\mathbf{S}^{n-1})=\mu(\mathbf{S}^{n-2})\int_0^{\pi}\sin^{n-2}\varphi\, d\varphi=$$
$$=\mu(\mathbf{S}^{n-2})B((n-1)/2,1/2)=\mu(\mathbf{S}^{n-2})\frac{\Gamma((n-1)/2)\Gamma(1/2)}{\Gamma(n/2)}.$$

From the recurrent relation

$$\mu(\mathbf{S}^{n-1})=\mu(\mathbf{S}^{n-2})\sqrt{\pi}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}$$

we get

(10.1.9) $$\mu(\mathbf{S}^{n-1})=\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

The Legendre function $P_\nu^\mu(z)$ satisfies the equation (see [2], volume 1, relation (3.2.1))

(10.1.10) $$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left[\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right]w=0.$$

We shall use the following integral representation of these special functions ([2] volume 1, relation (3.7.7))

$$P_\nu^\mu(\mathrm{ch}\,r)=$$
(10.1.11) $$=c_\mu(\mathrm{sh}\,r)^{-\mu}\int_0^{\pi}(\mathrm{ch}\,r-\mathrm{sh}\,r\cos\varphi)^{\nu+\mu}\sin^{-2\mu}\varphi\, d\varphi,$$

where

$$c_\mu = \frac{2^\mu}{\sqrt{\pi}\Gamma(1/2-\mu)}$$

and $\mathrm{Re}\mu < 1/2$.

Another solution of (10.1.10) are the associated Legendre functions $Q_\nu^\mu(z)$ of second kind. They are also solutions of (10.1.10) and satisfy the following relation

$$P_\nu^{-\mu}(z) =$$

$$(10.1.12) \quad = \frac{e^{-i\mu\pi}\Gamma(\nu-\mu+1)}{\pi\cos(\nu\pi)\Gamma(\nu+\mu+1)}\sin[\pi(\nu-\mu)][Q_\nu^\mu(z) - Q_{-\nu-1}^\mu(z)].$$

An integral representation of Q_ν^μ is given by (see [2] volume 1, relation (3.7.4))

$$(10.1.13) \quad Q_\nu^\mu(\mathrm{ch}r) = d_\mu(\mathrm{sh}r)^\mu \int_r^\infty e^{-(\nu+1/2)s}(\mathrm{ch}s - \mathrm{ch}r)^{-\mu-1/2}ds,$$

where

$$d_\mu = \sqrt{\frac{\pi}{2}}\frac{e^{i\mu\pi}}{\Gamma(1/2-\mu)}$$

and the above representation is valid for $\mathrm{Re}(\nu+\mu+1) > 0, \mathrm{Re}\mu < 1/2$. A differential equation satisfied by $P_\nu^\mu(z)$ is

$$(10.1.14) \qquad \frac{dP_\nu^\mu(z)}{dz} = \frac{(\nu+\mu)(\nu-\mu+1)}{\sqrt{z^2-1}}P_\nu^{\mu-1}(z) - \frac{\mu z}{z^2-1}P_\nu^\mu(z).$$

From these relations we shall establish the following.

Lemma 10.1.1. *The function*

$$(10.1.15) \qquad L^n(r) = L_\sigma^n(r) = (\mathrm{sh}r)^{-(n-2)/2}\, P_{-1/2-i\sigma}^{-(n-2)/2}(\mathrm{ch}r)$$

satisfies the recurrent relation

$$(10.1.16) \qquad L^n(r) = \frac{-1}{\sigma^2 + (\frac{n-3}{2})^2}\left(\frac{1}{\mathrm{sh}r}\partial_r\right)L^{n-2}(r).$$

Proof. From (10.1.14) we have

$$\partial_r\left(\mathrm{sh}^\mu r P_\nu^\mu(\mathrm{ch}r)\right) = (\nu+\mu)(\nu-\mu+1)\mathrm{sh}^\mu r P_\nu^{\mu-1}(\mathrm{ch}r).$$

With $\mu = -(n-4)/2$ and $\nu = -1/2 - i\sigma$ we obtain the desired relation.

Further, we shall obtain

Lemma 10.1.2. *For $n \geq 1$ odd the function*

$$L^n(r) = L^n_\sigma(r) = \operatorname{sh}^{-(n-2)/2} r P^{-(n-2)/2}_{-1/2-i\sigma}(\operatorname{ch} r)$$

takes the form

$$L^n(r) = (-1)^{(n-1)/2}\sqrt{\frac{2}{\pi}}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma+\frac{n-1}{2})|^2}(\frac{1}{\operatorname{sh} r}\partial_r)^{(n-1)/2}\cos\sigma r. \tag{10.1.17}$$

Proof. For $n = 1$ we use the relation (see [2]), identity (3.6.12))

$$P^{1/2}_\nu(z) = \frac{1}{\sqrt{2\pi}}(z^2-1)^{-1/4}[(z+\sqrt{z^2-1})^{\nu+1/2} + (z+\sqrt{z^2-1})^{-\nu-1/2}]$$

and we find

$$L^1(r) = \sqrt{\frac{2}{\pi}}\cos\sigma r. \tag{10.1.18}$$

Thus, the assertion is verified for $n = 1$. In case $n \geq 3$ odd from (10.1.16) we get

$$L^n(r) = \frac{(-1)^{(n-1)/2}}{\prod_{k=0}^{(n-3)/2}(\sigma^2+k^2)}(\frac{1}{\operatorname{sh} r}\partial_r)^{(n-1)/2}L^1(r). \tag{10.1.19}$$

On the other hand, we have

$$\prod_{k=0}^{(n-3)/2}(\sigma^2+k^2) = \frac{|\Gamma(i\sigma+\frac{n-1}{2})|^2}{|\Gamma(i\sigma)|^2} \tag{10.1.20}$$

and hence this relation combined with (10.1.18) and (10.1.19) imply the desired identity (10.1.17).

The Lemma is proved.

Lemma 10.1.3. *For $n \geq 2$ even the function*

$$L^n(r) = L^n_\sigma(r) = (\operatorname{sh} r)^{-(n-2)/2}\, P^{-(n-2)/2}_{-1/2-i\sigma}(\operatorname{ch} r)$$

takes the form

$$L^n(r) = \frac{(-1)^{n/2}\sqrt{2}}{\pi}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma+\frac{n-1}{2})|^2}\int_r^\infty \frac{\operatorname{sh} s}{\sqrt{\operatorname{ch} s - \operatorname{ch} r}}(\frac{1}{\operatorname{sh} s}\partial_s)^{n/2}(\cos\sigma s)ds. \tag{10.1.21}$$

Proof. Set

$$K^n(r) = K^n_\sigma(r) =$$

$$= \int_r^\infty \frac{\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}} (\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds. \tag{10.1.22}$$

Integrating by parts, we find

$$K^n(r) = 2\int_r^\infty \partial_s(\sqrt{\mathrm{ch}s - \mathrm{ch}r})(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds =$$
$$= -2\int_r^\infty (\sqrt{\mathrm{ch}s - \mathrm{ch}r})\partial_s(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds.$$

Hence, for $r \neq 0$ we have

$$\frac{1}{\mathrm{sh}r}\partial_r K^n(r) = \int_r^\infty \frac{1}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}\partial_s(\frac{1}{\mathrm{sh}s}\partial_s)^{n/2}(\cos\sigma s)ds = K^{n+2}(r).$$

Thus for $n \geq 2$ even we have

$$K^n(r) = (\frac{1}{\mathrm{sh}r}\partial_r)^{(n-2)/2}K^2(r). \tag{10.1.23}$$

Now we have to compute

$$K^2(r) = -\sigma\int_r^\infty \frac{\sin s\sigma}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}ds.$$

If we apply (10.1.13) with $\mu = 0, \nu = -1/2 + i\sigma$, then we get

$$\int_r^\infty \frac{\mathrm{e}^{-is\sigma}}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}ds = \sqrt{2}Q^0_{-1/2+i\sigma}(\mathrm{ch}r).$$

Applying (10.1.12), we obtain further

$$\int_r^\infty \frac{\sin s\sigma}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}ds = \frac{\sqrt{2}}{2}\pi\ \mathrm{th}(\sigma\pi)P^0_{-1/2+i\sigma}(\mathrm{ch}r).$$

Therefore, we have

$$K^2(r) = -\frac{\pi}{\sqrt{2}}\sigma\ \mathrm{th}(\sigma\pi)P^0_{-1/2+i\sigma}(\mathrm{ch}r) =$$
$$= -\frac{\pi}{\sqrt{2}}\sigma\ \mathrm{th}(\sigma\pi)L^2(r). \tag{10.1.24}$$

Now we can apply Lemma 10.1.1 and get for $n \geq 2$ even

$$\frac{(-1)^{n/2}}{\sqrt{2}}\pi\sigma\ \mathrm{th}(\sigma\pi)(\sigma^2 + (\frac{n-3}{2})^2)...(\sigma^2 + (\frac{1}{2})^2)L^n_\sigma(r) =$$
$$= (\frac{1}{\mathrm{sh}r}\partial_r)^{(n-2)/2}K^2_\sigma(r). \tag{10.1.25}$$

Comparing this relation with (10.1.23), we obtain

$$\frac{(-1)^{n/2}}{\sqrt{2}}\pi\sigma\ \mathrm{th}(\sigma\pi)(\sigma^2+(\frac{n-3}{2})^2)...(\sigma^2+(\frac{1}{2})^2)L^n_\sigma(r)=$$

(10.1.26)
$$=K^n_\sigma(r).$$

On the other hand, for $n \geq 2$ even we can apply the following relations for the Gamma function (see [2], vol.1, relations (1.2.5) and (1.2.7))

$$\Gamma(z)\Gamma(-z)=-\frac{\pi}{z\sin\pi z}\ \ ,\ \ \Gamma(1/2+z)\Gamma(1/2-z)=\frac{\pi}{\cos\pi z}.$$

Then we get

$$\frac{\Gamma(1/2+i\sigma)}{\Gamma(i\sigma)}=\sigma\mathrm{th}(\pi\sigma)\frac{\Gamma(-i\sigma)}{\Gamma(1/2-i\sigma)}$$

so

(10.1.27)
$$\left|\frac{\Gamma(1/2+i\sigma)}{\Gamma(i\sigma)}\right|^2=\sigma\mathrm{th}(\pi\sigma).$$

From $\Gamma(1+z)=z\Gamma(z)$ together with (10.1.27) we derive

(10.1.28)
$$\left|\frac{\Gamma((n-1)/2+i\sigma)}{\Gamma(i\sigma)}\right|^2=$$

$$=\sigma\mathrm{th}(\pi\sigma)\left(\left(\frac{n-3}{2}\right)^2+\sigma^2\right)...\left(\left(\frac{1}{2}\right)^2+\sigma^2\right).$$

From (10.1.25) we get

$$L^n(r)=\frac{(-1)^{n/2}\sqrt{2}}{\pi}\frac{|\Gamma(i\sigma)|^2}{|\Gamma(i\sigma+\frac{n-1}{2})|^2}K^n_\sigma(r).$$

This completes the proof of the Lemma.

10.2 Some properties of Bessel function

(see [2].)

The Bessel function $J_\nu(z)$ is a solution of the ordinary differential equation

(10.2.1)
$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0.$$

For $z\in\mathbf{C}$ close to 0 we have the following series expansion

(10.2.2)
$$J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)}.$$

The Bessel function satisfies the following recurrence relations

$$zJ'_\nu(z)+\nu J_\nu(z)=zJ_{\nu-1}(z) \tag{10.2.3}$$

or equivalently

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right]=z^\nu J_{\nu-1}(z). \tag{10.2.4}$$

Moreover, we have

$$zJ'_\nu(z)-\nu J_\nu(z)=-zJ_{\nu+1}(z). \tag{10.2.5}$$

The Wronskian of two Bessel functions is given by

$$W(w_1,w_2)=w_1w'_2-w_2w'_1.$$

For the case,when $w_1=J_\nu$, $w_2=J_{-\nu}$ the corresponding Wronskian is

$$W[J_\nu(z),J_{-\nu}(z)]=-\frac{2}{\pi z}\ \sin(\pi\nu). \tag{10.2.6}$$

The following integral representations by Poisson's integral shall be of special interest in our considerations

$$\Gamma(\nu+1/2)J_\nu(z)=\frac{2z^\nu}{\sqrt{\pi}2^\nu}\int_0^1(1-t^2)^{\nu-1/2}\cos(zt)dt. \tag{10.2.7}$$

For the special cases $\nu=\pm1/2$ we have

$$J_{1/2}(z)=\frac{\sqrt{2}}{\sqrt{\pi z}}\sin z,\ J_{-1/2}(z)=\frac{\sqrt{2}}{\sqrt{\pi z}}\cos z. \tag{10.2.8}$$

Chapter 11

Appendix I: Stationary phase method

11.0.1 Stationary phase method

In this section we shall give a brief review of the methods to study the asymptotic behavior of oscillatory integrals of type

$$I(R) = \int_{\mathbf{R}^n} e^{iR\phi(x)} f(x)dx \tag{11.0.1}$$

as $R > 0$ tends to infinity.

Here $f(x), \phi(x)$ are smooth functions defined on $\mathbb{R}^n$ with $\phi(x)$ being real-valued.

First, we consider the case, when the phase function $\varphi(x)$ has no critical points. More precisely, we consider the case, when there exist $\delta > 0, \delta \leq 1$ and $C > 0$ so that

$$|\nabla\phi(x)| \geq C^{-1} < x >^{\delta}, \ < x >^2 = 1 + |x|^2, \tag{11.0.2}$$

$$|\partial_x^\alpha \nabla\phi(x)| \leq C < x >^{\delta - |\alpha|} \tag{11.0.3}$$

for any $x \in \operatorname{supp} f$.

Lemma 11.0.1. *Suppose the assumptions (11.0.2), (11.0.3) are fulfilled and $f(x)$ is a smooth function with compact support. Then for any integer $N \geq 0$ and for any $\varepsilon > 0$ we have*

$$|I(R)| \leq \frac{C}{R^N} \sum_{|\alpha| \leq N} \| < x >^{-N\delta - N + |\alpha| + n/2 + \varepsilon} \partial^\alpha f \|_{L^2(\mathbb{R}^n)}.$$

Proof. Given any first order differential operator

$$L(x,\partial_x) = (\sum_{j=1}^{n} a_j(x)\partial_{x_j}) + b(x),$$

we denote by L^* its adjoint operator with respect to the inner product in $L^2(\mathbb{R}^n)$, i.e.

$$L^*(x,\partial_x) = -(\sum_{j=1}^{n} \overline{a_j(x)}\partial_{x_j}) + \overline{b(x)} + \sum_{j=1}^{n} \partial_{x_j}\overline{a_j(x)}.$$

Therefore, for any couple f, g of smooth compactly supported functions on $\mathbb{R}^n$ we have

$$(Lf, g)_{L^2(\mathbb{R}^n)} = (f, L^* g)_{L^2(\mathbb{R}^n)}. \tag{11.0.4}$$

Let $L(x,\partial_x)$ be the differential operator, such that its adjoint is

$$L^* = i^{-1} \sum_{k=1}^{n} \frac{\partial_{x_k}\phi}{|\nabla\phi|^2}\partial_{x_k},$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$.

It is clear that

$$L^*(\mathrm{e}^{iR\phi}) = R\mathrm{e}^{iR\phi}.$$

Then (11.0.4) implies that

$$I(R) = \frac{1}{R^N}\int_{\mathbb{R}^n} \mathrm{e}^{iR\phi} L^N(f)\,dx.$$

In order to evaluate $L^N(f)$, we shall establish inductively with respect to N that L^N can be represented as

$$L^N = \sum_{|\alpha|\leq N} a_\alpha^N(x)\partial_x^\alpha, \tag{11.0.5}$$

where the coefficients satisfy suitable decay estimates. To formulate precisely this statement, given any real number m we denote by S^m the class of all smooth functions $a(x)$ such that for any multiindex β there exists $C = C(\beta)$ so that

$$|(< x > \partial_x)^\beta a(x)| \leq C < x >^m .$$

Our goal is to show that the coefficients in (11.0.5) satisfy

$$a_\alpha^N(x) \in S^{-\delta N - N + |\alpha|}. \tag{11.0.6}$$

For $N = 1$ we have

$$L = i^{-1} \sum_{k=1}^{n} \frac{\partial_{x_k} \phi}{|\nabla \phi|^2} \partial_{x_k} + b(x),$$

where $b(x)$ is constant times

$$\sum_{k=1}^{n} \partial_{x_k} (\partial_{x_k} \phi / |\nabla \phi|^2).$$

Therefore, we have to show that

$$\frac{\nabla \phi(x)}{|\nabla \phi(x)|^2} \in S^{-\delta}. \tag{11.0.7}$$

Indeed, consider the function

$$v \in \mathbb{R}^n \setminus 0 \to \chi(v) = v/|v|^2.$$

Then the function in (11.0.7) can be represented as $\chi(\nabla\phi)$. Moreover, for any multiindex α we can represent $\partial_x^\alpha \chi(\nabla \phi)$ as a linear combination of terms of type

$$(\partial_v^\beta \chi)(\nabla\phi)(\partial_x^{\gamma_1} \nabla\phi)...(\partial_x^{\gamma_{|\beta|}} \nabla\phi)$$

with $|\beta| \leq |\alpha|$ and

$$\gamma_1 + ... + \gamma_{|\beta|} = \alpha.$$

Since

$$|\partial_v^\beta \chi(v)| \leq C|v|^{-1-|\beta|}$$

and $|v| = |\nabla\phi| \geq C^{-1} < x >^\delta$, we have

$$|(\partial_v^\beta \chi)(\nabla\phi)| \leq C_1 < x >^{-\delta(1+|\beta|)} .$$

Applying (11.0.3), we find

$$|\partial_x^\alpha \nabla\phi| \leq C < x >^{\delta - |\alpha|}$$

so (11.0.7) is established. Using the trivial property

$$a \in S^m \Longrightarrow \partial_x^\alpha a \in S^{m-\alpha}, \tag{11.0.8}$$

we obtain (11.0.6) and this implies

$$|L^N(f)(x)| \leq C < x >^{-\delta N - N + |\alpha|} \sum_{|\alpha| \leq N} |\partial_x^\alpha f(x)|.$$

Applying the Cauchy inequality, we complete the proof of the Lemma.

As an application we shall consider the oscillatory integral

$$A(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-i(z,\eta)} a(x+z,\xi+\eta) b(x,z,\xi,\eta)\, dz d\eta, \tag{11.0.9}$$

where $a(x,\xi)$ is a smooth function on $\mathbf{R}^n \times \mathbf{R}^n$ belonging to the class of symbols $S^{m,k}$, defined as follows.

Definition 11.0.1. *A smooth function $a(x,\xi)$ belongs to $S^{m,k}$, (m, k are real numbers) if for any integer $N \geq 0$ one can find a constant $C = C(N)$ so that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C < \xi >^{m-|\beta|} < x >^{k-|\alpha|}$$

for $|\alpha| + |\beta| \leq N$.

Further, we set

$$S^{-\infty,-\infty} = \cap_{m,k} S^{m,k}.$$

Moreover,

$$b(x,z,\xi,\eta) = (1-\varphi(\eta/(1+|\xi|)))(1-\varphi(z/(1+|x|)))$$

where the cut-off function $\varphi(x)$ in (11.0.9) is such that $\varphi(x) = 1$ for $|x| \leq 1/4$ and $\varphi(x) = 0$ for $|x| \geq 1/2$.

We shall establish the following.

Lemma 11.0.2. *If $a \in S^{m,k}$, then the oscillatory integral A in (11.0.9) belongs to $S^{-\infty,-\infty}$.*

Proof. Taking $\phi(y,z) = (y,z), R = 1$, we see that the oscillatory integral A has the form (11.0.1). Taking $\delta = 1$, we see that the assumptions (11.0.2) and (11.0.3) are fulfilled. Thus, for any integer $N \geq 1$ we have

$$|A(x,\xi)| \leq C \sum_{|\alpha|+|\beta| \leq N} \int\int B(x,z,\xi,\eta)\, dz d\eta,$$

where

$$B(x,z,\xi,\eta) = (1+|z|)^{-2N-2\delta+n+2\varepsilon} \times \times (1+|\eta|)^{-2N-2\delta+n+2\varepsilon} |\partial_z^\alpha \partial_\eta^\beta a(x+z,\xi+\eta)|^2. \tag{11.0.10}$$

The integration above is over $|z| \geq (1+|x|)/4$ and $|\eta| \geq (1+|\xi|)/4$. This observation implies that for any integer $N_1 \geq 1$ we have the estimate

$$|A(x,\xi)| \leq C(1+|x|)^{-N_1}(1+|\xi|)^{-N_1}.$$

In a similar way we estimate the derivatives of A and get

$$|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)| \leq C(1+|x|)^{-N_1}(1+|\xi|)^{-N_1}.$$

This completes the proof of the lemma.

