Elements of Partial Differential Equations

Elliptic equations

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Chapter 1

1 - D Maximum principle

1.1 Introduction

Let the operator L is defined by

$$L(u)(x) = -u'' + V(x)u'(x).$$

The we can consider the Sturm problem

$$L(u)(x) = f(x), a < x < b. \tag{1.1.1}$$

Here and below $V(x)$ is $C[a,b]$ function. Given any $f \in C(a,b)$ we look for classical solutions to (1.1.1) $u \in C[a,b] \cap C^2(a,b)$.

1.2 Easy and weak maximum principles

Lemma 1.2.1. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (1.1.1), then we have the following properties*

a) *(EASY MIN principle) if $f(x)$ is continuous POSITIVE function ($f(x) > 0$ for any $x \in (a,b)$), then*

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

b) *(EASY MAX principle) if $f(x)$ is continuous NEGATIVE function ($f(x) > 0$ for any $x \in (a,b)$), then*

$$\max(u(a), u(b)) = \max_{[a,b]} u(x).$$

Proof. We shall prove b) only. If

$$u(c) = \max_{[a,b]} u(x)$$

for some $c \in (a,b)$, then in the point c we have

$$u'(c) = 0, u''(c) \leq 0.$$

□

Lemma 1.2.2. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (1.1.1), then we have the following properties*

a) *(WEAK MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in (a,b)$), then*

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

b) *(WEAK MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in (a,b)$), then*

$$\max(u(a), u(b)) = \max_{[a,b]} u(x).$$

Remark 1.2.1. *A function satisfying $L(u)(x) \leq 0$ is called a subsolution. We are thus asserting a subsolution attains its maximum on the boundary of $[a,b]$. Similarly, if $L(u)(x) \leq 0$ holds, u is a supersolution and attains its minimum on the boundary of $[a,b]$.*

Proof. We shall prove only the assertion a). We shall modify $u(x)$ as follows

$$w_\varepsilon(x) = u(x) - \varepsilon z(x),$$

where

$$z(x) = e^{Rx}.$$

Then w_ε tends uniformly to u, so $\min_{[x_1,x_2]} w_\varepsilon(x)$ tends to $\min_{[x_1,x_2]} u(x)$ for any interval $[x_1, x_2] \subseteq [a,b]$.

Choose $R > 0$ so large that

$$L(w_\varepsilon) \geq -\varepsilon z'' + \varepsilon V(x) z'(x) = \varepsilon(R^2 - VR)e^{Rx} > 0$$

for any $x \in (a,b)$. Then the Easy MIN principle implies

$$\min(w_\varepsilon(a), w_\varepsilon(b)) = \min_{[a,b]} w_\varepsilon(x).$$

Taking the limit as $\varepsilon > 0$ tends to zero we get

$$\min(u(a), u(b)) = \min_{[a,b]} u(x).$$

□

1.3 Hopf lemma and strong maximum principle

In order to prepare strong maximum principle we shall start with Hopf Lemma

Lemma 1.3.1. *(Hopf's Lemma). Assume* $u \in C^2(a,b)) \cap C^1([a,b])$, *and*

$$Lu = f \leq 0 \quad in\ (a,b).$$

Then we have

a) *if*

$$u(b) > u(x) \quad \text{for all } x \in (a,b) \tag{1.3.2}$$

then

$$u'(b) > 0.$$

b) *if*

$$u(a) > u(x) \quad \text{for all } x \in (a,b) \tag{1.3.3}$$

then

$$u'(a) < 0.$$

Proof. We shall prove only a). Without loss of generality we can assume

$$a < 0 < b.$$

Then we take

$$w_\varepsilon(x) = u(x) + \varepsilon z(x),$$

where

$$z(x) = e^{-Rx^2} - e^{-Rb^2}$$

Then $z(b) = 0$ and

$$L(z)(x) = e^{-Rx^2}\left(-4R^2x^2 + 2R - VxR\right) < 0$$

for R big enough and $x \in (b/2, b)$. In view of (1.3.3) we can find $\varepsilon > 0$ so that

$$w_\varepsilon(x_0) = u(x_0) > u(b/2) + \varepsilon z(b/2) = w_\varepsilon(b/2).$$

We have also

$$w_\varepsilon(x_0) = u(x_0) \geq u(b) + \varepsilon z(b) = w_\varepsilon(b).$$

Applying the weak maximum principle for $w_\varepsilon(x) - w_\varepsilon(b)$ and the interval $[b/2, b]$ and we find

$$\max_{[b/2,b]} w_\varepsilon - w_\varepsilon(b) \leq 0$$

and hence

$$w_\varepsilon'(b) \geq 0.$$

In this way we find

$$u'(b) \geq -\varepsilon z'(b) = \varepsilon R b e^{-Rb^2} > 0.$$

□

Lemma 1.3.2. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of (1.1.1), then we have the following properties*

a) *(STRONG MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in (a,b)$) and f has minimum in internal point $x_0 \in (a,b)$, then is a constant;*

b) *(STRONG MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in (a,b)$)and f has maximum in internal point $x_0 \in (a,b)$, then is a constant.*