In a similar way, we can consider the oscillatory integral

$$A(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-i(z,\eta)} a_1(x,z,\xi,\eta)\, dz\, d\eta, \tag{11.0.11}$$

where

$$a_1(x,z,\xi,\eta) = a(x+z,\xi+\eta)(1-\varphi(\eta/(1+|\xi|)))\varphi(z/(1+|x|)).$$

Now we can use the argument of the proof of Lemma 11.0.1 and use the operator

$$L = (\frac{\eta}{i|\eta|^2}, \nabla_z).$$

Then integrating by parts as it was done in Lemma 11.0.1, we get

$$|A(x,\xi)| \leq C \sum_{|\alpha|=N} \int\int (1+|\eta|)^{-N} |\partial_z^\alpha (\varphi(z/(1+|x|))a(x+z,\xi+\eta))| dz\, d\eta.$$

Here the integration is over $|z| \leq (1+|x|)/2$ so on the integration domain the weights $1+|x+z|$ and $1+|x|$ are equivalent. Then the definition 11.0.1 shows that we have the estimate

$$|\partial_z^\alpha (\varphi(z/(1+|x|))a(x+z,\xi+\eta))| \leq C < x >^{k-|\alpha|}.$$

Choosing $N \geq 1$ sufficiently large, we get

Lemma 11.0.3. *If $a \in S^{m,k}$, then the oscillatory integral A in (11.0.11) belongs to $S^{-\infty,-\infty}$.*

Our next step is to consider the case when

$$I(R) = \int_{\mathbf{R}^n} \mathrm{e}^{iR(Qx,x)} f(x)\, dx, \tag{11.0.12}$$

where Q is a constant symmetric invertible matrix. Then the assumption (11.0.2) is not satisfied. For this case stationary phase method gives the following.

Lemma 11.0.4. *For any real number $s > n/2$ we have the estimate*

$$|I(R)| \le CR^{-n/2}\|f\|_{H^s}.$$

Proof. We have seen in (8.1.7) that the Fourier transform of the distribution $e^{iR(Qx,x)}$ is constant times

$$R^{-n/2}e^{-i(Q^{-1}\xi,\xi)/4R}.$$

Therefore, applying Plancherel identity, we get

$$|I(R)| \le CR^{-n/2}\int_{\mathbb{R}^n}|\hat{f}(\xi)|d\xi.$$

Applying the Cauchy inequality, we complete the proof.

We can obtain asymptotic expansion for $I(R)$. In fact, we have the expansion

$$e^{-i(Q^{-1}\xi,\xi)/4R} = \sum_{k=0}^{N-1}(-i/4R)^k(Q^{-1}\xi,\xi)^k/k! + r_N,$$

where the remainder $r_N(\xi)$ satisfies the estimate

$$|r_N(\xi)| \le C_N|(Q^{-1}\xi,\xi)|^N R^{-N}.$$

Therefore, we have the asymptotic expansion

$$I(R) = \sum_{k=0}^{N-1} I_k(R) + \sigma_N(R), \tag{11.0.13}$$

where

$$I_k(R) = \frac{Ci^{-k}}{k!(4R)^{k+n/2}}(Q^{-1}D_x, D_x)f(0)$$

and the remainder $\sigma_N(R)$ satisfies the estimate

$$|\sigma_N(R)| \le \frac{C_N}{R^{N+n/2}}\|f\|_{H^{2N+s}}$$

with $s > n/2$.

Chapter 12

Appendix II: Richiami sulla trasformata di Fourier, distribuzioni e convoluzioni

12.1 Definizione e prime proprietà

Sia $f : [-\pi;\pi] \longrightarrow \mathbb{R}$ una funzione tale che

$$f(x) = \sum_{k\in\mathbf{Z}} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(y) e^{-iky} dy \right) e^{ikx}.$$

Se $f : [-L\pi; L\pi] \longrightarrow \mathbb{R}$, allora

$$f(x) = \sum_{k\in\mathbf{Z}} \frac{1}{2\pi L} \left(\int_{-L\pi}^{L\pi} f(y) e^{-i\frac{k}{L}y} dy \right) e^{i\frac{k}{L}x}. \tag{12.1.1}$$

Ricordiamo che, per ogni funzione H appartenente a $L^1(\mathbb{R})$, si ha:

$$\lim_{L\to\infty} \sum_{k\in\mathbf{Z}} \frac{1}{L} H\left(\frac{1}{L}\right) = \int H(\xi)\, d\xi. \tag{12.1.2}$$

Facendo il limite per $L \to \infty$ nell'identità (12.1.1) si ha:

$$f(x) = \frac{1}{2\pi} \int d\xi\, e^{i\xi x} \int dy\, f(y) e^{-i\xi y}. \tag{12.1.3}$$

Definiamo la **trasformata di Fourier** di una funzione f:

$$\hat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int f(y) e^{-i\xi y} dy \tag{12.1.4}$$

e la funzione $f \overset{\mathscr{F}}{\longmapsto} \hat{f}$ sarà detta **trasformazione di Fourier**.

Innanzitutto osserviamo che la definizione di trasformata di Fourier ha senso ogni volta che f appartiene a $L^1(\mathbb{R})$ e $\hat{f}$ appartiene, così, a $L^\infty(\mathbb{R})$. Nel seguito, tuttavia, sarà più comodo considerare la trasformazione di Fourier come un'applicazione dello spazio $\mathscr{D}(\mathbb{R})$ (dove $\mathscr{D}(\mathbb{R})$ denota lo spazio vettoriale delle funzioni di classe C^∞ a supporto compatto), ovvero $\mathscr{D}(\mathbb{R}) \overset{\mathscr{F}}{\longrightarrow} L^\infty(\mathbb{R})$.

Proposition 12.1.1 (Proprietà elementari della trasformata di Fourier)**.** *Sia f una funzione di classe C^∞ a supporto compatto, allora,*

$$\widehat{\partial_x^\alpha f(x)}(\xi) = i^\alpha \xi^\alpha \hat{f}(\xi), \qquad \widehat{x^\alpha f(x)}(\xi) = i^\alpha \partial_\xi^\alpha \hat{f}(\xi).$$

Dimostrazione. Per induzione, anzitutto quando $\alpha = 1$,

$$\widehat{f'(x)}(\xi) = \int f'(x) e^{-i\xi x} dx = i\xi \int f(x) e^{-i\xi x} dx = i\xi \hat{f}(\xi),$$

in generale,

$$\begin{aligned}
\mathscr{F}\left[\partial_x^{\alpha+1} f(x)\right](\xi) &= \int \left(\partial_x^{\alpha+1} f(x)\right) e^{-i\xi x} dx \\
&= i\xi \int \left(\partial_x^{\alpha} f(x)\right) e^{-i\xi x} dx \\
&= (i\xi)^{\alpha+1} \int f(x) e^{-i\xi x} dx \\
&= i^{\alpha+1} \xi^{\alpha+1} \hat{f}(\xi).
\end{aligned}$$

Per la seconda uguaglianza, per $\alpha = 1$,

$$\widehat{xf(x)}(\xi) = \int x f(x) e^{-i\xi x} dx = \int f(x) i\partial_\xi e^{-i\xi x} dx = i\partial_\xi \int f(x) e^{-i\xi x} dx = i\partial_\xi \hat{f}(\xi),$$

in generale,

$$\begin{aligned}
\mathscr{F}\left[x^{\alpha+1} f(x)\right](\xi) &= \int x^{\alpha+1} f(x) e^{-i\xi x} dx \\
&= i\partial_\xi \int x^{\alpha} f(x) e^{-i\xi x} dx \\
&= i^{\alpha+1} \partial_\xi^{\alpha+1} \hat{f}(\xi),
\end{aligned}$$

che conclude la dimostrazione. □

12.2 Spazio di Schwarz

Definiamo lo **spazio di Schwarz** come

$$\mathscr{S} = \mathscr{S}(\mathbb{R}) = \left\{ f \in C^{\infty}(\mathbb{R}) : \forall (n,k) \in \mathbb{N} \times \mathbb{N}, \quad x^k \partial^n f \in L^{\infty}(\mathbb{R}) \right\}. \tag{12.2.5}$$

e chiameremo le funzioni f appartenenti alla classe $\mathscr{S}$, **funzioni rapidamente decrescenti**.

Osserviamo, innanzitutto, che $\mathscr{S} \subset Lip(\mathbb{R})$ per il teorema del valor medio di Lagrange ($k = 1, n = 1$); inoltre $\mathscr{S} \subset L^1(\mathbb{R})$. Infine $\mathscr{S} \subset L^p(\mathbb{R})$ per ogni $p \in [1;\infty]$.

Teorema 12.2.1. *Se $f \in \mathscr{S}$, allora $\hat{f} \in \mathscr{S}$, ovvero $\mathscr{F} : \mathscr{S} \longrightarrow \mathscr{S}$.*

Dimostrazione. Per ipotesi, per ogni coppia di interi positivi k e m, vale

$$\left| x^k D^m f \right| < c_{k,m}.$$

Dalle proprietà della trasformata di Fourier:

$$\left| \xi^m D^k \hat{f} \right| = \left| \widehat{D^m x^k f} \right|.$$

Dico che $D^m x^k f \in \mathscr{S}$; basta provare, infatti, che $|x^j D^m x^k f| < C$, per ogni terna di numeri interi positivi j, m e k:

$$\begin{aligned} \left| x^j D^m x^k f \right| &= \left| x^j \sum_{l=0}^{m} \binom{m}{l} D^l f D^{m-l} x^k \right| \\ &= \left| x^j \sum_{l=0}^{m} \binom{m}{l} \frac{k!}{(m-l)!} x^{k-m-l} D^l x \right| \\ &\leq \sum_{l=0}^{m} \frac{k!}{(m-l)!} \binom{m}{l} \left| x^{k-m-l+j} D^l f \right| < C \end{aligned}$$

essendo $f \in \mathscr{S}$ e la somma finita. Ne segue che $\left| \widehat{D^m x^k f} \right| < \infty$ in quanto $D^m x^k f$ è una funzione rapidamente decrescente e quindi appartenente a L^1. □

Lemma 12.2.1 (teorema di Riemann-Lebesgue)**.** *Per ogni funzione f appartenente a $L^1(\mathbb{R})$ vale*

$$\lim_{\xi \to \infty} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx = 0.$$

Dimostrazione. Per densità delle funzioni semplici in $L^1(\mathbb{R})$ possiamo supporre che f sia semplice. Per linearità possiamo supporre che f sia una indicatrice di un intervallo limitato. Allora

$$\int_{\mathbb{R}} I_{[a;b]} e^{-i\xi x} dx = \int_a^b e^{-i\xi x} dx = -\frac{1}{i\xi} [e^{-i\xi b} - e^{-i\xi a}]$$

che converge verso zero al tendere di ξ all'infinito dato che $e^{i\xi x}$ è limitata. □

Lemma 12.2.2. *Per ogni numero positivo R, vale*

$$I = \int_{\mathbb{R}} \frac{senRx}{x} dx = \pi. \tag{12.2.6}$$

Dimostrazione. Facendo il cambiamento di variabili $Rx = y$,

$$I = \int_{\mathbb{R}} \frac{seny}{y} dy.$$

Vogliamo provare che

$$\int_{\mathbb{R}} \frac{seny}{y} dy = \lim_{\substack{M\to\infty \\ \epsilon\to 0}} \int_{\Gamma} \frac{senz}{z} dz$$

dove Γ è definita come segue:
ossia $\Gamma = \Gamma_M \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon$, dove $\Gamma_M = \left\{Me^{i\theta} : \theta \in [0;\pi]\right\}$, $\Gamma_1 = [\epsilon, M]$, $\Gamma_2 = [-M, -\epsilon]$ e $\Gamma_\epsilon = \left\{\epsilon e^{i(\pi-\theta)} : \theta \in [0;\pi]\right\}$.

Essendo $senz = \frac{e^{iz}-e^{-iz}}{2i}$, segue che basta calcolare

$$\int_{\Gamma} \frac{e^{iz}}{z} dz.$$

Si ha:

$$\begin{aligned} \int_{\Gamma_M} \frac{e^{iz}}{z} dz &= \int_0^{\pi} \frac{e^{iM\cos\theta} e^{-Msen\theta}}{Me^{i\theta}} iMe^{i\theta} d\theta \\ &= \int_0^{\pi} ie^{iM\cos\theta} e^{-Msen\theta} d\theta, \end{aligned}$$

dunque,

$$\begin{aligned} \left|\int_{\Gamma_M} \frac{e^{i\theta}}{z} dz\right| &\leq \int_0^{\pi} e^{-Msen\theta} d\theta \leq 2\int_0^{\frac{\pi}{2}} e^{-Msen\theta} d\theta \\ &\leq 2\int_0^{\frac{\pi}{2}} e^{\frac{2}{\pi}M\theta} d\theta \leq 2\int_0^{\infty} e^{\frac{2}{\pi}M\theta} d\theta = \frac{\pi}{M} \end{aligned}$$

che converge verso zero al tendere di M all'infinito. Inoltre, essendo 0 un polo semplice,

$$\int_{\Gamma_\epsilon} \frac{e^{iz}}{z} dz = \int_{\Gamma_\epsilon} \left[\frac{1}{z} + h(z)\right] dz = \int_{\Gamma_\epsilon} \frac{1}{z} dz + \int_{\Gamma_\epsilon} h(z)\, dz = i\pi + \int_{\Gamma_\epsilon} h(z)\, dz$$

dove $h(z)$ è olomorfa e quindi, al tendere di ϵ a zero, l'integrale converge verso $i\pi$. Ne segue che

$$\int_{\mathbb{R}} \frac{seny}{y} = \lim_{\substack{M\to\infty \\ \epsilon\to 0}} \int_{\Gamma} \frac{senz}{z} dz = \frac{1}{2i} \lim_{\substack{M\to\infty \\ \epsilon\to 0}} \int_{\Gamma} \frac{e^{iz} - e^{-iz}}{z} dz = \frac{1}{i} \lim_{\substack{M\to\infty \\ \epsilon\to 0}} \int_{\Gamma} \frac{e^{iz}}{z} dz = \pi.$$

□

Teorema 12.2.2. *Per ogni funzione f appartenente a $\mathscr{S}$ si ha:*

$$f(x) = \frac{1}{2\pi}\int d\xi\, e^{i\xi x}\int dy\, f(y)e^{-i\xi y}. \tag{12.2.7}$$

Dimostrazione. Dobbiamo provare che

$$f(x) = \lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R} d\xi\, e^{i\xi x}\int_{\mathbb{R}} dy\, f(y)e^{-i\xi y}.$$

Sia

$$I(R) = \frac{1}{2\pi}\int_{-R}^{R} d\xi\, e^{i\xi x}\int_{\mathbb{R}} dy\, f(y)e^{-i\xi y};$$

per il teorema di Fubini abbiamo,

$$I(R) = \frac{1}{2\pi}\int_{\mathbb{R}} dy\, f(y)\int_{-R}^{R} d\xi\, e^{i\xi(x-y)}.$$

Siccome:

$$\int_{-R}^{R} e^{i\xi z}d\xi = \left.\frac{e^{i\xi z}}{iz}\right|_{\xi=-R}^{\xi=R} = \frac{1}{z}\cdot\frac{e^{izR}-e^{-izR}}{i} = \frac{2senRz}{z},$$

ne segue che

$$I(R) = \frac{1}{\pi}\int senR(x-y)\frac{f(y)}{x-y}\,dy.$$

Inoltre, per il lemma 12.2.2, si ha:

$$f(x) = \frac{1}{\pi}\int senR(x-y)\frac{f(x)}{x-y}\,dy.$$

Pertanto:

$$f(x) - I(R) = \frac{1}{\pi}\int senR(x-y)\frac{f(x)-f(y)}{x-y}\,dy = \frac{1}{\pi}\left\{\int_{|y-x|<\delta} + \int_{|y-x|\geq\delta}\right\}.$$

Per il primo integrale:

$$\left|\frac{1}{\pi}\int_{|x-y|<\delta} senR(x-y)\frac{f(x)-f(y)}{x-y}\,dy\right| \leq \frac{1}{\pi}\int_{|x-y|<\delta}\frac{|f(y)-f(x)|}{|y-x|}\,dy \leq \frac{1}{\pi}L\delta$$

dove L è la costante di Lipschitz di f. Per il secondo integrale: dal teorema di Riemann-Lebesgue (lemma 12.2.1) segue che:

$$\lim_{R\to\infty}\int_{|x-y|\geq\delta} senR(x-y)\frac{f(x)-f(y)}{x-y}\,dy = 0.$$

Ne segue che

$$\lim_{R\to\infty} I(R) = f(x).$$

□

La dimostrazione precedente pone un altro problema. La quantitá

$$I_R(f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dy\, f(y) \int_{-R}^{R} d\xi\, e^{i\xi(x-y)}$$

si puo presentare come

$$I_R(f)(x) = D_R * f(x), D_R(y) = \frac{\sin Ry}{\pi y}. \tag{12.2.8}$$

Problema 12.2.1. *$I_R(f)$ tende a f in $L^p(\mathbb{R})$ per ogni $f \in L^p(\mathbb{R})$, se e sole se esiste una costante C_p tale che*

$$\|I_R(f)\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}. \tag{12.2.9}$$

Problema 12.2.2. *Verificare* (12.2.9) *per* $1 < p < \infty$ *e generalizzare* (12.2.10) *per* $\mathbb{R}^n, n \geq 2$.

Soggerimento. Usare la disequazione di Young con spazi $L^{1,\infty}$

Possiamo studiare la convergenza di Cesaro (che era il teorema di Fejer per le serie di Fourier). Introducendo l'operatore

$$J_R(f)(x) = \frac{1}{R}\int_0^R I_t(f)(x)dt, \tag{12.2.10}$$

abbiamo

$$J_R(f)(x) = F_R * f(x), \;\; F_R(y) = \frac{1}{R}\int_0^R I_t(f)(y)dt = \frac{2\sin^2(Ry/2)}{\pi R y^2}$$

Vale la seguente proprieta'

Problema 12.2.3.

$$\|F_R\|_{L^1(\mathbb{R})} = 1.$$

Sommabilita' in senso di Abel - Poisson significa introdurre un fattore $\exp(-t|\xi|)$ dove $t > 0$. Poniamo

$$u_f(t,x) = \int_{\mathbb{R}} e^{ix\xi} e^{-t|\xi|} \widehat{f}(\xi) d\xi. \tag{12.2.11}$$

Problema 12.2.4. *(nucleo di Poisson)*

$$u_f(t,x) = P_t * f(x),$$

dove

$$P_t(x) = c\frac{t}{t^2+x^2}.$$

Vedere se

$$\lim_{t\to 0+} u_f(t,x) = f$$

in $L^p(\mathbb{R})$.

Problema 12.2.5. *Generalizzare il Problema 12.2.4 per* $\mathbb{R}^n$, $n \geq 2$..

Sommabilita' in senso di Gauss - Weierstrass significa introdurre un fattore $\exp(-t|\xi|^2)$ dove $t > 0$. Poniamo

$$w_f(t,x) = \int_{\mathbb{R}} e^{ix\xi} e^{-t|\xi|^2} \widehat{f}(\xi) d\xi. \tag{12.2.12}$$

Problema 12.2.6. *(nucleo di Gauss - Weierstrass)*

$$w_f(t,x) = W_t * f(x),$$

dove

$$W_t(x) = c\frac{e^{-|\xi|^2/4t}}{t}.$$

Vedere se

$$\lim_{t\to 0+} w_f(t,x) = f$$

in $L^p(\mathbb{R})$.

Problema 12.2.7. *Generalizzare il Problema 12.2.6 per* $\mathbb{R}^n$, $n \geq 2$..

Proposition 12.2.1. *Siano* f *e* ψ *appartenenti a* $\mathscr{S}$. *Allora*

$$\int f(x+z)\hat{\psi}(z)\,dz = \int \hat{f}(\xi)\psi(\xi)e^{ix\xi}d\xi \qquad \big(\forall x \in \mathbb{R}\big). \tag{12.2.13}$$

Dimostrazione. Siccome f e ψ appartengono a $\mathscr{S}(\mathbb{R})$, in particolare esse appartengono allo spazio di Lebesgue $L^1(\mathbb{R})$. L'applicazione

$$\mathbb{R}\times\mathbb{R} \ni (x,\xi) \longrightarrow f(y)\psi(\xi)e^{ix\xi} \in \mathbb{C}$$

appartiene a $L^1(\mathbb{R})$. Per il teorema di Fubini,

$$\begin{aligned} \int \hat{f}(\xi)\psi(\xi)e^{ix\xi}d\xi &= \iint f(y)\psi(\xi)e^{i(x-y)\xi}\,dyd\xi \\ &= \int dy\,f(y)\int d\xi\,\psi(\xi)e^{-i(y-x)\xi} \\ &= \int dy\,f(y)\hat{\psi}(y-x). \end{aligned}$$

Facendo il cambiamento di variabile $z = y - x$ si ha la formula (12.2.13). □

Teorema 12.2.3 (di Plancharel)**.** *Per ogni coppia di funzioni f e ψ appartenenti allo spazio di Schwarz $\mathscr{S}$, vale la formula*

$$\int \hat{f}(\xi)\psi(\xi)\,d\xi = \int f(x)\hat{\psi}(x)\,dx \tag{12.2.14}$$

In particolare, se f e g appartengono allo spazio di Schwarz $\mathscr{S}$,

$$\int f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \tag{12.2.15}$$

ossia la trasformazione di Fourier $\mathscr{F} : \mathscr{S} \longrightarrow \mathscr{S}$ è un'isometria (a meno di un fattore moltiplicativo costante 2π) di $\mathscr{S}$ in sé come sottospazio di $L^2(\mathbb{R})$.

Dimostrazione. Consideriamo la formula (12.2.13): al limite per $x \to 0$,

$$f(x+z)\hat{\psi}(z) \xrightarrow{q.o.} f(z)\hat{\psi}(z)$$

e inoltre la convergenza è dominata:

$$\left|f(x+z)\hat{\psi}(z)\right| \le \|f\|_\infty \left|\hat{\psi}(z)\right| \in L^1(\mathbb{R})$$

e analogamente $\hat{f}(\xi)\psi(\xi)e^{ix\xi} \xrightarrow{q.o.} \hat{f}(\xi)\psi(\xi)$ in maniera dominata:

$$\left|\hat{f}(\xi)\psi(\xi)e^{ix\xi}\right| \le \|\psi\|_\infty \left|\hat{f}(\xi)\right| \in L^1(\mathbb{R})$$

e quindi la formula (12.2.14) segue dalla formula (12.2.13) passando al limite per $x \to 0$.

Per la seconda equazione si consideri la formula (12.2.14): si prenda $\psi(\xi) = \overline{\frac{1}{2\pi}\hat{g}(\xi)}$. Per la formula di inversione, si ha:

$$\hat{\psi}(x) = \frac{1}{2\pi}\int \overline{\hat{g}(\xi)}e^{-ix\xi}\,d\xi = \overline{\frac{1}{2\pi}\int \hat{g}(\xi)e^{ix\xi}\,d\xi} = \overline{g(x)}$$

e, sostituita nella (12.2.14) si ottiene proprio la formula (12.2.15). □

12.3 Regolarizzazione mediante convoluzione

Consideriamo la funzione $\varphi_0 : \mathbb{R} \longrightarrow \mathbb{R}$ cosi definita

$$\varphi_0(x) = \begin{cases} e^{-\frac{1}{x}} & \text{se } x > 0, \\ 0 & \text{se } x \le 0. \end{cases}$$

Tale funzione è di classe $C^\infty(\mathbb{R})$.

A partire da φ_0 costruiamo la seguente funzione $\psi_0 : \mathbb{R} \longrightarrow \mathbb{R}$

$$\psi_0(x) = \varphi_0(x) \cdot \varphi_0(1-x).$$

Essa ha supporto compatto $[0;1]$ ed è di classe $C^\infty(\mathbb{R})$.

Posto

$$c = \int_{\mathbb{R}} \psi_0(x)\,dx.$$

consideriamo, infine, la funzione $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ ottenuta da ψ_0 nel seguente modo

$$\psi(x) = \frac{1}{2c}\psi_0\left(\frac{1}{2}x + \frac{1}{2}\right).$$

Essa gode delle proprietà seguenti:

i) $\psi \in C^\infty(\mathbb{R})$;

ii) $\psi(x) > 0$ per ogni $x \in \mathbb{R} \,:\, |x| < 1$ e $\psi(x) = 0$ per ogni $x \in \mathbb{R} \,:\, |x| \geq 1$; pertanto, ψ è una funzione non negativa, a supporto compatto e risulta

$$\operatorname{supp}\psi = [-1;1]$$

iii)

(12.3.16) $$\int_{\mathbb{R}} \psi(x)\,dx = \int_{-1}^{1} \psi(x)\,dx = 1.$$

Servendoci della funzione ψ, possiamo costruire la seguente successione di funzioni $\{\psi_n\}$

(12.3.17) $$\psi_n(x) = n\psi(nx) \qquad \forall\ n \in \mathbb{N} \quad \text{e} \quad \forall\ x \in \mathbb{R}$$

Ogni funzione ψ_n è non negativa, di classe $C^\infty(\mathbb{R})$, ha supporto compatto $\operatorname{supp}\psi_n = \left[-\frac{1}{n}; \frac{1}{n}\right]$ ed, inoltre,

(12.3.18) $$\int_{\mathbb{R}} \psi_n(x)\,dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} n\psi(nx)\,dx = \int_{-1}^{1} \psi(\xi)\,d\xi = 1.$$

Poiché

$$\operatorname{supp}\psi_n = \left[-\tfrac{1}{n}; \tfrac{1}{n}\right] \subset (-\pi, \pi) \quad \forall\ n \in \mathbb{N},$$

ψ_n é identicamente nulla fuori dell'intervallo $(-\pi, \pi)$. Se consideriamo la restrizione di ciascuna ψ_n a tale intervallo e la prolunghiamo, per periodicita a tutto $\mathbb{R}$, otteniamo delle funzioni ancora di classe $C^\infty(\mathbb{R})$ e 2π-periodiche.

Sia, ora, $f \in L^2(0;2\pi)$ e consideriamo la successione di funzioni $\{f_n\}$, cosi definita

$$f_n = f * \psi_n \qquad \forall\ n \in \mathbb{N}. \tag{12.3.19}$$

Le funzioni f_n risultano 2π-periodiche e di classe $C^\infty(\mathbb{R})$; vale, inoltre, la seguente proposizione.

Proposition 12.3.1. *Per ogni $f \in L^2(0;2\pi)$, la successione di funzioni $\{f_n\}$ converge ad f in norma $L^2(0,2\pi)$, cioe*

$$\lim_{n\to+\infty} \|f_n - f\|_2 = \lim_{n\to+\infty} \|f * \psi_n - f\|_2 = 0\,. \tag{12.3.20}$$

DIMOSTRAZIONE. Tenendo conto della (12.3.16) e della (12.3.18), per ogni $f \in L^2(0;2\pi)$

$$\begin{aligned} 0 \le \|f_n - f\|_2 &= \|f*\psi_n - f\|_2 = \left\| \int_{-\pi}^{\pi} f(t-s)\psi_n(s)\,ds - f(t)\right\|_2 \\ &= \left\| \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t-s)\psi(ns)n\,ds - f(t)\int_{-1}^{1}\psi(\sigma)\,d\sigma\right\|_2 \qquad \forall\ n \in \mathbb{N} \end{aligned}$$

Posto $ns = \sigma$, si ha

$$\begin{aligned} 0 \le \|f_n - f\|_2 &= \left\| \int_{-1}^{1} f\left(t - \frac{\sigma}{n}\right)\psi(\sigma)\,d\sigma - \int_{-1}^{1} f(t)\,\psi(\sigma)\,d\sigma \right\|_2 \\ &= \left\| \int_{-1}^{1} \left[f\left(t-\frac{\sigma}{n}\right) - f(t)\right]\psi(\sigma)\,d\sigma\right\|_2 \\ &\le \int_{-1}^{1} \left\| f\left(t-\frac{\sigma}{n}\right) - f(t)\right\|_2 |\psi(\sigma)|\,d\sigma \qquad \forall\ n \in \mathbb{N} \end{aligned}$$

dove nell'ultimo disuguaglianza abbiamo tenuto conto della disuguaglianza di Minkowski generalizzata.

Poiche per ipotesi $f \in L^2(0;2\pi)$, il teorema di Lebesgue ci assicura che

$$\lim_{n\to+\infty} \left\| f\left(t-\frac{\sigma}{n}\right) - f(t)\right\|_2 = 0\,.$$

Osservato che

$$\int_{-1}^{1} \left\| f\left(t-\frac{\sigma}{n}\right) - f(t)\right\|_2 |\psi(\sigma)|\,d\sigma \le 4\,\|f\|_2 \max_{\sigma\in[-1,1]} |\psi(\sigma)| \qquad \forall\ n \in \mathbb{N}$$

per il teorema della convergenza dominata di Lebesgue risulta

$$\lim_{n\to+\infty}\int_{-1}^{1}\left\|f\left(t-\frac{\sigma}{n}\right)-f(t)\right\|_2 |\psi(\sigma)|\,d\sigma = 0$$

Dal teorema del confronto segue, infine, la (12.3.20). □

Dalla proposizione precedente si ricava, come corollario, il seguente teorema.

Teorema 12.3.1. *Lo spazio delle funzioni 2π-periodiche e di classe $C^\infty(\mathbb{R})$ è denso in $L^2(0,2\pi)$.*

Osservazione 12.3.1. *Piu in generale, risulta che $C_c^\infty(\Omega)$ è denso in $L^p(\Omega)$, per ogni $p\in[1,+\infty)$ e per ogni aperto $\Omega\subseteq\mathbb{R}$.*

12.4 Approsimazione dell'unitá

Se $\phi(x)$ é una funzione in $\mathbb{R}^n$ integrabile e tale che

$$\int_{\mathbb{R}^n}\phi(x)\,dx = 1,$$

allora consideriamo la funzione

$$\phi_t(x) = t^{-n}\phi(tx),\ \ t>0.$$

Problema 12.4.1. *Per ogni $p, 1\le p\le\infty$,*

$$\|\phi_t * f - f\|_{L^p} = 0,$$

quando $t>0$ tende a 0.

12.4.1 Approximation of Josida

Approximation of Josida is defied for any $f\in L^p(\mathbb{R}^n)$ as follows

(12.4.21) $$f_\varepsilon = (I-\varepsilon\Delta)^{-1}f.$$

We shall prove later on the following

Lemma 12.4.1. *For any $p\in[1,\infty]$ we have*

(12.4.22) $$\|f_\varepsilon\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

(12.4.23) $$\lim_{\varepsilon\searrow 0}\|f_\varepsilon - f\|_{L^p(\mathbb{R}^n)} = 0.$$

The proof of (12.4.22) is based on the representation formula

$$(I-\varepsilon\Delta)^{-1}f(x)=\int_{\mathbb{R}^n}K_\varepsilon(x-y)f(y)dy$$

with

$$\|K_\varepsilon(x)\|_{L^1}\lesssim 1.$$

12.5 Esempi di calcolo per trasformata di Fourier

Integrale di Fresnel si puo scrivere come

$$\int_0^\infty \sin t^2 dt \tag{12.5.24}$$

Problema 12.5.1. *Verificare le indentitá*

$$\int_0^\infty \sin t^2 dt=\int_0^\infty \cos t^2 dt=\sqrt{\frac{\pi}{8}}.$$

Esempio 12.5.1. *Calcoliamo la trasformata di Fourier di*

$$f(x)=\frac{1}{(x-i)^2}.$$

Dobbiamo calcolare

$$\int_{-\infty}^\infty \frac{e^{-i\xi x}}{(x-i)^2}dx.$$

Introduciamo la funzione complessa

$$g(z)=\frac{e^{-i\xi z}}{(z-i)^2}.$$

Quando $\xi>0$, allora

$$\int_{-\infty}^\infty \frac{e^{-i\xi x}}{(x-i)^2}dx=\lim_{M\to\infty}\int_{\Gamma_M}\frac{e^{-i\xi z}}{(z-i)^2}dz$$

dove Γ_M è definita come in figura:

ossia $\Gamma_M=[-M;M]\cup\gamma_M$ e $\gamma_M=\left\{Me^{i\theta}:\theta\in[\pi;2\pi]\right\}$. Con procedimento analogo al lemma 12.2.2 si ha:

$$\lim_{M\to\infty}\int_{\gamma_M}g(z)dz=0.$$

Dunque

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi z}}{(z-i)^2}\, dz = \lim_{M\to\infty} \int_{\Gamma_M} \frac{e^{-i\xi z}}{(z-i)^2}\, dz = 0 \qquad (\xi > 0)$$

per il teorema di Cauchy.

Quando $\xi < 0$, *analogamente, consideriamo* $\Gamma'_M = [-M; M] \cup \gamma'_M$ *con* $\gamma'_M = \{Me^{i\theta} : \theta \in [0;\pi]\}$.

Allora

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{(x-i)^2}\, dx = \lim_{M\to\infty} \int_{\Gamma'_M} \frac{e^{-i\xi z}}{(z-i)^2}\, dz = 2\pi i Res_i \frac{e^{-i\xi z}}{(z-i)^2}.$$

Basta calcolare il residuo di g *in* i*:*

$$\frac{e^{-i\xi z}}{(z-i)^2} = e^{\xi} \frac{e^{-i\xi(z-i)}}{(z-i)^2} = e^{\xi} \sum_{n\geq 0} \frac{(-i\xi)^n (z-i)^{n-2}}{n!},$$

per cui:

$$Res_i \frac{e^{-i\xi z}}{(x-i)^2} = -e^{\xi} i\xi.$$

Ne segue che

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{(x-i)^2}\, dx = 2\pi\xi e^{\xi} \qquad (\xi < 0).$$

12.6 Fractional powers of operators and some integral representation

We start with the identity

Lemma 12.6.1.

(12.6.25)
$$\int_0^\infty \frac{t^{s-1}dt}{1+t} = \frac{\pi}{\sin \pi s}$$

for any $s \in (0,1)$.

Proof. Using the integral representation of the Gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt,$$

we get the representation

(12.6.26)
$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-At} dt, \forall s \in (0,1), A > 0.$$

Using the relation
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, s \in (0,1),$$
we find
$$\int_0^\infty \frac{t^{s-1}dt}{1+t} = \int_0^\infty t^{s-1} \int_0^\infty e^{-\lambda(1+t)} d\lambda dt =$$
$$\int_0^\infty \int_0^\infty \left(t^{s-1} e^{-\lambda t}\right) dt e^{-\lambda} d\lambda = \int_0^\infty \int_0^\infty \left(\tilde{t}^{s-1} e^{-\tilde{t}}\right) d\tilde{t} \lambda^{-s} e^{-\lambda} d\lambda =$$
$$= \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

□

Make change of variables $t = \tau^2$ and get

$$I \equiv \int_0^\infty \frac{t^{s-1}dt}{1+t} = 2\int_0^\infty \frac{\tau^{2s-1}d\tau}{1+\tau^2}.$$

12.7 Spazio delle Distribuzioni Temperate

Consideriamo lo spazio di Schwarz $\mathscr{S}$. Si ha:

$$g \in \mathscr{S} \qquad \Longleftrightarrow \qquad \|g\|_{k,j} = \sup_{x \in \mathbb{R}} \left| x^k D^j g(x) \right| \le c_{k,j} < \infty.$$

Vogliamo trovare una topologia su $\mathscr{S}$. Si consideri la famiglia di seminorme

$$\mathscr{Y} = \left\{ \|\cdot\|_N : N \in \mathbb{N} \right\}$$

definite come

$$\|g\|_N = \sum_{k+j \le N} \sup_{x \in \mathbb{R}} \left| x^k D^j g(x) \right|.$$

La famiglia $\mathscr{Y}$ induce su $\mathscr{S}$ una topologia dove un sistema fomdamentale di intorni di zero è, per ogni intero positivo N:

$$V_N = \left\{ g \in \mathscr{S} : \|g\|_N < \frac{1}{N} \right\}. \tag{12.7.27}$$

Inoltre questa topologia è metrizzabile e la metrica è

$$d(f;g) = \sum_{N=0}^{\infty} \frac{\|f-g\|_N}{1+\|f-g\|_N} 2^{-N}. \tag{12.7.28}$$

In questo modo resta definita la convergenza in $\mathscr{S}$ come:

$$g_n \xrightarrow{\mathscr{S}} 0 \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} d(g_n;0) = 0 \tag{12.7.29}$$

equivalentemente,

$$g_n \xrightarrow{\mathscr{S}} 0 \qquad \Longleftrightarrow \qquad \left(\forall N \in \mathbb{N}\right) \lim_{n\to\infty} \|g_n\|_N = 0. \tag{12.7.30}$$

Con la convergenza appena definita, *lo spazio $\mathscr{S}$ è completo.* Infatti, se (g_n) è una successione di Cauchy in $\mathscr{S}$, allora essa è di Cauchy in $C^\infty(\mathbb{R})$ e quindi esiste una funzione $g \in C^\infty(\mathbb{R})$ tale che, per ogni intero positivo k,

$$g_n^{(k)} \xrightarrow{\text{unif.}} g^{(k)}.$$

Basta dimostrare che g è una funzione rapidamente decrescente:

$$\|g\|_{k,j} \le \|g - g_n\|_{k,j} + \|g_n\|_{k,j}.$$

Siccome $\|g_n - g_m\|_{k,j} < \epsilon$, ne segue che, per ogni $x \in \mathbb{R}$,

$$x^k D^j \left(g_n(x) - g_m(x)\right) < \epsilon$$

e, passando al limite per $n \to \infty$, $D^j g_n \xrightarrow{\text{unif.}} D^j g$, ossia:

$$x^k D^j \big(g_n(x) - g(x)\big) \le \epsilon \qquad \big(\forall x \in \mathbb{R}\big)$$

che significa: $\|g - g_n\|_{k,j} \le \epsilon$ ossia

$$\|g\|_{k,j} \le \|g - g_n\|_{k,j} + \|g_n\|_{k,j} < \epsilon + C_{k,j} < \infty$$

che prova che $g \in \mathscr{S}$.

Lo spazio duale (dei funzionali lineari e continui) di $\mathscr{S}$ si denota con $\mathscr{S}' = \mathscr{S}'(\mathbb{R})$ e si chiama **spazio delle distribuzioni temperate**. Data una distribuzione temperata $v \in \mathscr{S}'$, scriveremo talvolta $\lambda v, \varphi\rangle$ in luogo di $v(\varphi)$.

Se $v \in \mathscr{S}$ è una funzione rapidamente decrescente, per ogni funzione test $\varphi \in \mathscr{S}$,

$$v(\varphi) = \int v(x)\varphi(x)\, dx$$

è un funzionale lineare, quindi $\mathscr{S} \subset \mathscr{S}'$.

La topologia di $\mathscr{S}'$ è indotta dalla topologia di $\mathscr{S}$. In particolare, se (v_n) è una successione di distribuzioni temperate,

$$v_n \xrightarrow{\mathscr{S}'} 0 \qquad \Longleftrightarrow \qquad \big(\forall \varphi \in \mathscr{S}\big) \qquad \lim_{n\to\infty} v_n(\varphi) = 0. \tag{12.7.31}$$

Esempio 12.7.1. *Costanti e polinomi appartengono allo spazio delle distribuzioni temperate.*

Esempio 12.7.2. *Il funzionale lineare e continuo*

$$\delta_0 : \mathscr{S} \ni \varphi \longrightarrow \delta_0(\varphi) = \varphi(0) \in \mathbb{R}$$

è una distribuzione temperata e si chiama ***delta di Dirac****.*

Esempio 12.7.3. *Per ogni $\epsilon > 0$ consideriamo la funzione $\frac{1}{x+i\epsilon}$. Per ogni funzione test $\varphi \in \mathscr{S}$, si ha:*

$$\left\langle \frac{1}{x+i\epsilon}; \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = \underbrace{\int_{|x|\ge\epsilon}}_{I_1} + \underbrace{\int_{|x|<\epsilon}}_{I_2}$$

Dapprima consideriamo l'integrale I_1; esso si suddivide ancora come:

$$\int_{|x|\ge\epsilon} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = \int_{\epsilon}^{\infty} + \int_{-\infty}^{-\epsilon}$$

e il secondo di questi integrali si può scrivere come:

$$\int_{-\infty}^{-\epsilon} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = \int_{\epsilon}^{\infty} \frac{1}{-x+i\epsilon} \cdot \varphi(-x)\, dx$$

e quindi

$$\begin{aligned} \int_{|x|\geq\epsilon} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx &= \int_{\epsilon}^{\infty} \left[\frac{1}{x+i\epsilon}\varphi(x) + \frac{1}{-x+i\epsilon}\varphi(-x) \right] dx \\ &= \int_{\epsilon}^{\infty} \frac{1}{x+i\epsilon} [\varphi(x) - \varphi(-x)]\, dx + \\ &\quad + \int_{\epsilon}^{\infty} \varphi(-x) \left[\frac{1}{x+i\epsilon} + \frac{1}{-x+i\epsilon} \right] dx. \end{aligned}$$

Essendo $\varphi(x) - \varphi(-x) = O(x)$, il primo integrale converge, per $\epsilon \to 0$, verso

$$\int_0^{\infty} \frac{1}{x} [\varphi(x) - \varphi(-x)]\, dx;$$

il secondo integrale,

$$-\int_{\epsilon}^{\infty} \varphi(-x) \frac{2i\epsilon}{x^2+\epsilon^2}\, dx = -2i \int_1^{\infty} \frac{\varphi(-\epsilon y)}{1+y^2}\, dy,$$

converge, al tendere di ϵ a zero, verso

$$-\frac{2i\pi}{4}\varphi(0) = -\frac{i\pi}{2}\delta_0(\varphi).$$

Quanto all'integrale I_2,

$$\int_{|x|<\epsilon} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = \varphi(\eta) \int_{-\epsilon}^{\epsilon} \frac{dx}{x+i\epsilon} = \varphi(\eta) \int_{-1}^{1} \frac{dy}{y+i},$$

dove $|\eta| < \epsilon$. Moltiplicando e dividendo per $y - i$, si ha:

$$\int_{-1}^{1} \frac{dy}{y+1} = \int_{-1}^{1} \frac{y-i}{y^2+1} dy = \int_{-1}^{1} \frac{y}{y^2+1} dy - i \int_{-1}^{1} \frac{1}{y^2+1} dy = -\frac{i\pi}{2}.$$

Quindi,

$$\int_{|x|<\epsilon} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = -\frac{i\pi}{2}\varphi(\eta) \qquad \text{con } |\eta| < \epsilon.$$

Pertanto, al tendere di ϵ a zero,

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{x+i\epsilon} \cdot \varphi(x)\, dx = \int_0^{\infty} \frac{1}{x} [\varphi(x) - \varphi(-x)]\, dx - i\pi\delta_0(\varphi).$$

Definiamo, allora ***valore principale di*** $\frac{1}{x}$ *la distribuzione temperata* $\mathrm{Pv}\left(\frac{1}{x}\right) \in \mathscr{S}'$ *tale che:*

$$\left\langle \mathrm{Pv}\left(\frac{1}{x}\right); \varphi \right\rangle = \int_0^\infty \frac{1}{x}[\varphi(x) - \varphi(-x)]\, dx. \tag{12.7.32}$$

In tal modo, definita

$$\frac{1}{x+i0} = \lim_{\epsilon \to 0} \frac{1}{x+i\epsilon},$$

si ha che $\frac{1}{x+i0}$ *è una distribuzione temperata, che coincide con*

$$\frac{1}{x+i0} = \mathrm{Pv}\left(\frac{1}{x}\right) - i\pi\delta_0.$$

In modo del tutto analogo possiamo trovare la distribuzione temperata

$$\frac{1}{x-i0} = \overline{\frac{1}{x+i0}} = \lim_{\epsilon \to 0} \frac{1}{x-i\epsilon} = \mathrm{Pv}\left(\frac{1}{x}\right) + i\pi\delta_0. \tag{12.7.33}$$

Le formule

$$\frac{1}{x \pm i0} = \mathrm{Pv}\left(\frac{1}{x}\right) \mp i\pi\delta_0 \tag{12.7.34}$$

sono le formule di Sokhotski – Plemelj.

Esempio 12.7.4. *La funzione*

$$\theta(x) = \begin{cases} 1 & \textit{se } x \geq 0 \\ 0 & \textit{se } x < 0 \end{cases}$$

è una distribuzione temperata che prende il nome di ***distribuzione di Heaviside****. Essa agisce su ogni* $\varphi \in \mathscr{S}$ *come:*

$$\theta(\varphi) = \int \theta(x)\varphi(x)\, dx = \int_0^\infty \varphi(x)\, dx.$$

Esempio 12.7.5. *Per ogni intervallo* $[a;b] \subseteq \mathbb{R}$*, l'indicatrice* $I_{[a;b]}$ *è una distribuzione temperata e*

$$I_{[a;b]}(\varphi) = \int \varphi(x) I_{[a;b]}(x)\, dx = \int_a^b \varphi(x)\, dx.$$

12.8 Trasformata di Fourier di Distribuzioni Temperate

Data una distribuzione temperata $v \in \mathscr{S}'$, si definisce la **trasformazione di Fourier** $\mathscr{F} : \mathscr{S}' \ni v \longrightarrow \hat{v} \in \mathscr{S}'$, di v come:

(12.8.35) $$\langle \hat{v};\varphi \rangle = \langle v;\hat{\varphi} \rangle \qquad (\forall \varphi \in \mathscr{S}).$$

Tale definizione è ben posta, dato che la trasformazione di Fourier $\mathscr{F} : \mathscr{S} \longrightarrow \mathscr{S}$ è un automorfismo di $\mathscr{S}$.

Esempio 12.8.1. *Supponiamo che v sia una funzione rapidamente decrescente. Allora*

$$\begin{aligned}
\langle \hat{v};\varphi \rangle &= \langle v;\hat{\varphi} \rangle = \int v(x)\hat{\varphi}(x)\,dx \\
&= \int dx\, v(x) \int dy\, \varphi(y) e^{-ixy} \\
&= \int dy\, \varphi(y) \int dx\, v(x) e^{-ixy} \\
&= \int \hat{v}(x)\varphi(x)\,dx
\end{aligned}$$

che significa che la trasformata di Fourier distribuzionale coincide con la trasformata di Fourier (in senso classico) per le funzioni rapidamente decrescenti.

Esempio 12.8.2. *Comunque presa una funzione test $\varphi \in \mathscr{S}$,*

$$\hat{\delta}_0(\varphi) = \delta_0(\hat{\varphi}) = \hat{\varphi}(0) = \int \varphi(x)\,dx = \lambda 1;\varphi\rangle$$

che significa:

$$\boxed{\hat{\delta}_0 = 1}$$

Poiché, per le funzioni rapidamente decrescenti, vale la formula di inversione:

$$\varphi^* = \frac{1}{2\pi}\hat{\hat{\varphi}},$$

dove $\varphi^*(x) = \varphi(-x)$; comunque data una distribuzione temperata $v \in \mathscr{S}'$,

$$\langle \hat{\hat{v}};\varphi \rangle = \langle \hat{v};\hat{\varphi} \rangle = \langle v;\hat{\hat{\varphi}} \rangle = \langle v; 2\pi\varphi^* \rangle = 2\pi \langle v^*;\varphi \rangle$$

ossia

(12.8.36) $$\boxed{v^* = \frac{1}{2\pi}\hat{\hat{v}}}$$

dove $\langle v^*;\varphi \rangle = \langle v;\varphi^* \rangle$. Ciò significa che **la trasformazione di Fourier è un automorfismo di $\mathscr{S}'$**.

12.8.1 Fourier transform of Heaviside function

Let $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.

We have the relations

$$\widehat{\theta}(\xi) = \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-ix\xi - \varepsilon x} dx$$

Using the relations

$$\int_0^\infty e^{-ix\xi - \varepsilon x} = -i \frac{1}{\xi - i\varepsilon}$$

and therefore

(12.8.37) $$\widehat{\theta}(\xi) = \lim_{\varepsilon \searrow 0} = -i \frac{1}{\xi - i0}.$$

Using (12.7.34) we find

(12.8.38) $$\widehat{\theta}(\xi) = -i \frac{1}{\xi - \mathrm{i}0} = \pi\delta(\xi) - \mathrm{iPv}\left(\frac{1}{\xi}\right).$$

We can use the relation

$$\widehat{f_-}(\xi) = \widehat{f}(-\xi),$$

where

$$f_-(x) = f(-x).$$

Indeed,

$$\int_R f(-x) e^{-\mathrm{i}x\xi} dx = \int_R f(x) e^{\mathrm{i}x\xi} dx.$$

In this way we get

(12.8.39) $$\widehat{\theta_-}(\xi) = -i \frac{1}{-\xi - \mathrm{i}0} = \pi\delta(-\xi) + \mathrm{iPv}\left(\frac{1}{\xi}\right).$$

12.8.2 Fourier transform of sgn - function

Now we can find the Fourier transform of the function

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 1; \\ -1, & x < 0. \end{cases}$$

Obviously, we have the relation

$$sgn(x) = \theta(x) - \theta(-x)$$

and hence from (12.8.38) and (12.8.39) we find

(12.8.40) $$\widehat{\mathrm{sgn}}(\xi) = -2\mathrm{i}\mathrm{Pv}\left(\frac{1}{\xi}\right).$$

If we define the translated function

$$sgn_k(x) = \mathrm{sgn}(x-k)$$

then we can use

$$\int_{\mathrm{R}} f(x-k)e^{-\mathrm{i}x\xi}\,dx = \int_{\mathrm{R}} f(x-k)e^{-\mathrm{i}x\xi-\mathrm{i}k\xi}\,dx = e^{-\mathrm{i}k\xi}\widehat{f}(\xi)$$

and find

(12.8.41) $$\widehat{\mathrm{sgn_k}}(\xi) = -2\mathrm{i}e^{-\mathrm{i}k\xi}\mathrm{Pv}\left(\frac{1}{\xi}\right).$$

12.8.3 Fourier transform of $Pv(1/x)$

Using the formula for the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{\mathrm{R}} f(\xi)e^{ix\xi}\,d\xi$$

and (12.8.41), we find

$$\left(\mathrm{Pv}\left(\frac{1}{x}\right), e^{-ix\xi}\right) = \pi\mathrm{i}\,\mathrm{sgn}(-\xi) = -(\pi\mathrm{i})\mathrm{sgn}(\xi).$$

Hence we can write (in sense of distributions)

(12.8.42) $$\widehat{(\mathrm{Pv}(1/\cdot))}(\xi) = -\pi\mathrm{i}\ \mathrm{sgn}(\xi).$$

12.9 Prodotto di una distribuzione per una funzione

Sia f una funzione di classe C^∞ a supporto compatto e sia v una distribuzione temperata. Definiamo la distribuzione temperata fv come segue:

(12.9.43) $$\langle fv;\varphi\rangle = \langle v; f\varphi\rangle \qquad \big(\forall\varphi\in\mathscr{S}\big).$$

In particolare, quando v sia una funzione,

$$\langle fv;\varphi\rangle = \langle v; f\varphi\rangle = \int v(x)f(x)\varphi(x)\,dx$$

e quindi fv coincide con il prodotto tra due funzioni in senso classico.

12.10 Derivata distribuzionale

Data una distribuzione temperata $v \in \mathscr{S}'$, la **derivata** (**distribuzionale**) di v è definita come:

$$\langle \partial v, \varphi \rangle = -\langle v, \partial \varphi \rangle \qquad \big(\forall \varphi \in \mathscr{S}\big).$$

In generale, per ogni intero positivo n,

$$\langle \partial^n v, \varphi \rangle = (-1)^n \langle v, \partial^n \varphi \rangle \qquad \big(\forall \varphi \in \mathscr{S}\big). \tag{12.10.44}$$

Esempio 12.10.1. *Se v è una funzione (p.es. polinomio o funzione rapidamente decrescente),*

$$\langle v', \varphi \rangle = -\langle v, \varphi' \rangle = -\int v(x)\varphi'(x)\,dx = \int v'(x)\varphi(x)\,dx$$

che significa che, per le funzioni, la derivata distribuzionale coincide con la derivata in senso classico.

Esempio 12.10.2. *Sia H la distribuzione di Heaviside,*

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = \int_0^{\infty} \varphi'(x)\,dx = \varphi(0) = \delta_0(\varphi),$$

pertanto:

$$\boxed{H' = \delta_0}$$

Esempio 12.10.3. *Per ogni intero positivo k,*

$$\langle \partial^k \delta_0, \varphi \rangle = (-1)^k \delta_0\big(\partial^k \varphi\big) = (-1)^k \partial^k \varphi(0).$$

Esempio 12.10.4. *Per ogni intervallo $[a;b]$,*

$$\langle \partial I_{[a;b]}, \varphi \rangle = \delta_a(\varphi) - \delta_b(\varphi).$$

La derivata distribuzionale soddisfa delle proprietà analoghe a quelle della derivata classica: anzitutto è ovvio che

$$\boxed{v \in \mathscr{S}' \quad \Longrightarrow \quad \partial^n v \in \mathscr{S}'.}$$

Vale la **formula di Leibniz**:

$$\boxed{\partial(f v) = (\partial f) v + f \partial v.} \tag{12.10.45}$$

In effetti, per ogni funzione test $\varphi \in \mathscr{S}$,

$$\begin{aligned}
\langle \partial(f\,v), \varphi \rangle &= -\langle f\,v, \varphi' \rangle = -\lambda\,v, f\varphi' \rangle \\
&= -\lambda\,v, \partial(f\varphi) - (\partial f)\varphi \rangle \\
&= -\lambda\,v, \partial(f\varphi) \rangle + \lambda\,v, (\partial f)\varphi \rangle \\
&= \lambda \partial v, f\varphi \rangle + \lambda(\partial f)\,v, \varphi \rangle \\
&= \lambda f \partial v, \varphi \rangle + \lambda(\partial f)\,v, \varphi \rangle = \lambda f \partial v + (\partial f)\,v, \varphi \rangle.
\end{aligned}$$

Infine,

(12.10.46) $$\boxed{v \in \mathscr{S}', \quad \partial v = 0 \qquad \Longrightarrow \qquad v \equiv c \text{ è una costante.}}$$

Infatti, sia $\varphi \in \mathscr{S}$ una funzione test. Supponiamo che $\int \varphi(x)\,dx = 0$. Allora, detta

$$\psi(x) = \int_{-\infty}^{x} \varphi(t)\,dt,$$

è evidente che $\psi \in \mathscr{S}$ e $\psi' = \varphi$. Pertanto,

$$\lambda\,v, \varphi \rangle = \lambda\,v, \psi' \rangle = -\lambda\,v', \psi \rangle = 0.$$

Se, invece, $\int \varphi(x)\,dx = k \neq 0$, sia $\psi_0 \in \mathscr{S}$ una funzione positiva e tale che $\int \psi_0(x)\,dx = 1$. Allora, detta

$$\varphi_1(x) = \varphi(x) - k\psi_0(x),$$

si può applicare quanto provato prima a φ_1, ossia,

$$0 = \lambda\,v, \varphi_1 \rangle = \lambda\,v, \varphi \rangle - k\lambda\,v, \psi_0 \rangle,$$

detta $c = \lambda\,v, \psi_0 \rangle$, si ha:

$$\lambda\,v, \varphi \rangle = \int c\varphi(x)\,dx$$

che significa che v coincide con una costante c.

La trasformata di Fourier in senso delle distribuzioni è una vera estensione della trasformata di Fourier in senso classico, valgono, cioè, le proprietà:

(12.10.47) $$\widehat{\partial^n v} = i^n \xi^n \widehat{v}.$$

Infatti, fissata una funzione test $\varphi \in \mathscr{S}$,

$$\begin{aligned}
\lambda \widehat{\partial^n v}, \varphi \rangle &= \lambda \partial^n v, \widehat{\varphi} \rangle \\
&= (-1)^n \lambda\,v, \partial^n \widehat{\varphi} \rangle \\
&= (-1)^n \lambda\,v, (-i)^n \widehat{\xi^n \varphi} \rangle \\
&= i^n \lambda \widehat{v}, \xi^n \varphi \rangle \\
&= \lambda i^n \xi^n \widehat{v}, \varphi \rangle.
\end{aligned}$$

Analogamente:

$$\widehat{x^n v} = i^n \partial^n \widehat{v}. \tag{12.10.48}$$

Infatti, fissata una funzione test $\varphi \in \mathscr{S}$,

$$\begin{aligned}
\lambda \widehat{x^n v}, \varphi\rangle &= \lambda x^n v, \widehat{\varphi}\rangle \\
&= \lambda v, x^n \widehat{\varphi}\rangle \\
&= \lambda v, (-i)^n \widehat{\partial^n \varphi}\rangle \\
&= (-i)^n \lambda \widehat{v}, \partial^n \varphi\rangle \\
&= i^n \lambda \partial^n \widehat{v}, \varphi\rangle = \lambda i^n \partial^n \widehat{v}, \varphi\rangle.
\end{aligned}$$

Esempio 12.10.5. *Calcoliamo la trasformata di Fourier della distribuzione di Heaviside H. Fissata una funzione test $\varphi \in \mathscr{S}$,*

$$\begin{aligned}
\langle \widehat{H}, \varphi\rangle = \langle H, \widehat{\varphi}\rangle &= \int_0^\infty \widehat{\varphi}(\xi)\, d\xi \\
&= \lim_{M\to\infty} \int_0^M d\xi \int dx\, \varphi(x) e^{-ix\xi} \\
&= \lim_{M\to\infty} \int dx\, \varphi(x) \int_0^M d\xi\, e^{-ix\xi}.
\end{aligned}$$

Consideriamo

$$\begin{aligned}
I_M &= \int dx\, \varphi(x) \int_0^M d\xi\, e^{-ix\xi} \\
&= \int \varphi(x) \Big(e^{-iMx} - 1\Big) \frac{1}{-ix} dx \\
&= \int i \Big(e^{-iMx} - 1\Big) \frac{\varphi(x)}{x} dx = \int_0^\infty + \int_{-\infty}^0 .
\end{aligned}$$

Il secondo integrale si trasforma come segue:

$$\int_{-\infty}^0 i\Big(e^{-iMx} - 1\Big) \frac{\varphi(x)}{x} dx = \int_\infty^0 i\Big(e^{iMy} - 1\Big) \frac{\varphi(-y)}{y} dy = -\int_0^\infty i\Big(e^{iMy} - 1\Big) \frac{\varphi(-y)}{y} dy$$

e quindi

$$\begin{aligned}
I_M &= \int_0^\infty i \left\{ \Big(e^{-iMx} - 1\Big) \frac{\varphi(x)}{x} - \Big(e^{iMx} - 1\Big) \frac{\varphi(-x)}{x} \right\} dx \\
&= i \int_0^\infty \frac{1}{x} \big(\varphi(-x) - \varphi(x)\big) dx + i \int_0^\infty \left\{ \frac{\varphi(x)}{x} e^{-iMx} - \frac{\varphi(-x)}{x} e^{iMx} \right\} dx.
\end{aligned}$$

Osserviamo che

$$\int_0^{\infty}\left\{\frac{\varphi(x)}{x}e^{-iMx}-\frac{\varphi(-x)}{x}e^{iMx}\right\}dx=$$
$$=\int_0^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\cos Mx\,dx-i\int_0^{\infty}\frac{\varphi(x)+\varphi(-x)}{x}senMx\,dx.$$

In primo integrale converge verso zero, al tendere di M all'infinito, per il teorema di Riemann-Lebesgue (lemma 12.2.1). Quanto al secondo integrale:

$$\int_0^{\infty}\frac{\varphi(x)+\varphi(-x)}{x}senMx\,dx=\int_0^{\infty}\left[\varphi\left(\frac{y}{M}\right)+\varphi\left(-\frac{y}{M}\right)\right]\frac{seny}{y}dy$$

al tendere di M all'infinito converge verso

$$2\varphi(0)\int_0^{\infty}\frac{seny}{y}dy=\pi\varphi(0).$$

Pertanto,

$$\langle\widehat{H},\varphi\rangle=\lim_{M\to\infty}I_M=-i\mathrm{Pv}\left(\frac{1}{x}\right)(\varphi)+\pi\delta_0(\varphi),$$

cioè,

$$\boxed{\widehat{H}=-i\mathrm{Pv}\left(\frac{1}{x}\right)+\pi\delta_0.}$$

Usually the Fourier transform of $f\in S'8\mathbb{R})$ will be denoted by

$$\mathfrak{F}(f)(\xi)=\widehat{f}(\xi).$$

Problema 12.10.1. *Show that*

$$\mathfrak{F}\left(\mathrm{Pv}\left(\frac{1}{x}\right)\right)(\xi)=i\pi\mathrm{sgn}(\xi).$$

Problema 12.10.2. *Show that*

$$\mathfrak{F}\left(x_+^{\lambda}\right)(\xi)=ie^{i\lambda\pi/2}\Gamma(\lambda+1)(\xi+i0)^{-\lambda-1}$$

$\lambda\neq-1,-2,\cdots$. *Here* $x_+=\max(x,0)$.

One can compute the Fourier transform of

$$\frac{1}{\xi+i\varepsilon},\xi\in\mathbb{R}$$

using Cauchy's residue theorem and compute

Lemma 12.10.1. *Let $\varepsilon > 0$. If $x > 0$, then*

(12.10.49)
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{\xi + i\varepsilon} d\xi = -e^{-\varepsilon x}.$$

If $x < 0$, then

(12.10.50)
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{\xi + i\varepsilon} d\xi = 0.$$

We have also the identity (in distributional sense)

(12.10.51)
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{\xi + i\varepsilon} d\xi = -e^{-\varepsilon x} H(x),$$

where $H(x)$ is the Heviside distribution.

Taking one derivative (in distributional sense), we get

Lemma 12.10.2.

(12.10.52)
$$\lim_{\varepsilon \searrow 0} \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ix\xi}\ \xi}{\xi + i\varepsilon} d\xi = \delta(x),$$

where $\delta(x)$ is the Delta distribution.

Another application of the residue theorem is the following.

Lemma 12.10.3. *Let $\varepsilon > 0$ and A, B are real numbers with $A \neq B$. If $x > 0$, then there exists $c \neq 0$ so that*

(12.10.53)
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(\xi + A + i\varepsilon)(\xi + B + i\varepsilon)} d\xi = c\frac{\sin(x(A-B))}{A-B}.$$

If $x < 0$, then

(12.10.54)
$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(\xi + A + i\varepsilon)(\xi + B + i\varepsilon)} d\xi = 0.$$

12.11 Free resolvent kernel via Fourier transform

Lemma 12.11.1. *If $\lambda > 0$, then the Fourier transform*

$$\mathscr{F}_0^*(f)(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{\mathrm{i}x\xi} \frac{d\xi}{\xi^2 + \lambda}$$

of

$$f(\xi)=\frac{1}{\lambda+\xi^2}$$

is

(12.11.55) $$(2\pi)^{-1}\int_{\mathbb{R}} e^{\mathrm{i}x\xi}\frac{d\xi}{\xi^2+\lambda}=\frac{1}{2\sqrt{\lambda}}e^{-\sqrt{\lambda}|x|}.$$

Proof. We have to compute

$$\int_{\mathbb{R}} e^{\mathrm{i}x\xi}\frac{d\xi}{\xi^2+\lambda}$$

Introduce the complex function

$$g(z)=\frac{e^{\mathrm{i}xz}}{\lambda+z^2}=\frac{e^{\mathrm{i}xz}}{(z+\mathrm{i}\sqrt{\lambda})(z-\mathrm{i}\sqrt{\lambda})}.$$

If $x>0$, then we have

$$\int_{\mathbb{R}} e^{\mathrm{i}x\xi}\frac{d\xi}{\xi^2+\lambda}=\lim_{M\to\infty}\int_{-M}^{M} e^{\mathrm{i}xz}\frac{dz}{z^2+\lambda}.$$

and we can introduce the closed contour

$$\Gamma_M=[-M;M]\cup\gamma_M$$

with

$$\gamma_M=\left\{Me^{i\theta}:\theta\in[0;\pi]\right\}.$$

For $\theta\in(\varepsilon,\pi-\varepsilon)$

$$\mathrm{Im}Me^{i\theta}=M\sin\theta,\implies\lim_{M\to\infty}\left|\frac{e^{\mathrm{i}xMe^{i\theta}}}{\lambda+M^2e^{2\mathrm{i}\theta}}\right|\le\lim_{M\to\infty}\frac{e^{-xM\sin\varepsilon}}{M^2-\lambda}=0.$$

Applying the Lebesgue convergence theorem we get

$$\lim_{M\to\infty}\int_{\gamma_M}g(z)\,dz=0.$$

Hence

$$\mathscr{F}_0^*(f)(x)=\lim_{M\to\infty}\int_{\Gamma_M}g(z)\,dz=i\mathrm{Res}_{\mathrm{i}\sqrt{\lambda}}g(z)=\frac{e^{-x\sqrt{\lambda}}}{2\sqrt{\lambda}}.$$

□

Corollary 12.11.1. *(Resolvent* $(\lambda-\Delta)^{-1}$ *in dimension one) For* $\lambda>0$, $f(x)\in S(\mathbb{R})$ *we have the representation*

$$(\lambda-\Delta)^{-1}f(x)=\frac{1}{2\sqrt{\lambda}}\int_{\mathbb{R}}e^{-\sqrt{\lambda}|x-y|}f(y)dy. \tag{12.11.56}$$

The next case is to compute the resolvent

$$(\lambda+i\varepsilon+\Delta)^{-1},\lambda>0,$$

in the case of dimension one. Therefore we aim at finding the Fourier transform

$$\mathscr{F}_0^*(f)(x)=(2\pi)^{-1}\int_{\mathbb{R}}e^{\mathrm{i}x\xi}f(\xi)d\xi$$

with

$$f(\xi)=\frac{1}{\lambda+i\varepsilon-\xi^2}.$$

Lemma 12.11.2. *If* $\lambda>0$, *then we have*

$$(2\pi)^{-1}\int_{\mathbb{R}}e^{\mathrm{i}x\xi}\frac{d\xi}{\lambda+\mathrm{i}\varepsilon-\xi^2}=-\frac{1}{2\sqrt{-\lambda-\mathrm{i}\varepsilon}}e^{-\sqrt{-\lambda-\mathrm{i}\varepsilon}|x|}, \tag{12.11.57}$$

where here and below

$$\sqrt{-\lambda-\mathrm{i}\varepsilon}=e^{(\mathrm{Log}(-\lambda-\mathrm{i}\varepsilon))/2}$$

and $\mathrm{Log}(-\lambda-\mathrm{i}\varepsilon)$ *is the principle branch of the logarithm.*

Proof. We have to compute

$$\int_{\mathbb{R}}e^{\mathrm{i}x\xi}\frac{d\xi}{\lambda+\mathrm{i}\varepsilon-\xi^2}$$

Introduce the complex function

$$g(z)=\frac{e^{\mathrm{i}xz}}{\lambda+\mathrm{i}\varepsilon-z^2}=\frac{-e^{\mathrm{i}xz}}{(z+\mathrm{i}\sqrt{-\lambda-\mathrm{i}\varepsilon})(z-\mathrm{i}\sqrt{-\lambda-\mathrm{i}\varepsilon})}.$$

If $x>0$, then we have

$$\int_{\mathbb{R}}e^{\mathrm{i}x\xi}\frac{d\xi}{\lambda+\mathrm{i}\varepsilon-\xi^2}=\lim_{M\to\infty}\int_{-M}^{M}e^{\mathrm{i}xz}\frac{dz}{\lambda+\mathrm{i}\varepsilon-z^2}.$$

and we can introduce the closed contour

$$\Gamma_M=[-M;M]\cup\gamma_M$$

with

$$\gamma_M = \{Me^{i\theta} : \theta \in [0;\pi]\}.$$

For $\theta \in (\varepsilon, \pi - \varepsilon)$

$$\mathrm{Im} Me^{i\theta} = M\sin\theta, \implies \lim_{M\to\infty} \left| \frac{e^{\mathrm{i}xMe^{i\theta}}}{\lambda + \mathrm{i}\varepsilon - M^2 e^{2\mathrm{i}\theta}} \right| \le \lim_{M\to\infty} \frac{e^{-xM\sin\varepsilon}}{M^2 - \lambda} = 0.$$

Applying the Lebesgue convergence theorem we get

$$\lim_{M\to\infty} \int_{\gamma_M} g(z)\, dz = 0.$$

Hence

$$\mathscr{F}_0^*(f)(x) = \lim_{M\to\infty} \int_{\Gamma_M} g(z)\, dz = i\mathrm{Res}_{\mathrm{i}\sqrt{-\lambda-\mathrm{i}\varepsilon}}\, g(z) = -\frac{e^{-x\sqrt{-\lambda-\mathrm{i}\varepsilon}}}{2\sqrt{-\lambda-\mathrm{i}\varepsilon}}.$$

□

In a similar way we can deduce the following

Lemma 12.11.3. *If* $\lambda > 0$, *then we have*

$$(2\pi)^{-1} \int_{\mathbb{R}} e^{\mathrm{i}x\xi} \frac{d\xi}{\lambda - \mathrm{i}\varepsilon - \xi^2} = -\frac{1}{2\sqrt{-\lambda + \mathrm{i}\varepsilon}} e^{-\sqrt{-\lambda+\mathrm{i}\varepsilon}|x|}, \tag{12.11.58}$$

where here and below

$$\sqrt{-\lambda - \mathrm{i}\varepsilon} = e^{(\mathrm{Log}(-\lambda - \mathrm{i}\varepsilon))/2}$$

and $\mathrm{Log}(-\lambda - \mathrm{i}\varepsilon)$ *is the principle branch of the logarithm.*

Proof. We have to compute

$$\int_{\mathbb{R}} e^{\mathrm{i}x\xi} \frac{d\xi}{\lambda - \mathrm{i}\varepsilon - \xi^2}$$

Introduce the complex function

$$g(z) = \frac{e^{\mathrm{i}xz}}{\lambda - \mathrm{i}\varepsilon - z^2} = \frac{-e^{\mathrm{i}xz}}{(z + \mathrm{i}\sqrt{-\lambda + \mathrm{i}\varepsilon})(z - \mathrm{i}\sqrt{-\lambda + \mathrm{i}\varepsilon})}.$$

If $x > 0$, then we have

$$\int_{\mathbb{R}} e^{\mathrm{i}x\xi} \frac{d\xi}{\lambda + \mathrm{i}\varepsilon - \xi^2} = \lim_{M\to\infty} \int_{-M}^{M} e^{\mathrm{i}xz} \frac{dz}{\lambda + \mathrm{i}\varepsilon - z^2}.$$

and we can introduce the closed contour

$$\Gamma_M = [-M; M] \cup \gamma_M$$

with

$$\gamma_M = \left\{ M e^{i\theta} : \theta \in [0;\pi] \right\}.$$

For $\theta \in (\varepsilon, \pi - \varepsilon)$

$$\operatorname{Im} M e^{i\theta} = M \sin\theta, \implies \lim_{M\to\infty} \left| \frac{e^{\mathrm{i} x M e^{i\theta}}}{\lambda + \mathrm{i}\varepsilon - M^2 e^{2\mathrm{i}\theta}} \right| \le \lim_{M\to\infty} \frac{e^{-xM\sin\varepsilon}}{M^2 - \lambda} = 0.$$

Applying the Lebesgue convergence theorem we get

$$\lim_{M\to\infty} \int_{\gamma_M} g(z)\,dz = 0.$$

Hence

$$\mathscr{F}_0^*(f)(x) = \lim_{M\to\infty} \int_{\Gamma_M} g(z)\,dz = i\operatorname{Res}_{\mathrm{i}\sqrt{-\lambda+\mathrm{i}\varepsilon}} g(z) = -\frac{e^{-x\sqrt{-\lambda+\mathrm{i}\varepsilon}}}{2\sqrt{-\lambda+\mathrm{i}\varepsilon}}.$$

□

Corollary 12.11.2. *(Limiting absorption principle for the free Laplace operator) If $\lambda > 0$, then for any $f \in S(\mathbb{R})$ the limit*

$$(\lambda + \mathrm{i}0 + \Delta)^{-1} f(x) = \lim_{\varepsilon \searrow 0} (\lambda + \mathrm{i}\varepsilon + \Delta)^{-1} f(x)$$

is a well defined limit in L^∞ and we have the representation formula

$$(\lambda + \mathrm{i}0 + \Delta)^{-1} f(x) = \tag{12.11.59}$$
$$= \lim_{\varepsilon \searrow 0} (2\pi)^{-1} \int_{\mathbb{R}} e^{\mathrm{i}x\xi} \mathscr{F}_0(f)(\xi) \frac{d\xi}{\lambda + \mathrm{i}\varepsilon - \xi^2} =$$
$$= \frac{1}{2\mathrm{i}\sqrt{\lambda}} \int_{\mathbb{R}} e^{\mathrm{i}\sqrt{\lambda}|x-y|} f(y)\,dy,$$

where

$$\mathscr{F}_0(f)(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}x\xi} f(x)\,dx$$

is the Fourier transform of f.

Corollary 12.11.3. *(Limiting absorption principle for the free Laplace operator) If $\lambda > 0$, then for any $f \in S(\mathbb{R})$ the limit*

$$(\lambda - \mathrm{i}0 + \Delta)^{-1} f(x) = \lim_{\varepsilon \searrow 0} (\lambda - \mathrm{i}\varepsilon + \Delta)^{-1} f(x)$$

is a well defined limit in L^∞ and we have the representation formula

$$(\lambda - \mathrm{i}0 + \Delta)^{-1} f(x) = \tag{12.11.60}$$
$$= \lim_{\varepsilon \searrow 0} (2\pi)^{-1} \int_{\mathbb{R}} e^{\mathrm{i}x\xi} \mathscr{F}_0(f)(\xi) \frac{d\xi}{\lambda - \mathrm{i}\varepsilon - \xi^2} =$$
$$= -\frac{1}{2\mathrm{i}\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\mathrm{i}\sqrt{\lambda}|x-y|} f(y)\, dy,$$

where

$$\mathscr{F}_0(f)(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}x\xi} f(x)\, dx$$

is the Fourier transform of f.

12.12 Teoria della Trasformata di Fourier in più variabili

In modo deltutto analogo al caso di una variabile possiamo definire lo **spazio di Schwarz** n-dimensionale, come:

$$\mathscr{S}_n = \mathscr{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \left(\forall \alpha, \beta \in \mathbb{N}^n\right) \qquad \mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta f(\mathbf{x}) \in L^\infty(\mathbb{R}^n) \right\}.$$

Si dimostra in maniera analoga al caso unidimensionale ($n = 1$) che l'applicazione:

$$\mathscr{F} : \mathscr{S}_n \ni f \longrightarrow \widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\mathbf{x} \in \mathscr{S}_n \tag{12.12.61}$$

è un automorfismo di $\mathscr{S}_n$ che prende il nome di **trasformazione di Fourier**; allo stesso modo che nel caso unidimensionale si dimostra, inoltre che $\mathscr{F}$ è un'isometria (a meno di un fattore costante $(2\pi)^n$) di $\mathscr{S}_n$ in sé (teorema di Plancharel, vedi (12.2.15)). La formula di inversione di $\mathscr{F}$ è:

$$f^* = \frac{1}{(2\pi)^n} \widehat{\widehat{f}} \tag{12.12.62}$$

con la notazione ovvia $f^*(\mathbf{x}) = f(-\mathbf{x})$. Non è difficile provare che, per la trasformata di Fourier multidimensionale, valgono le stesse proprietà formali dimostrate precedentemente:

$$\widehat{\partial_{\mathbf{x}}^\alpha f(\mathbf{x})} = i^{|\alpha|} \boldsymbol{\xi}^\alpha \widehat{f}(\boldsymbol{\xi}), \qquad \text{e} \qquad \widehat{\mathbf{x}^\alpha f(\mathbf{x})} = i^{|\alpha|} \partial_{\boldsymbol{\xi}}^\alpha \widehat{f}(\boldsymbol{\xi}). \tag{12.12.63}$$

Su $\mathscr{S}_n$ è possibile introdurre una topologia in modo deltutto analogo a $\mathscr{S}$ $(= \mathscr{S}_1)$. Tale topologia è quella generata dalla famiglia di seminorme:

$$\mathscr{Y} = \left\{ \|\cdot\|_N : N \in \mathbb{N} \right\},$$

dove

$$\|f\|_N = \sum_{k+|\alpha| \le N} \sup_{\mathbf{x}\in\mathbb{R}^n} \left| \mathbf{x}^k \partial_{\mathbf{x}}^\alpha f(\mathbf{x}) \right|.$$

In tal modo ha senso considerare lo spazio duale di $\mathscr{S}_n$, che prende il nome di **spazio delle distribuzioni temperate** e si denota con $\mathscr{S}'(\mathbb{R}^n) = \mathscr{S}_n'$.

Lo spazio delle distribuzioni in $\mathbb{R}^n$ ha le stesse proprietà di $\mathscr{S}'$, in particolare esiste un operatore di derivazione

$$\partial^\alpha : \mathscr{S}_n' \ni v \longrightarrow \partial^\alpha v \in \mathscr{S}_n'$$

definito come:

$$\lambda \partial^\alpha v, \varphi\rangle = (-1)^{|\alpha|} \lambda v, \partial^\alpha \varphi\rangle \qquad \left(\forall \varphi \in \mathscr{S}_n\right)$$

che estende l'operatore di derivata in senso classico e verifica le proprietà "naturali" di derivata:

(1) per ogni distribuzione temperata $v \in \mathscr{S}'_n$, $\partial^\alpha v \in \mathscr{S}'_n$;

(2) per ogni funzione f di classe C^∞ a supporto compatto e ogni distribuzione temperata $v \in \mathscr{S}'_n$, vale la formula di Leibniz:

$$\partial(f \cdot v) = (\partial f)v + f\partial v; \tag{12.12.64}$$

dove $f \cdot v$ è la distribuzione che agisce al modo ovvio:

$$\langle f \cdot v, \varphi\rangle = \langle v, f\varphi\rangle \qquad (\forall \varphi \in \mathscr{S}_n);$$

(3) se $v \in \mathscr{S}'_n$ è una distribuzione temperata tale che $\partial v = 0$, allora esiste una costante $c \in \mathbb{R}$ tale che $v \equiv c$.

Infine, la trasformata di Fourier di una distribuzione $v \in \mathscr{S}'_n$, $\mathscr{F} : \mathscr{S}'_n \ni v \longrightarrow \hat{v} \in \mathscr{S}'_n$, è definita al modo ovvio:

$$\langle \hat{v}, \varphi\rangle = \langle v, \hat{\varphi}\rangle \qquad (\forall \varphi \in \mathscr{S}_n) \tag{12.12.65}$$

ed è un automorfismo di $\mathscr{S}'_n$. Essa verifica, inoltre, le stesse proprietà formali della trasformata di Fourier in $\mathscr{S}_n$:

$$\widehat{\partial^\alpha v} = i^{|\alpha|}\boldsymbol{\xi}^\alpha \hat{v} \qquad \text{e} \qquad \widehat{\mathbf{x}^\alpha v} = i^{|\alpha|}\partial^\alpha \hat{v}.$$

12.13 Applicazione: simmetria e la trasformata di Fourier

Sia $\mathbf{y} \in \mathbb{R}^n$. Definiamo l'applicazione:

$$\tau_{\mathbf{y}} : \mathscr{S}_n \ni f \longrightarrow \tau_{\mathbf{y}} f \in \mathscr{S}_n$$

tale che

$$\tau_{\mathbf{y}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{y}) \qquad (\forall \mathbf{x} \in \mathbb{R}^n).$$

In particolare, se $v \in \mathscr{S}_n$ e $\varphi \in \mathscr{S}_n$,

$$\langle \tau_{\mathbf{y}} v, \varphi\rangle = \int \tau_{\mathbf{y}} v(\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x} = \int v(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\, d\mathbf{x} = \int v(\mathbf{z})\varphi(\mathbf{z}+\mathbf{y})\, d\mathbf{z} = \langle v, \tau_{-\mathbf{y}}\varphi\rangle.$$

Pertanto ha senso definire l'applicazione

$$\tau_{\mathbf{y}} : \mathscr{S}'_n \ni v \longrightarrow \tau_{\mathbf{y}} v \in \mathscr{S}'_n$$

definita da

$$\langle \tau_{\mathbf{y}} v, \varphi\rangle = \langle v, \tau_{-\mathbf{y}}\varphi\rangle \qquad (\forall \varphi \in \mathscr{S}_n).$$

Sia, adesso, $A \in \mathbf{GL}_n(\mathbb{R})$. Definiamo

$$\varrho_A : \mathscr{S}_n \ni f \longrightarrow \varrho_A f \in \mathscr{S}_n$$

definita da

$$\varrho_A f(\mathbf{x}) = f(A\mathbf{x}) \qquad \big(\forall \mathbf{x} \in \mathbb{R}^n\big).$$

In particolare, se $v \in \mathscr{S}_n$ e $\varphi \in \mathscr{S}_n$,

$$\langle \varrho_A v, \varphi \rangle = \int v(A\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x} = \int v(\mathbf{y})\varphi(A^{-1}\mathbf{y})|\det A^{-1}| d\mathbf{y} = \frac{1}{|\det A|}\langle v, \varrho_{A^{-1}}\varphi \rangle.$$

Pertanto ha senso definire l'applicazione

$$\varrho_A : \mathscr{S}'_n \ni v \longrightarrow \varrho_A v \in \mathscr{S}'_n$$

definita da

$$\langle \varrho_A v, \varphi \rangle = \frac{1}{|\det A|}\langle v, \varrho_{A^{-1}}\varphi \rangle \qquad \big(\forall \varphi \in \mathscr{S}_n\big).$$

Infine, per ogni $\lambda > 0$, definiamo

$$\theta_\lambda : \mathscr{S}_n \ni f \longrightarrow \theta_\lambda f \in \mathscr{S}_n$$

definita da

$$\theta_\lambda f(\mathbf{x}) = f(\lambda \mathbf{x}) \qquad \big(\forall \mathbf{x} \in \mathbb{R}^n\big).$$

In particolare, se $v \in \mathscr{S}_n$ e $\varphi \in \mathscr{S}_n$,

$$\langle \theta_\lambda v, \varphi \rangle = \int v(\lambda\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x} = \int v(\mathbf{y})\varphi(\lambda^{-1}\mathbf{y})|\lambda|^{-n} d\mathbf{y} = |\lambda|^{-n}\langle v, \theta_{\lambda^{-1}}\varphi \rangle.$$

Pertanto definiamo l'applicazione

$$\theta_\lambda \mathscr{S}'_n \ni v \longrightarrow \theta_\lambda v \in \mathscr{S}'_n$$

definita da

$$\langle \theta_\lambda v, \varphi \rangle = |\lambda|^{-n}\langle v, \theta_{\lambda^{-1}}\varphi \rangle \qquad \big(\forall \varphi \in \mathscr{S}_n\big).$$

Quando $\lambda = -1$, $\theta_{-1} v$ coincide con v^*. Diremo che una distribuzione $v \in \mathscr{S}'_n$ è **pari** se

$$v = v^*.$$

Diremo, inoltre, che una distribuzione $v \in \mathscr{S}'_n$ è **omogenea di ordine** k, se

$$\theta_\lambda v = \lambda^k v \qquad \big(\forall \lambda > 0\big).$$

Osserviamo che, per ogni $\mathbf{y} \in \mathbb{R}^n$,

$$\widehat{\tau_{\mathbf{y}} v} = e^{-i\mathbf{x}\cdot\mathbf{y}}\hat{v} \qquad \text{e} \qquad \widehat{e^{-i\mathbf{x}\cdot\mathbf{y}} v} = \tau_{\mathbf{y}}\hat{v}, \tag{12.13.66}$$

Consideriamo una funzione rapidamente decrescente $f \in \mathscr{S}_n$, per ogni $S \in \mathbf{SO}(n)$,

$$\varrho_S f(\mathbf{x}) = f(S\mathbf{x}) = f(x) \qquad \Longleftrightarrow \qquad f(\mathbf{x}) = f(|\mathbf{x}|).$$

Infatti, ($\Longleftarrow$) segue dal fatto che, per ogni $S \in \mathbf{SO}(n)$, $|S\mathbf{x}| = |\mathbf{x}|$. ($\Longrightarrow$): basta prendere una rotazione $S \in \mathbf{SO}(n)$ che porti $\mathbf{x}$ in $(|\mathbf{x}|, 0, \cdots, 0)$. Allora

$$f(|\mathbf{x}|) = f(S\mathbf{x}) = f(\mathbf{x}).$$

Problema 12.13.1. *Se $f(\mathbf{x}) = f(|\mathbf{x}|)$, allora $\hat{f}(\boldsymbol{\xi}) = \hat{f}(|\boldsymbol{\xi}|)$.*

Soluzione. Dalla definizione, fissata $S \in \mathbf{SO}(n)$,

$$\begin{aligned}
\hat{f}(S\boldsymbol{\xi}) &= \int f(|\mathbf{x}|)e^{-i\mathbf{x}\cdot S\boldsymbol{\xi}}\, d\mathbf{x} \\
&= \int f(|\mathbf{x}|)e^{-iS^{-1}\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} \\
&= \int f(|S\mathbf{y}|)e^{-\mathbf{y}\cdot\boldsymbol{\xi}} |\det S| d\mathbf{y} \\
&= \int f(|\mathbf{y}|)e^{-i\mathbf{y}\cdot\boldsymbol{\xi}}\, d\mathbf{y} = \hat{f}(\boldsymbol{\xi}).
\end{aligned}$$

che prova l'asserto. □

Problema 12.13.2. *Se $f \in \mathscr{S}_n$ è una funzione rapidamente decrescente omogenea di ordine k, allora $\hat{f}$ è omogenea di ordine $-n-k$.*

Soluzione. Dalla definizione, per ogni $\lambda > 0$:

$$\begin{aligned}
\hat{f}(\lambda\boldsymbol{\xi}) &= \int f(\mathbf{x})e^{-i\mathbf{x}\cdot\lambda\boldsymbol{\xi}} d\mathbf{x} \\
&= \int f\left(\lambda^{-1}\mathbf{y}\right) e^{-i\mathbf{y}\cdot\boldsymbol{\xi}}\lambda^{-n} d\mathbf{y} \\
&= \lambda^{-n}\int \lambda^{-k} f(\mathbf{y})e^{-i\mathbf{y}\cdot\boldsymbol{\xi}} d\mathbf{y} \\
&= \lambda^{-n-k}\hat{f}(\boldsymbol{\xi})
\end{aligned}$$

che prova l'asserto. □

Osserviamo che, per ogni distribuzione $v \in \mathscr{S}'_n$, ogni $\lambda > 0$ e ogni $S \in \mathbf{SO}(n)$,

$$\widehat{\theta_\lambda v} = \lambda^{-n}\theta_{\lambda^{-1}}\hat{v} \qquad \text{e} \qquad \widehat{\varrho_S v} = \varrho_S\hat{v} \tag{12.13.67}$$

le proposizioni precedenti valgono anche per le distribuzioni temperate:

Problema 12.13.3. *Sia $v \in \mathscr{S}'_n$ una distribuzione temperata. Allora:*

(1) *se $\varrho_S v = v$ allora $\varrho_S \hat{v} = \hat{v}$;*

(2) *se v è omogenea di ordine k, allora $\hat{v}$ è omogenea di ordine $-n-k$.*

12.14 Applicazioni: l'equazione di Laplace e l'equazione delle onde

Vogliamo studiare la soluzione generale dell'equazione di Laplace in $\mathbb{R}^3$:

$$\Delta u = \delta.$$

Applicando la trasformazione di Fourier,

$$-|\boldsymbol{\xi}|^2 \hat{u} = \widehat{\Delta u} = \hat{\delta} = 1,$$

pertanto

$$\hat{u} = -\frac{1}{|\boldsymbol{\xi}|^2} \in L^1_{\text{loc}}(\mathbb{R}^3).$$

Dal problema precedente segue che u deve essere omogenea di ordine -1 ed invariante per rotazioni, quindi

$$u(\mathbf{x}) = \frac{c}{|\mathbf{x}|}$$

dove $c \in \mathbb{R}$.

In $\mathbb{R}^n$ $(n \geq 3)$, in modo deltutto analogo, al soluzione generale di

$$\Delta u = \delta$$

con $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ è:

$$u(\mathbf{x}) = \frac{c}{|\mathbf{x}|^{n-2}}.$$

The one dimensional wave equation has the form

$$\partial_t^2 u - \partial_x^2 u = 0 \tag{12.14.68}$$

with initial data

$$u(0,x) = 0,\ \partial_t u(0,x) = f(x).$$

The classical solution is given by d'Alembert's formula

$$u(t,x) = \frac{1}{2}\int_{|x-y|<t} f(y)dy. \tag{12.14.69}$$

Using Fourier transform in x we can see that

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}} u(t,x)e^{-ix\xi}\,dx$$

satisfies the ODE

$$\partial_t^2 \widehat{u} + \xi^2 \widehat{u} = 0 \tag{12.14.70}$$

with initial data

$$\widehat{u}(0,\xi) = 0,\ \partial_t \widehat{u}(0,\xi) = \widehat{f}(\xi).$$

The solution is give by

$$\widehat{u}(t,\xi) = \frac{\sin t\xi}{\xi}\widehat{f}(\xi). \tag{12.14.71}$$

Using the inverse Fourier transform

$$u(t,x) = \frac{1}{2\pi}\int \widehat{u}(t,\xi)e^{ix\xi}\,d\xi$$

we can derive

Lemma 12.14.1. *If $t > 0$, then*

$$\int_{\mathbb{R}} e^{\pm ix\xi}\,\frac{\sin(t\xi)}{\xi}\,d\xi = \pi H(|x| < t), \tag{12.14.72}$$

where $H(|x| < T)$ is the characteristic function of the set $\{x, |x| < t\}$.

Chapter 13

Appendix IV: Sobolev spaces

13.1 Michlin - Hörmander theorem

13.1.1 Laplace operator in $\mathbb{R}^n$; fractional powers of $(1-\Delta)$

For any differential operator with constant coefficients of order m

$$P(\partial_x) = \sum_{|\alpha|\le m} a_\alpha \partial_x^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is multiindex,

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$$

we can use the Fourier transform and for any $f \in S(\mathbb{R}^n)$ we can write

$$\widehat{P(f)}(\xi) = \sum_{|\alpha|\le m} \mathrm{i}^{|\alpha|} a_\alpha \xi^\alpha \partial_x^\alpha$$

with

$$|\alpha| = \alpha_1 + \cdots \alpha_n.$$

Therefore inverting the Fourier transform we have

$$P(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} Q(\xi) \widehat{f}(\xi) d\xi,$$

where

$$Q(\xi) = \sum_{|\alpha|\le m} \mathrm{i}^{|\alpha|} a_\alpha \xi^\alpha$$

with

$$\xi = (\xi_1, \cdots, \xi_n) \in R^n, \;\; \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

is called the symbol of the operator P.

Using the notations

$$D_{x_j} = \frac{1}{\mathrm{i}}\partial_{x_j},\ \mathrm{D}_{\mathrm{x}}^{\alpha} = \mathrm{D}_{\mathrm{x}_1}^{\alpha_1}\cdots\mathrm{D}_{\mathrm{x}_\mathrm{n}}^{\alpha_\mathrm{n}},$$

we can consider the differential operator

$$P(D_x) = \sum_{|\alpha|\le m} a_\alpha D_x^\alpha,$$

so that its symbol is exactly

$$P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$$

and we have the representation

$$P(D)(f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} P(\xi)\widehat{f}(\xi)d\xi \tag{13.1.1}$$

In particular, the Laplace operator

$$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_1}^2 = -D_{x_1}^2 - \cdots - D_{x_n}^2$$

has symbol

$$-|\xi|^2 = -\xi_1^2 - \cdots - \xi_n^2$$

and therefore

$$\Delta(f)(x) = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}|\xi|^2\widehat{f}(\xi)d\xi \tag{13.1.2}$$

This relation can be generalized as follows

$$(1-\Delta)^\ell(f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}(1+|\xi|^2)^\ell\widehat{f}(\xi)d\xi \tag{13.1.3}$$

for any integer $\ell \ge 0$.

Now we can give the DEFINITION of the fractional powers $(1-\Delta)^{s/2}$ for any real s

$$(1-\Delta)^{s/2}(f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}(1+|\xi|^2)^{s/2}\widehat{f}(\xi)d\xi \tag{13.1.4}$$

This integral is well -defined for example for $f \in S(\mathbb{R}^n)$. The above identities suggest to consider operators of the form

$$A(D)(f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} A(\xi)\widehat{f}(\xi)d\xi \tag{13.1.5}$$

with symbol A in a class larger that the class of polynomials. In general operators of this type can be called pseudodifferential operators of convolution type.

For example, we can take

$$A(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} b(x) dx,$$

where $b \in L^1(\mathbb{R}^n)$. Then using the fact that the product of two Fourier transforms is Fourier transform of their convolution, i.e.

$$\widehat{b}(\xi)\widehat{f}(\xi) = \widehat{(b * f)}(\xi)$$

we find

$$A(D)(f)(x) = b * f(x) = \mathfrak{F}^{-1}(A) * f$$

and then the Young inequality implies that

$$A(D) : L^p(R^n) \to L^p(R^n)$$

is a bounded operator for $p \in [1,\infty]$.

Let ρ be a tempered distribution. ρ is called a Fourier multiplier on L^p if the convolution

$$\mathfrak{F}^{-1}\rho * f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} \rho(\xi)\widehat{f}(\xi) d\xi \in L^p(\mathbb{R}^n)$$

for all $f \in S(\mathbb{R}^n)$, and if

$$\sup_{\|f\|_{L^p}=1} \|\mathfrak{F}^{-1}\rho * f\|_{L^p}$$

is finite. The linear space of all such ρ is denoted by M^p; the norm on M^p is the above supremum.

Teorema 13.1.1. *(The Michlin – Hörmander multiplier theorem). Assume that $\rho(\xi)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ and that*

$$\sum_{|\alpha|\le L} |\xi|^{|\alpha|} \left|\partial_\xi^\alpha \rho(\xi)\right| \le C_0.$$

for some integer $L > n/2$. Then there exists a constant C_1 depending on $p \in (1,\infty)$ and C_0 so that for any $f \in L^p(\mathbb{R}^n)$ we have

$$\| \left(\mathfrak{F}^{-1}\rho\right) * f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

The proof is based on the application of properties of Calderon - Zygmund operators and we shall skip the proof refering to [13].

13.2 Sobolev spaces on $\mathbb{R}^n$

13.2.1 Sobolev spaces H^s (via Fourier transform)

As is customary, we denote the Sobolev spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ also by $H^k(\Omega)$ and $H_0^k(\Omega)$, respectively. Recall that when $\Omega = \mathbb{R}^N$, the spaces $H^k(\Omega)$ can be equivalently defined via Fourier Transform since $H^k\left(\mathbb{R}^N\right)$ is the space of functions $u \in L^2\left(\mathbb{R}^N\right)$ such that

$$\left(\int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^k |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \tag{13.2.1}$$

is finite. Here $\hat{u}$ denotes the Fourier Transform $\mathfrak{F}[u]$ of a function $u \in L^2\left(\mathbb{R}^N\right)$ defined by

$$\hat{u}(\xi) = \mathfrak{F}[u](\xi) = \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} dx$$

Recall that (13.2.1) defines a norm in $H^k\left(\mathbb{R}^N\right)$ equivalent to the standard one since

$$\|u\|^2_{H^k(\mathrm{R}^N)} = \sum_{|\alpha|\le k} \int_{\mathbb{R}^N} \left|D^\alpha u\right|^2 dx = \int_{\mathbb{R}^N} \sum_{|\alpha|\le k} \left|\xi^\alpha\right|^2 |\hat{u}(\xi)|^2 d\xi$$

The previous definitions extend to the case of non-integer order of smoothness and allow to define the whole scale of spaces $H^s\left(\mathbb{R}^N\right), s > 0$ simply by replacing k by s in (13.2.1).

Let $\Omega \subseteq R^n$ be an open set with continuous boundary $\partial\Omega$. Denote by $\mathscr{T}$ the trace operator, namely the linear operator defined by the uniformly continuous extension of the operator of restriction to $\partial\Omega$ for functions in $C_0^\infty(\overline{\Omega})$, that is the space of functions $C_0^\infty(\mathbb{R}^n)$ restricted [6] to $\bar{\Omega}$.

Now, for any $x = \left(x', x_n\right) \in \mathbb{R}^n$ and for any $u \in \mathfrak{S}(\mathrm{R}^n)$, we denote by $v \in \mathfrak{S}\left(\mathrm{R}^{\mathrm{n}-1}\right)$ the restriction of u on the hyperplane $x_n = 0$, that is

$$v\left(x'\right) = u\left(x', 0\right) \quad \forall x' \in \mathbf{R}^{n-1}. \tag{13.2.2}$$

Then, we have

$$\mathfrak{F}v\left(\xi'\right) = (2\pi)^{-1} \int_{\mathbb{R}} \mathfrak{F}u\left(\xi', \xi_n\right) d\xi_n \quad \forall \xi' \in \mathbf{R}^{n-1}, \tag{13.2.3}$$

where, for the sake of simplicity, we keep the same symbol $\mathfrak{F}$ for both the Fourier transform in $n-1$ and in n variables. To check (13.2.3), we write

$$\begin{aligned} \mathfrak{F}v\left(\xi'\right) &= \int_{\mathbb{R}=1} e^{-i\xi'\cdot x'} v\left(x'\right) dx' \\ &= \int_{\mathbb{R}^{-1}} e^{-i\xi'\cdot x'} u\left(x', 0\right) dx' \end{aligned} \tag{13.2.4}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathfrak{F}u\left(\xi',\xi_n\right) d\xi_n\\
&\quad = \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{-i\left(\xi'\cdot\xi_n\right)\cdot\left(x',x_n\right)} u\left(x',x_n\right) dx' dx_n d\xi_n\\
&\quad = \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi_n\cdot x_n} u\left(x',x_n\right) dx_n d\xi_n\right] dx'\\
&= 2\pi \int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} \left[u\left(x',0\right)\right] dx'
\end{aligned}$$

where the last equality follows by transforming and anti-transforming u in the last variable, and this coincides with (13.2.4).

We have the following.

Proposition 13.2.1. *Let $s > 1/2$, then any function $u \in H^s(\mathbb{R}^n)$ has a trace v on the hyperplane $\{x_n = 0\}$, such that $v \in H^{s-\frac{1}{2}}\left(\mathbb{R}^{n-1}\right)$. Also, the trace operator T is surjective from $H^s(\mathbb{R}^n)$ onto $H^{s-\frac{1}{2}}\left(\mathbb{R}^{n-1}\right)$*

Proof. In order to prove the first claim, it suffices to show that there exists an universal constant C such that, for any $u \in \mathscr{S}(\mathbb{R}^n)$ and any v defined as in (13.2.2)

$$\|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \le C\|u\|_{H^s(\mathbb{R}^n)}. \tag{13.2.5}$$

By taking into account (13.2.3), the Cauchy-Schwarz inequality yields

$$\left|\mathfrak{F}v\left(\xi'\right)\right|^2 \le \left(\int_{\mathbb{R}} \left(1+|\xi|^2\right)^{\mathbf{a}} \left|\mathfrak{F}u\left(\xi',\xi_n\right)\right|^2 d\xi_n\right)\left(\int_{\mathbb{R}} \frac{d\xi_n}{\left(1+|\xi|^2\right)^s}\right). \tag{13.2.6}$$

Using the changing of variable formula by setting $\xi_n = t\sqrt{1+|\xi'|^2}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{d\xi_n}{\left(1+|\xi|^2\right)^s} &= \int_{\mathbb{R}} \frac{\left(1+\left|\xi'\right|^2\right)^{1/2}}{\left(\left(1+|\xi'|^2\right)\left(1+t^2\right)\right)^s} dt = \int_{\mathbb{R}} \frac{\left(1+\left|\xi'\right|^2\right)^{\frac{1}{2}-s}}{\left(1+t^2\right)^a} dt\\
&= C(s)\left(1+\left|\xi'\right|^2\right)^{1/2-s}
\end{aligned} \tag{13.2.7}$$

where $C(s) := \int_{\mathbb{R}} \frac{dt}{(1+t^2)^s} < +\infty$ since $s > 1/2$ Combining (13.2.6) with (13.2.7) and integrating in $\xi' \in \mathbb{R}^{n-1}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \left(1+\left|\xi'\right|^2\right)^{s-\frac{1}{2}} \left|\mathfrak{F}v\left(\xi'\right)\right|^2 d\xi'\\
&\qquad\qquad \le C(s) \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left(1+|\xi|^2\right)^s \left|\mathfrak{F}u\left(\xi',\xi_n\right)\right|^2 d\xi_n d\xi'
\end{aligned}$$

that is (13.2.5). Now, we will prove the surjectivity of the trace operator T. For this, we show that for any $v \in H^{s-\frac{1}{2}}\left(\mathrm{R}^{n-1}\right)$ the function u defined by

$$\mathfrak{F}u\left(\xi',\xi_n\right) = \mathfrak{F}v\left(\xi'\right)\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\frac{1}{\sqrt{1+|\xi'|^2}} \tag{13.2.8}$$

with $\varphi \in C_0^\infty(\mathrm{R})$ and $\int_{\mathbb{R}}\varphi(t)dt = 1$, is such that $u \in H^s\left(\mathrm{R}^n\right)$ and $Tu = v$ Indeed, we integrate (3.22) with respect to $\xi_n \in \mathbb{R}$, we substitute $\xi_n = t\sqrt{1+|\xi'|^2}$ and we obtain

$$\begin{aligned}\int_{\mathbb{R}}\mathfrak{F}u\left(\xi',\xi_n\right)d\xi_n &= \int_{\mathbb{R}}\mathfrak{F}v\left(\xi'\right)\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\frac{1}{\sqrt{1+|\xi'|^2}}d\xi_n \\ &= \int\mathfrak{F}v\left(\xi'\right)\varphi(t)dt = \mathfrak{F}v\left(\xi'\right)\end{aligned} \tag{13.2.9}$$

and this implies $v = \mathscr{T}u$ because of (13.2.3) The proof of the H^n -boundedness of u is straightforward. In fact, from (13.2.8), for any $\xi' \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(1+|\xi|^2\right)^s\left|\mathfrak{F}u\left(\xi',\xi_n\right)\right|^2 d\xi_n \\ &= \int_{\mathbb{R}}\left(1+|\xi|^2\right)^s\left|\mathfrak{F}v\left(\xi'\right)\right|^2\left|\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\right|^2\frac{1}{1+|\xi'|^2}d\xi_n \\ &= C\left(1+\left|\xi'\right|^2\right)^{s-\frac{1}{2}}\left|\mathfrak{F}v\left(\xi'\right)\right|^2\end{aligned} \tag{13.2.10}$$

where we used again the changing of variable formula with $\xi_n = t\sqrt{1+|\xi'|^2}$ and the constant C is given by $\int_{\mathbb{R}}\left(1+t^2\right)^s|\varphi(t)|^2dt$. Finally, we obtain that $u \in H^s\left(R^n\right)$ by integrating (13.2.10) in $\xi' \in \mathbb{R}^{n-1}$.

□

13.2.2 Sobolev spaces $W_p^\ell(\mathbb{R}^n)$ of integer order

The space of test functions $C_0^\infty(\mathbf{R}^n)$ is too strong, while the space of distribution is too weak to describe the space of solutions of partial differential equations. As intermediate spaces one can consider the Sobolev spaces $W_p^\ell(\mathbf{R}^n)$ defined for integer $\ell \geq 0$ and any $p \in [1,\infty]$ as follows. A distribution $f \in S'(\mathbf{R}^n)$ belongs to $W_p^l(\mathbf{R}^n)$ if the norm

$$\|f\|_{W_p^\ell} = \sum_{|\alpha|\leq\ell}\|\partial^\alpha f\|_{L^p} \tag{13.2.11}$$

is finite.

Lemma 13.2.1. *For any* $\ell \geq 0$ *and* $p \in [1,\infty]$ *the space* $W_p^\ell(\mathbf{R}^n)$ *is a Banach space.*

Idea of the proof. Follows from the fact that $L^p(R^n)$ is a Banach space.

□

Basic properties of Sobolev spaces are given in the next

Lemma 13.2.2. *Let* $\ell \geq 1$ *and let* $p \in [1,\infty]$. *Then we have*

- *for any multiindex* α, $|\alpha| \leq \ell$,

$$\partial_x^\alpha : W_p^\ell(\mathbb{R}^n) \to W_p^{\ell-|\alpha|}(\mathbb{R}^n)$$

- *if* $f \in C_0^\infty(\mathbb{R}^n)$ *and* $g \in W_p^\ell(\mathbb{R}^n)$, *then* $fg \in W_p^\ell(\mathbb{R}^n)$ *and we have the Leibniz rule*

$$\partial_x^\alpha(fg) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial_x^\beta(f)\partial_x^{\alpha-\beta}(g)$$

- *if* $b > a >$ *are positive numbers,* $1 \leq p < \infty$ *and* $f(x) \in C_0^\infty(\mathbb{R}^n)$ *is such that*

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq a; \\ 0, & \text{if } |x| \geq b, \end{cases}$$

then for any $g \in W_p^\ell(\mathbb{R}^n)$,

$$\lim_{R\to\infty} \|g_R - g\|_{W_p^\ell(\mathbb{R}^n)} = 0, \tag{13.2.12}$$

where

$$g_R(x) = f\left(\frac{x}{R}\right) g(x).$$

Idea of the proof of (13.2.12). One ca apply Lebesgue convergence theorem to show

$$\lim_{R\to\infty} \|g_R - g\|_{L^p(\mathbb{R}^n)} = 0, \tag{13.2.13}$$

the remaining part follows from the Leibniz rule, the definition of Sobolev space $W_p^\ell(\mathbb{R}^n)$ and the estimate

$$\left|\partial_x^\alpha f\left(\frac{x}{R}\right)\right| \lesssim \frac{1}{R^{|\alpha|}}.$$

□

Finally, we can show that $C_0^\infty(\mathbb{R}^n)$ is dense in the Sobolev space $W_p^\ell(\mathbb{R}^n)$.

Lemma 13.2.3. *If* $g \in W_p^\ell(\mathbb{R}^n)$, *then for any* $\varepsilon > 0$ *there exists* $h \in C_0^\infty(\mathbb{R}^n)$ *so that*

$$\|h - g\|_{W_p^\ell(\mathbb{R}^n)} \le \varepsilon. \tag{13.2.14}$$

Proof. The property (13.2.12) shows that we can assume g compactly supported. Take $\psi \in C_0^\infty(\mathbb{R}^n)$, such that $\int_{\mathbb{R}^n} \psi = 1$, $\psi(x) > 0$ for $|x| < 1$ and $\operatorname{supp}\psi = \{x; |x| \le 1\}$. Then setting

$$\psi_\varepsilon(x) = \varepsilon^{-n}\psi\left(\frac{|x|}{\varepsilon}\right)$$

we define

$$h_\varepsilon(x) = \psi_\varepsilon * g(x) = \int_{\mathbb{R}^n} \varepsilon^{-n}\psi\left(\frac{x-y}{\varepsilon}\right) g(y)dy =$$

$$= \int_{\mathbb{R}^n} \psi(z)\, g(x + \varepsilon z) dz.$$

For any $\varepsilon > 0$ the function

$$h_\varepsilon(x) = \psi_\varepsilon * g(x) = \int_{\mathbb{R}^n} \varepsilon^{-n}\psi\left(\frac{x-y}{\varepsilon}\right) g(y)dy$$

is smooth in x and has compact support. Indeed if

$$\operatorname{supp} g \subseteq \{x; |x| \le A\},$$

then

$$\operatorname{supp} h_\varepsilon \subseteq \{x; |x| \le A + \varepsilon\}$$

Further we have

$$\|\psi_\varepsilon\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \varepsilon^{-n}\psi\left(\frac{x}{\varepsilon}\right) dx = \int_{R^n} \psi = 1.$$

Hence, the Young inequality implies

$$\|\psi_\varepsilon * g\|_{L^p(\mathbb{R}^n)} \le C\|g\|_{L^p(\mathbb{R}^n)}$$

with constant C independent of ε.

Further we can apply the Lebesgue differentiation theorem and deduce

$$\lim_{\varepsilon\to 0} \int_{|z|\le 1} g(x + \varepsilon z)\psi(z)dz = g(x) \int_{\mathbb{R}^n} \psi(z)dz = g(x)$$

for almost every $x \in \mathbb{R}^n$.

Since

$$h_\varepsilon(x) = \int_{\mathbb{R}^n} \psi(z)\, g(x + \varepsilon z)dz,$$

we can write

$$h_\varepsilon(x) - g(x) = \int_{\mathbb{R}^n} \left[\psi(z)\, g(x+\varepsilon z) - g(x)\right] dz \to 0$$

for almost every x. Then the Lebesgue convergence theorem implies

$$\left\| h_\varepsilon - g \right\|_{L^p(\mathbb{R}^n)} = 0. \tag{13.2.15}$$

This proves (13.2.14) for $\ell = 0$. The remaining part follows from the definition of $W_p^\ell(\mathbb{R}^n)$ and the identity

$$\partial_x^\alpha h_\varepsilon = \psi_\varepsilon * \partial_x^\alpha g.$$

□

The above Lemma shows that the Sobolev space $W_p^\ell(\mathbb{R}^n)$ can be defined also as the closure of the space $C_0^\infty(\mathbf{R}^n)$ of test functions with respect to the norm (13.2.1).

On the other hand, for any integer $\ell \geq 0$ one can define the Sobolev space $H_p^\ell(\mathbf{R}^n)$ as completition of the space of smooth functions with compact support with respect to the norm

$$\|f\|_{H_p^\ell} = \|(1-\Delta)^{\ell/2} f\|_{L^p} \tag{13.2.16}$$

Lemma 13.2.4. *For any integer $\ell \geq 0$, $p \in (1,\infty)$ we have*

- $W_p^\ell(R^n) = H_p^\ell(R^n)$;
- *there exists a constant $C > 0$ so that for any $f \in W_p^\ell(R^n)$ we have*

$$C^{-1}\|f\|_{W_p^\ell} \leq \|f\|_{H_p^\ell} \leq C\|f\|_{W_p^\ell} \tag{13.2.17}$$

Proof. It is sufficient to verify (13.2.17) for f in the Schwartz class, or in $C_0^\infty(\mathbb{R}^n)$. The inequality

$$\|f\|_{W_p^\ell} \lesssim \|(1-\Delta)^{\ell/2} f\|_{L^p}$$

follows easily from

$$\|\partial_x^\alpha f\|_{L^p} \lesssim \|(1-\Delta)^{\ell/2} f\|_{L^p},\ |\alpha| \leq \ell.$$

This inequality in turns follows from Michlin - Hörmander theorem 13.1.1, since the assumption

$$\sum_{|\beta|\leq L} |\xi|^{|\beta|} \left|\partial_\xi^\beta \rho(\xi)\right| \leq C_0 \tag{13.2.18}$$

can be checked easily for the symbol

$$\rho(\xi) = \frac{\xi^\alpha}{(1+|\xi|)^2}, \quad |\alpha| \le \ell.$$

This proves the left estimate in (13.2.17).

The right one can be checked using the representation

$$(1+|\xi|^2)^{\ell/2} = \psi_0(\xi) + \sum_{j=1}^{n} \xi_j^\ell \psi_j(\xi), \tag{13.2.19}$$

where $\psi_0 \in C_0^\infty(\mathbb{R}^n)$, and ψ_j are smooth functions, supported in $\{|\xi| > 1\}$ and satisfying the estimate (13.2.18) needed to apply the Michlin - Hörmander theorem 13.1.1.

To make the proof of (13.2.19) more clear let us take $n = 2$. □

13.2.3 Sobolev spaces of fractional order

13.2.4 Sobolev spaces of integer order

The space of test functions $C_0^\infty(\mathbf{R}^n)$ is too strong, while the space of distribution is too weak to describe the space of solutions of partial differential equations. As intermediate spaces one can consider the Sobolev spaces $W_p^l(\mathbf{R}^n)$ defined for integers $l \ge 0$ and real numbers $1 \le p < \infty$ as follows. A distribution $f \in S'(\mathbf{R}^n)$ if the norm

$$\|f\|_{W_p^l} = \sum_{|\alpha| \le l} \|\partial^\alpha f\|_{L^p} \tag{13.2.1}$$

are finite.

Using Friedrich's molifiers, one can see that for any

$$u \in W_p^l$$

there exists a sequence of smooth compactly supported functions tending to u with respect to the norm (13.2.1).

This fact shows that the Sobolev space can be defined also as the closure of the space $C_0^\infty(\mathbf{R}^n)$ of test functions with respect to the norm (13.2.1).

The fundamental property of these Sobolev spaces is the inclusion

$$W_p^l \subseteq W_q^r \tag{13.2.2}$$

when $1 < p < q < \infty, 1/p - 1/q \le (l-r)/n > 0$. The inclusion follows directly from the Sobolev inequality

$$\|f\|_{W_q^r} \le C\|f\|_{W_p^l} \tag{13.2.3}$$

The proof of the Sobolev inequality (13.2.3) , we shall present, is based on suitable Sobolev identity. For the purpose we follow the approach from the book of Maz'ya [38]. Namely, we shall use the following Sobolev identity.

Lemma 13.2.5. *For any integer $l, 0 < l \leq n$ and any real numbers $D > \delta > 0$ one can find smooth functions φ_β (β is multi index with $|\beta| < l$) supported in the unit ball and smooth functions ψ_α such that for any smooth function $u(x)$ and for $|x| < D$ we have the representation*

$$u(x) = \delta^{-n} \sum_{|\beta|<l} \left(\frac{x}{\delta}\right)^\beta \int_{|y|\leq\delta} \varphi_\beta\left(\frac{y}{\delta}\right) u(y) dy +$$

$$+ \sum_{|\alpha|=l} \int_{|y|\leq 2D} \psi_\alpha(x,r,\theta) \partial^\alpha u(y) \frac{dy}{r^{n-l}} \tag{13.2.4}$$

where $r = |x-y|, \theta = (x-y)/r$ and the functions $\psi_\alpha(x,r,\theta)$ satisfy the estimate

$$|\psi_\alpha(x,r,\theta)| \leq C\left(\frac{D}{\delta}\right)^{n-1} \tag{13.2.5}$$

Proof. A scaling argument shows it is sufficient to consider the case $\delta = 1$. Let us take a smooth nonnegative function $\omega(y)$ supported in the unit ball so that

$$\int_{|y|\leq 1} \omega(y) dy = 1. \tag{13.2.6}$$

Introduce functions

$$f(x;r,\theta) = -\frac{r^{l-1}}{(l-1)!} \int_r^\infty \omega(x+t\theta) t^{n-1} dt$$

$$F(x;r,\theta) = \sum_{k=0}^{l-1} (-1)^k \frac{\partial^k}{\partial r^k} u(x+r\theta) \frac{\partial^{l-1-k}}{\partial r^{l-1-k}} f(x;r,\theta) \tag{13.2.7}$$

We lose no generality assuming u has a compact support in $|x| \leq 2D$. Since all the derivatives with respect to r of the function f at $r = 0$ up to order $l-2$ are zero we see that

$$F(x;0,\theta) = -u(x) \int_0^\infty \omega(x+t\theta) t^{n-1} dt$$

Now we apply the formula

$$0 = F(x;0,\theta) + \int_0^\infty \frac{\partial F}{\partial r}(x;r,\theta) dr$$

valid in view of the fact that u is compactly supported.

On the other hand, from (13.2.7) we get

$$\frac{\partial F}{\partial r}(x;r,\theta) = u\frac{\partial^l f}{\partial r^l} + (-1)^{l-1}\frac{\partial^l u}{\partial r^l} f.$$

Hence we get the identity

$$u(x)\int_0^\infty \omega(x+t\theta)\,t^{n-1}\,dt = \int_0^\infty u(x+r\theta)\frac{\partial^l f}{\partial r^l}(x;r,\theta)\,dr +$$
$$+(-1)^{l-1}\int_0^\infty \frac{\partial^l u}{\partial r^l}(x+r\theta) f(x;r,\theta)\,dr$$

Integrating this identity over $\theta \in \mathbf{S}^{n-1}$, we obtain

$$u(x) = \int u(y)\frac{\partial^l f}{\partial r^l}(x;r,\theta)\frac{dy}{r^{n-1}} +$$
$$+(-1)^{l-1}\int \frac{\partial^l u}{\partial r^l}(y) f(x;r,\theta)\frac{dy}{r^{n-1}}$$

Having in mind that we have the representation

$$\frac{\partial^l f}{\partial r^l}(x;r,\theta)\frac{1}{r^{n-1}} = \sum_{|\beta|<l} x^\beta \varphi_\beta(y),$$

we arrive at Sobolev identity (13.2.4) with

$$\psi_\alpha = c_\alpha \int_r^\infty \omega(x+t\theta)\,t^{n-1}\,dt. \tag{13.2.8}$$

The estimate (13.2.5) follows directly from (13.2.8).

This completes the proof of the Sobolev identity.

To prove the inequality (13.2.3) it is sufficient to combine the Sobolev identity with the Sobolev inequality of Lemma 1.3.4.

In fact, Lemma 1.3.4 allows us to estimate the Riesz potential operator

$$I_\lambda f(x) = \int |x-y|^{-\lambda} f(y)\,dy.$$

In fact, for $1/p - 1/q + \lambda/n = 1$ and $1 < p, q < \infty$ we have the following Adam's estimate

$$\|I_\lambda(f)\|_{L^q} \le C\|f\|_{L^p}. \tag{13.2.9}$$

The Sobolev identity shows that

$$u(x) = u^I(x) + u^{II}(x),$$

where

$$u^I(x) = \sum_{|\beta|<l} \delta^{-n-|\beta|} x^\beta \int_{|y|\le\delta} \varphi_\beta(y/\delta) u(y) dy.$$

Let us assume u is a smooth function with a compact support inside the ball of radius R and center at 0. Taking $\delta = R$ and $D = 2R$ in the Sobolev identity we get

$$\|u^I\|_{L^q(|y|\le R)} \le C \sum_{|\beta|<l} R^{-n-|\beta|} \left(\int_{|y|\le R} |y|^{\beta q} dy\right)^{1/q} \int |u(y)| dy.$$

Applying the Hölder inequality, we get

$$\|u^I\|_{L^q(|y|\le R)} \le C R^{-n/p+n/q} \|u\|_{L^p},$$

where $C > 0$ is a constant independent of R. To estimate u^{II} we start with the estimate

$$|u^{II}(x)| \le C \sum_{|\alpha|=l} I_{n-l}(|\partial^\alpha f|)(x).$$

Now we are in situation to apply Adams estimate (13.2.9) and get

$$\|u^{II}\|_{L^q} \le C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p}$$

provided $1/q + 1 = 1/p + (n-l)/n$, i.e. for $1/p - 1/q = l/n$. Hence, we get

$$\|u\|_{L^q} \le C R^{-n/p+n/q} \|u\|_{L^p} + C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p},$$

where $C > 0$ is independent of $R > 0$. Taking $R \to \infty$, we get

$$\|u\|_{L^q(\mathbb{R}^n)} \le C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \tag{13.2.10}$$

provided u is a smooth compactly supported function and $l/n = 1/p - 1/q$.

Taking the closure of the space of smooth compactly supported functions with respect to the norm in W^l_p, we complete the proof of the Sobolev inequality (13.2.3).

We shall close this section with the following Sobolev inequality

$$\|f\|_{L^\infty} \le C \|f\|_{W^s_p} \tag{13.2.11}$$

for $1 < p < \infty, 1/p < s/n$. An easy proof can be found when $p = 2$ by the aid of the Fourier transform and Plancherel identity. For the general case, one can use the Sobolev identity and the Hölder inequality.

Lemma 13.2.6. *(Hardy, Littlewood) Let $m(\xi) = c|\xi|^{-t}$ and consider the operator*

$$I(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} m(\xi) \hat{f}(\xi) d\xi.$$

Prove that for $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q = t/n$ this operator can be extended as a bounded operator from L^p to L^q.

On the other hand, for any real s one can define the Sobolev space $H^s(\mathbf{R}^n)$ as completition of the space of smooth functions with compact support with respect to the norm

$$\|f\|_{H^s} = \|(1-\Delta)^{s/2} f\|_{L^2} \tag{13.2.12}$$

Applying the Plancherel identity, we see that this norm is equivalent to the following one

$$\|(1+|\xi|^2)^{s/2} \hat{f}\|_{L^2}.$$

It is clear that this is a Hilbert space with scalar product

$$(f,g)_{H^s} = \int (1+|\xi|^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi.$$

We have the following property

$$H^k = W_2^k.$$

Moreover, the dual space of H^s is H^{-s}.

Now we can formulate a result for existence and uniqueness of higher regularity solutions of the Cauchy problem

$$\begin{aligned} &(-\partial_t^2 + \Delta - M^2) u = 0, \\ &u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x) \end{aligned} \tag{13.2.13}$$

for the linear Klein-Gordon equation. Namely, if the initial data $f = f_0 \times f_1$ belongs to the Hilbert space

$$H^s \times H^{s-1}$$

with $s \geq 1$, then the Cauchy problem (13.2.13) has a unique solution

$$u(t,.) \in \cap_{m=0}^{[s]} C^m([0,\infty); H^{s-m}),$$

where $[s]$ denotes the integer part of the real number s.

13.2.5 Gagliardo - Nirenberg inequality

Another useful interpolation inequality is the following variant of Gagliardo - Nirenberg inequality.

Lemma 13.2.7. *Suppose* $f \in W^k_{q/k}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then for any integer* $k \geq 1$ *and any* $q \geq k$ *we have*

$$\|f\|_{W^{k-1}_{q/(k-1)}(\mathbb{R}^n)} \leq C\|f\|^{1-1/k}_{W^k_{q/k}(\mathbb{R}^n)} \|f\|^{1/k}_{L^\infty(\mathbb{R}^n)}.$$

Proof. Let $\varphi(x)$ be a smooth compactly supported function such that $\varphi(x) = 1$ near $x = 0$. Then for any $q > 1$ and any integer $k \geq 0$ we know that

$$\varphi(x/R)f(x)$$

tends to $f \in W^k_q$ as R tends to infinity. Moreover, for $f \in L^\infty$ we have the uniform estimate

$$|\varphi(x/R)f(x)| \leq C|f(x)|.$$

This argument shows that without loss of generality we can assume f is compactly supported. We can reduce the proof to the case f is smooth compactly supported function, using Friedrich's molifiers.

Moreover, the simple estimate

$$\|f\|_{L^{q/(k-1)}} \leq C\|f\|^{1-1/k}_{L^{q/k}} \|f\|^{1/k}_{L^\infty}$$

enables us to reduce the proof to the case $n = 1$. Then we shall proceed by means of induction with respect to k.

We shall consider in details only the case

$$q \geq 2k$$

and shall give only the idea for the case $k \leq q < 2k$.

The identity

$$|f^{(k)}(x)|^{q/k} = f^{(k)}(x)|f^{(k)}(x)|^{-2+q/k} f^{(k)}(x)$$

and integration by parts give

$$\left|\int_{-\infty}^{\infty} |f^{(k)}(x)|^{q/k} dx\right| \leq$$
$$C\int_{-\infty}^{\infty} |f^{(k+1)}(x)||f^{(k)}(x)|^{-2+q/k}|f^{(k-1)}(x)|dx.$$

Applying the Hölder inequality, we get

$$\|f^{(k)}\|_{L^{q/k}}^{q/k} \le C\|f^{(k+1)}\|_{L^{q/(k+1)}}\|f^{(k)}\|_{L^{q/k}}^{-2+q/k}\|f^{(k-1)}\|_{L^{q/(k-1)}}.$$

Applying the inductive assumption for $k-1$, we get the desired estimate for k.

For the case $k \le q < 2k$ we use the relation

$$|f^{(k)}(x)|^{q/k} = |f(x)|^A |f^{(k)}(x)|^{q/k} |f(x)|^{-A}$$

and applying Hölder inequality, we get an estimate of type

$$\int |f^{(k)}(x)|^{q/k} dx \le C(\int |f(x)|^{Ar} dx)^{1/r} (\int |f(x)|^{-As} |f^{(k)}(x)|^2 dx)^{1/s},$$

where $s = 2k/q$ and $r = 2k/(2k-q)$. Further, we make an integration by parts in

$$\int |f(x)|^{-sA} |f^{(k)}(x)|^2 dx$$

and use the inductive argument as it was done above in the case $q \ge 2k$.

This completes the proof of the Lemma.

The above estimate has few interesting corollaries.

Corollary 13.2.1. *Suppose $f \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \ge 1$. Then for any integers $k > l \ge 1$ we have*

$$\|f\|_{W^l_{pk/l}(\mathbb{R}^n)} \le C\|f\|_{W_p^k(\mathbb{R}^n)}^{l/k} \|f\|_{L^\infty(\mathbb{R}^n)}^{(k-l)/k}.$$

Corollary 13.2.2. *Suppose $f, g \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \ge 1$. Then we have*

$$\|fg\|_{W_p^k(\mathbb{R}^n)} \le C(\|f\|_{W_p^k(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{W_p^k(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)})$$

Proof. It is sufficient to apply Leibniz rule in combination with the estimate of the previous Corollary.

Corollary 13.2.3. *Suppose $f \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \ge 1$. Then for any $\lambda > k$ we have*

$$\||f|^\lambda\|_{W_p^k(\mathbb{R}^n)} \le C\|f\|_{W_p^k(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)}^{\lambda-1}.$$

Proof. It is sufficient to apply induction with respect to k in combination with Gagliardo - Nirenberg inequality.

In fact, due to Leibniz rule we have

$$\partial_x^\alpha F(f) = \sum c_{\alpha_1,...,\alpha_m} F^m(f) \partial_x^{\alpha_1} f, ..., \partial_x^{\alpha_m} f,$$

where the sum is over $m = 0, 1, ..., |\alpha|$ and $|\alpha_1| + ... + |\alpha_m| = |\alpha|$. Then with $|\alpha| = k$ we get

$$\| |f|^\lambda \|_{W_p^k(\mathbf{R}^n)} \leq \sum \| |f|^{\lambda - m} \|_{L^\infty} \| f \|_{W^{|\alpha_1|}_{pk/|\alpha_1|}} ... \| f \|_{W^{|\alpha_m|}_{pk/|\alpha_m|}}$$

provided

$$|\alpha_1|/pk + ... + |\alpha_m|/pk = 1/p.$$

Applying Corollary 13.2.1, we get the desired estimate.

This completes the proof.

13.2.6 Fractional powers of operators and some integral representation

We start with the identity

Lemma 13.2.8.

(13.2.1)
$$\int_0^\infty \frac{t^{s-1} dt}{1+t} = \frac{\pi}{\sin \pi s}$$

for any $s \in (0,1)$.

Proof. Using the integral representation of the Gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt,$$

we get the representation

(13.2.2)
$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-At} dt, \forall s \in (0,1), A > 0.$$

Using the relation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, s \in (0,1),$$

we find

$$\int_0^\infty \frac{t^{s-1} dt}{1+t} = \int_0^\infty t^{s-1} \int_0^\infty e^{-\lambda(1+t)} d\lambda dt =$$

$$\int_0^\infty \int_0^\infty \left(t^{s-1} e^{-\lambda t} \right) dt e^{-\lambda} d\lambda = \int_0^\infty \int_0^\infty \left(\tilde{t}^{s-1} e^{-\tilde{t}} \right) d\tilde{t} \lambda^{-s} e^{-\lambda} d\lambda =$$

$$= \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

□

Make change of variables $t = \tau^2$ and get

$$I \equiv \int_0^\infty \frac{t^{s-1}dt}{1+t} = 2\int_0^\infty \frac{\tau^{2s-1}d\tau}{1+\tau^2}.$$

For any real s one can define the Sobolev space $H_p^s(\mathbf{R}^n)$ as completition of the space of smooth functions with compact support with respect to the norm

$$\|f\|_{H_p^s} = \|(1-\Delta)^{s/2}f\|_{L^p}. \tag{13.2.3}$$

Chapter 14

Sobolev spaces $H^s(\Omega)$ in domains

Let Ω be an open domain in R^n with C^2 boundary $\partial\Omega$. Sobolev spaces on manifolds $H^s_p(\Omega), s \in \mathbb{R}, 1 < p < \infty$, can be defined similar to corresponding Euclidean spaces $H^s_p(\mathbb{R}^n)$, usually characterized via

$$H^s_p = (\mathrm{Id} - \Delta)^{-s/2} L_p$$

by replacing the Euclidean Laplacian Δ with the Laplace-Beltrami operator on (M, g) and using an auxiliary parameter ρ, see Section [3.1] The spaces $H^s_p(M)$ were introduced and studied in detail in [Str83] and generalize in a natural way classical Sobolev spaces on manifolds, $W^k_p(M)$, which contain all L_p functions on M having bounded covariant derivatives up to order $k \in \mathbb{N}$, cf. [Aub 76, Aub 82]. To avoid any confusion, let us emphasize that in this article we study exactly these fractional Sobolev spaces $H^s_p(M)$ defined by means of powers of Δ. But we shall use an alternative characterization of these spaces on manifolds with bounded geometry as definition - having in mind the proof of our main theorem. To be more precise, on manifolds with bounded geometry, see Definition [18, one can alternatively define fractional Sobolev spaces $H^s_p(M)$ via localization and pull-back onto $\mathbb{R}^n$, by using geodesic normal coordinates and corresponding fractional Sobolev spaces on $\mathbb{R}^n$, cf. [Tri92, Sections 7.2.2, 7.4.5] and also [Skr98, Definition 1]. Unfortunately, for some applications the choice of geodesic normal coordinates is not convenient, which is why we do not wish to restrict ourselves to these coordinates only. The main application we have in mind are traces on submanifolds N of M. But also for manifolds with symmetries, product manifolds or warped products, geodesic normal coordinates may not be the first and natural choice and one is interested in coordinates better suited to the problem at hand.

Therefore, we introduce in Definition [11 Sobolev spaces $H^{s,\mathcal{T}}_p(M)$ in a more general way, containing all those complex-valued distributions f on M such that

$$\|f\|_{H_p^{s,T}} := \left(\sum_{\alpha\in I} \left\|\left(h_\alpha f\right)\circ\kappa_\alpha\right\|^p_{H_p^s(\mathbb{R}^n)}\right)^{1/p}$$

is finite, where $\mathscr{T} = (U_\alpha, \kappa_\alpha, h_\alpha)_{\alpha\in I}$ denotes a trivialization of M consisting of a uniformly locally finite covering U_α, local coordinates $\kappa_\alpha : V_\alpha \subset \mathbb{R}^n \to U_\alpha \subset M$ (not necessarily geodesic normal coordinates) and a subordinate partition of unity h_α. Of course, the case of local coordinates κ_α being geodesic normal coordinates is covered but we can choose from a larger set of trivializations. Clearly, we are not interested in all $\mathscr{T}$ but merely the so called admissible trivializations $\mathscr{T}$, cf. Definition 12, yielding the coincidence

$$H_p^{s,\mathscr{T}}(M) = H_p^s(M)$$

cf. Theorem 14 As pointed out earlier, our main applications in mind are Trace Theorems. In [Skr90, Theorem 1], traces on manifolds were studied using the Sobolev norm (?) with geodesic normal coordinates. since these coordinates in general do not take into account the structure of the underlying submanifold where the trace is taken, one is limited to so-called geodesic submanifolds. This is highly restrictive, since geodesic submanifolds are very exceptional. Choosing coordinates that are more adapted to the situation will immediately enable us to compute the trace on a much larger class of submanifolds. In particular, we consider Riemannian manifolds (M, g) with submanifolds N such that (M, N) is of bounded geometry, see Definition [18, i.e., (M, g) is of bounded geometry, the mean curvature of N and its covariant derivatives are uniformly bounded, the injectivity radius of $\left(N, g_N\right)$ is positive and there is a uniform collar of N. The coordinates of choice for proving Trace Theorems are Fermi coordinates, introduced in Definition [20] We show in Theorem [26] that for a certain cover with Fermi coordinates there is a subordinated partition of unity such that the resulting trivialization is admissible. The main Trace Theorem itself is stated in Theorem [27, where we prove that if M is a manifold of dimension $n \geq 2, N$ a submanifold of dimension $k < n$, and (M, N) of bounded geometry, we have for $s > \frac{n-k}{p}$

$$\operatorname{Tr}_N H_p^s(M) = B_{p,p}^{s-\frac{n-k}{p}}(N)$$

i.e., there is a linear, bounded and surjective trace operator Tr $_N$ with a linear and bounded right inverse Ex_M from the trace space into the original space such that $\mathrm{Tr}_N \circ \mathrm{Ex}_M = \mathrm{Id}$, where Id denotes the identity on operator N. The spaces on the right hand side of (?) are Besov spaces obtained via real interpolation of the spaces H_p^s, cf. Remark [7. When just asking for Tr_N to be linear and bounded, one can reduce the assumptions on (M,N) further by replacing the existence of a collar of N with a uniform local collar, cf. Remark

cf. [Tri92, Section 1.3.2]. In particular, for $k \in \mathbb{N}_0$, these spaces coincide with the classical Sobolev spaces $W_p^k(\mathbb{R}^n)$

$$H_p^k\left(\mathbb{R}^n\right) = W_p^k\left(\mathbb{R}^n\right), \quad \text{i.e.,} \quad H_p^0\left(\mathbb{R}^n\right) = L_p\left(\mathbb{R}^n\right)$$

usually normed by

$$\|f\|_{W_p^k(\mathbb{R}^n)} = \left(\sum_{|a|\le k} \left\|\mathrm{D}^a f\right\|_{L_p(\mathbb{R}^n)}^p\right)^{1/p}$$

Furthermore, Besov spaces $B_{p,p}^s(\mathbb{R}^n)$ can be defined via interpolation of Sobolev spaces. In particular, let $(\cdot,\cdot)_{\Theta,p}$ stand for the real interpolation method, cf. [Tri92, Section 1.6.2]. Then for $s_0, s_1 \in \mathbb{R}, 1 < p < \infty$ and $0 < \Theta < 1$, we put $B_{p,p}^s(\mathbb{R}^n) := \left(H_p^{s_0}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n)\right)_{\Theta,p}$, where $s = \Theta s_0 + (1-\Theta)s_1$. Note that $B_{p,p}^s(\mathbb{R}^n)$ does not depend on the choice of s_0, s_1, Θ The following lemma about pointwise multipliers and diffeomorphisms may be found in [Tri92, Sections 4.2,4.3], where it was proven in a more general setting.

Lemma 14.0.1. *Lemma 1. Let* $s \in \mathbb{R}$ *and* $1 < p < \infty$. *(i) Let* $f \in H_p^s(\mathbb{R}^n)$ *and* φ *a smooth function on* $\mathbb{R}^n$ *such that for all a with* $|\mathfrak{a}| \le [s]+1$ *we have* $\left|\mathrm{D}^a\varphi\right| \le C_{|a|}$. *Then there is a constant C only depending on s, p, n and* $C_{|a|}$ *such that*

$$\|\varphi f\|_{H_p^s(\mathbb{R}^n)} \le C\|f\|_{H_p^s(\mathbb{R}^n)}$$

(ii) Let $f \in H_p^s(\mathbb{R}^n)$ *with* $\operatorname{supp} f \subset U \subset \mathbb{R}^n$ *for U open and let* $\kappa : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$ *be a diffeomorphism such that for all a with* $|\mathfrak{a}| \le [s]+1$ *we have* $|\mathrm{D}^a\kappa| \le C_{|a|}$. *Then there is a constant C only depending on s, p, n and* $C_{|a|}$ *such that*

$$\|f\circ\kappa\|_{H_p^s(\mathbb{R}^n)} \le C\|f\|_{H_p^s(\mathbb{R}^n)}$$

Vector-valued function spaces on $\mathbb{R}^n$. Let $\mathscr{D}(\mathbb{R}^n,\mathbb{F}^r)$ be the space of compactly supported smooth functions on $\mathbb{R}^n$ with values in $\mathbb{F}^r$ where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and $r \in \mathbb{N}$. Let $\mathscr{D}'(\mathbb{R}^n,\mathbb{F}^r)$ denote the corresponding distribution space. Then, $H_p^s(\mathbb{R}^n,\mathbb{F}^r)$ is defined in correspondence with $H_p^s(\mathbb{R}^n)$ from above, cf. [Triebel, Fractals and spectra, Section 15]. Moreover, Besov spaces $B_{p,p}^s(\mathbb{R}^n,\mathbb{F}^r)$ are defined as the spaces $B_{p,p}^s(\mathbb{R}^n)$ from above; $B_{p,p}^s(\mathbb{R}^n,\mathbb{F}^r) := \left(H_p^{s_0}(\mathbb{R}^n,\mathbb{F}^r), H_p^{s_1}(\mathbb{R}^n,\mathbb{F}^r)\right)_{\Theta,p}$ where $(\cdot,\cdot)\Theta,p$ again denotes the real interpolation method with $s_0, s_1 \in \mathbb{R}, 1 < p < \infty$, and $0 < \Theta < 1$ with $s = \Theta s_0 + (1-\Theta)s_1$ Lemma 2. The norms $\|\varphi\|_{H_p^s(\mathbb{R}^n,\mathbb{F}^r)}$ and $\left(\sum_{i=1}^r \|\varphi_i\|_{H_p^s(\mathbb{R}^n,\mathbb{F})}^p\right)^{\frac{1}{p}}$ are equivalent where $\varphi = \left(\varphi_1,\ldots,\varphi_r\right) \in H_p^s(\mathbb{R}^n,\mathbb{F}^r)$. The analogous statement is true for Besov spaces.

Bibliography

[1] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, AND H. HOLDEN, *Solvable Models in Quantum Mechanics*, Texts and Monographs in Physics, Springer-Verlag, New York, 1988.

[2] H. Bateman and A.Erdelyi, Higher transcendental functions, Vol.1 and Vol. 2, *Mc Graw-Hill Company, INC*, New York, Toronto, London, 1953 .

[3] J. Bergh and J. Löfström, Interpolation spaces, *Springer* Berlin, Heidelberg, New York, 1976.

[4] Ph. Brenner, $L^p - L^{p'}$ estimates for Fourier integral operators related to hyperbolic equations, *Math. Z.* **152** (1977) 273 - 286.

[5] H. Brezis, Analyse Functionelle - Theorie et applications, Masson Editeur, Paris, 1983.

[6] Y.Choquet - Bruhat and D.Christodoulou, Elliptic $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity *Acta Matematica* **146** (1981) p.129 - 150.

[7] Y. Choquet - Bruhat, C. De Witt - Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics, North - Holland, Amsterdam, 1982.

[8] R.Coifman and Y. Meyer, Au-delá des operateurs pseudo-différentiels, *Asterisques 57, Société de Mathématique de France* 1973

[9] D.Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.* **39** (1986), no. 2, 267–282.

[10] H. O. CORDES, *The Tecnique of Pseudodifferential Operators,* London Mathematical Society Lectures Note Series, **202**, Cambridge University Press, Cambridge (1995).

[11] I.Gelfand, G.Shilov, Generalized functions, vol I, Properties and Operations, Academic Press, 1964, New York.

[12] I. Gelfand, M. Graev and Ya. Vilenkin, Generalized Functions, vol. V, Integral Geometry and Representation Theory, Academic Press, 1966, New York.

[13] V.Georgiev, Lecture notes on Harmonic Analysis, 2019.

[14] V.Georgiev, Une estimation avec poids pour l'equation des ondes, *C.R.Acad.Sci. Paris* **322, Serie I** (1996) 829 – 834.

[15] V.Georgiev, Existence of global solutions to supercritical wave equations *Serdica Math. J.* **22** (1996) 125-164.

[16] V.Georgiev, H.Lindblad and C. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations, *Amer. J. Math.* **119 (6)** (1997) 1291 – 1319.

[17] V.Georgiev and S. Lucente, Decay for nonlinear Klein – Gordon equation, preprint 1999.

[18] J. Ginibre and G. Velo, The global Cauchy problem for the Nonlinear Klein - Gordon Equation, Math. Z. **189** (1985), 487 - 505.

[19] J.Ginibre and G.Velo, Generalized Strichartz inequalities for the wave equation, J. Func. Anal. **133** (1995), 50 - 68.

[20] R. Glassey, Existence in the large for $\Box u = F(u)$ in two dimensions *Math. Z.* **178** (1981) 233–261.

[21] N.Hayashi and P. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree Equations, *Amer. J.Math* **120** (1998) 369 – 391.

[22] S. Helgason, Groups and geometrical transformations, Academic Press, New York , 1984

[23] L. Hörmander, The analysis of linear partial differential operators, vol. I, Distribution Theory and Fourier analysis, Springer - Verlag, Berlin, New York, Tokyo 1983.

[24] L.Hörmander, The analysis of linear partial differential operators, vol. III, Pseudodifferential Operators, Springer - Verlag, Berlin, New York, Tokyo, 1985.

[25] L.Hörmander, On Sobolev spaces associated with some Lie algebras, Report No. 4, *Instutut Mittag Leffler*(1985)

[26] L. Hörmander, Lectures on Nonlinear Hyperbolic Diffferential Equations, Mathematiques et Applications 26 *Springer* (1997).

[27] F.John, Blow-up for quasi – linear wave equation in three space dimensions. *Comm. Pure Appl. Math.* **34** (1981), 29 – 51.

[28] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979) 235–265.

[29] T.Kato and G.Ponce, Commutator estimates and the Euler and Navier-Stokes equations, CPAM, **41** (1988) p. 891 - 907.

[30] S. Klainerman, Global existence of small amplitude solutions to non-linear Klein – Gordon equations in four space time dimensions, *Comm. Pure Appl. Math* **38** (1985), 631 – 641.

[31] S. Klainerman, Remarks on the global Sobolev inequalities in the Minkowski space $\mathbf{R}^{n+1}$. *Comm. Pure Appl. Math* **40** (1987), 111 – 117.

[32] H. Kubo, On the critical decay and power for semilinear wave equations in odd space dimensions *preprint*

[33] P. Lax and R. Philips, Scattering Theory, Academic Press, New York, 1967.

[34] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations *J. Funct. Anal.* **130** (1995) 357–426.

[35] H. Lindblad and C. D. Sogge, Long - time existence for small amplitude semilinear wave equation *Amer. J. Math.* **118** (1996) 1047–1135.

[36] H. Lindblad and C. D. Sogge, Restriction theorems and semilinear wave equations in $(1+3)$-dimensions *Duke Math. J.* **85, 1** (1996), 227-252 .

[37] Jacques-Louis Lions, and Enrico Magenes *Problemes aux limites non homogenes et applications.* Dunod, Paris, 1968.

[38] V. Maz'ya, Sobolev Spaces, Edition of Leningrad Univ., Leningrad, 1985.

[39] Benjamin Muckenhoupt, *Weighted norm inequalities for the Fourier transform Transactions A.M.S.*, 276:729–742, 1983.

[40] J. Peetre, Zur Interpolation von Operatorenräumen, *Arch. Math.* **21** (1970), 601 -608.

[41] J. Peetre, On the connection between the theory of interpolation spaces and approximation theory, *Proc. Conf. Constructive Theory Func. Budapest 1969*, Akademiai Kiado, Budapest 1971, 351-363.

[42] G.Ponce and T.Sideris, Local regularity of nonlinear wave equations in three space dimensions, *Comm. Part. Diff. Equations* **18** (1993) 169 -177.

[43] R. Racke, Lectures on nonlinear evolution equations. Initial value problems. *Aspects of Mathematics,* Friedr. Vieweg & Sohn, Braunschweig, 1992. viii+259 pp.

[44] M.Reed and B.Simon, Methods of Modern Mathematical Physics, vol. I, Functional Analysis, Academic Press, New York 1975.

[45] M. Reed and B.Simon, Methods of Modern Mathematical Physics, vol. II, Fourier Analysis and Self Adjointness, 1975 Academic Press, New York, San Francisco , London .

[46] W.Rudin, Functional analysis, McGraw-Hill Book Company, 1973, New York.

[47] T.Runst, Mapping properties of non-linear operators in spaces of Triebel-Lizorkin and Besov type, Anal. Math. **12** (1986) 313-346.

[48] T. RUNST AND W. SINCKEL, *Sobolev Spaces of Fractional Order, Nemitskij Operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin (1996).

[49] J. Schaeffer, The equation $\Box u = |u|^p$ for the critical value of p, *Proc. Royal Soc. Edinburgh* **101** (1985) 31–44.

[50] J. Shatah, M. Struwe, Geometric wave equations. *Courant Lecture Notes in Mathematics,* 2. New York University, Courant Institute of Mathematical Sciences, New York, 1998.

[51] T. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions *Comm. Partial Diff. Equations* **12** (1987) 378–406.

[52] I.Sigal, Nonlinear semi – groups. *Ann. of Math.* **78** No. 2 (1963) 339 – 364.

[53] E. Stein, Interpolation of linear operators, *Trans. Amer. Math. Soc.* **83** (1956) 482–492.

[54] E.Stein, Singular integrals and differential properties of functions, Princeton Mathematical Series, Princeton Univ. Press, Princeton 1970.

[55] W. Strauss, Nonlinear scattering theory, in Scattering theory in mathematical physics *Reidel* Dordrect (1979) 53– 79.

[56] W.Strauss, Nonlinear scattering at low energy, *J. Funct. Anal.* **41** (1981) 110-133.

[57] R. Strichartz, A priori estimates for the wave equation and some applications *J. Funct. Analysis* **5** (1970) 218–235.

[58] R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations *Duke Math. J.* **44** (1977) 705–714.

[59] R. Strichartz, Harmonic analysis as Spectral theory of Laplacians. *Journal of Functional Analysis* **87** (1989) 51 – 148.

[60] R.Strichartz, Multipliers on fractional Sobolev spaces, *J. Math. and Mech.,* **16**, 9, (1967), 1031-1060.

[61] M.Taylor, Partial Differential Equations I, Basic Theorey, Springer - Verlag, 1996, New York.

[62] M.Taylor, Pseudodifferential operators, Princeton Mathematical Series, Princeton Univ. Press, Princeton 1981.

[63] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation, preprint.

[64] H. Triebel, Interpolation theory, functional spaces, differential operators, *VEB Deutscher Verlag der Wissenschaften*, Berlin, 1978.

[65] W. von Wahl, L^p decay rates for homogeneous wave equation, Math. Z. **120** (1971) 93 – 106.

[66] Y. Zhou, Cauchy problem for semilinear wave equations with small data in four space dimensions *J. Diff. Equations* **8** (1995) 135–144.

[67] Eb. Zeidler, Applied Functional Analysis, Applications to Mathematical Physics, series: Applied Math. Siences, vol. 108, (Springer - Verlag, New York) 1995.