Proof. We shall prove only b). Set

$$M := \max_{[a,b]} u$$

and assuming u is not a constant we can decompose (a,b) as

$$(a,b) = C \cup V,$$

where

$$C := \{x \in (a,b) \mid u(x) = M\},$$

$$V := \{x \in (a,b) \mid u(x) < M\}.$$

Since V is an open set, it is a union of open intervals and we can take such interval $(\alpha, \beta) \subset V$ so that $\beta \in C$. In this way we can apply Hopf lemma and deduce $u'(\beta) > 0$ and this is a contradiction with the fact that $\beta \in (a,b)$ is maximum point. The contradiction shows that V is empty. □

Problem 1.3.1. *If $u \in C[a,b] \cap C^2(a,b)$ is a solution of*

$$u'' + V(x)u'(x) + W(x)u(x) = f(x), a < x < b, \tag{1.3.4}$$

$V(x)$ and $W(x)$ are $C[a,b]$ functions, $W(x) \leq 0$, $f(x)$ is any bounded NON-NEGATIVE function and if

$$u(c) = \max_{[a,b]} u(x)$$

is POSITIVE for some $c \in (a,b)$, then $u(x)$ is a constant.

Problem 1.3.2. *Show that the condition* $W(x) \leq 0$ *in the Problem (1.3.1) can not be removed.*

Problem 1.3.3. *Show that the condition* $u(c)$ *is POSITIVE in the Problem (1.3.1) can not be removed.*

Problem 1.3.4. *If* $u \in C[a,b] \cap C^2(a,b)$ *is a solution of (2.1.1)* $V(x)$ *and* $W(x)$ *are bounded functions,* $W(x) \leq 0$*,* $f(x)$ *is any bounded NON-NEGATIVE function and if*

$$u(a) = u(b) = 0,$$

then $u(x) < 0$ *in* (a,b) *or* $u(x) = 0$.

Chapter 2

Maximum principle in domains

2.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain with boundary $\partial\Omega$.

Let the operator L be defined by

$$L(u)(x) = -\Delta u(x) + \sum_{j=1}^{n} V_j(x)\partial_{x_j} u(x).$$

Then we can consider the problem

$$L(u)(x) = f(x), x \in \Omega. \tag{2.1.1}$$

Here and below $V_j(x)$ are $C(\overline{\Omega})$ function. Given any $f \in C(\overline{\Omega})$ we look for classical solutions to (2.1.1) $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

For simplicity we shall concentrate in this chapter to the case $n = 3$.

2.2 Easy and weak maximum principles

Lemma 2.2.1. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (2.1.1), then we have the following properties*

a) *(EASY MIN principle) if $f(x)$ is continuous POSITIVE function ($f(x) > 0$ for any $x \in \Omega$), then*

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

b) *(EASY MAX principle) if $f(x)$ is continuous NEGATIVE function ($f(x) > 0$ for any $x \in \Omega$), then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u(x).$$

Proof. We shall prove b) only. If

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

for some $x_0 \in \Omega$, then in the point x_0 we have

$$\nabla u(x_0) = 0, \Delta u(x_0) \leq 0.$$

□

Lemma 2.2.2. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (2.1.1), then we have the following properties*

a) *(WEAK MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \geq 0$ for any $x \in \Omega$), then*

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

b) *(WEAK MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \leq 0$ for any $x \in \Omega$), then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u.$$

Remark 2.2.1. *A function satisfying $L(u)(x) \leq 0$ is called a subsolution. We are thus asserting a subsolution attains its maximum on the boundary of Ω. Similarly, if $L(u)(x) \leq 0$ holds, u is a supersolution and attains its minimum on the boundary of Ω.*

Proof. We shall prove only the assertion a). We shall modify $u(x)$ as follows

$$w_\varepsilon(x) = u(x) - \varepsilon z(x),$$

where

$$z(x) = e^{Rx_1}.$$

Then w_ε tends uniformly to u, so $\min_{\overline{U}} w_\varepsilon(x)$ tends to $\min_{\overline{U}} u(x)$ for any open $U \subseteq \Omega$.

Choose $R > 0$ so large that

$$L(w_\varepsilon) \geq -\varepsilon\Delta z + \varepsilon \sum_{j=1}^{n} V_j(x)\partial_{x_j} z(x) = \varepsilon(R^2 - VR)e^{Rx_1} > 0$$

for any $x \in \Omega$. Then the Easy MIN principle implies

$$\min_{\partial\Omega} w_\varepsilon = \min_{\overline{\Omega}} w_\varepsilon(x).$$

Taking the limit as $\varepsilon > 0$ tends to zero we get

$$\min_{\partial\Omega} u = \min_{\overline{\Omega}} u.$$

□

2.3 Hopf lemma and strong maximum principle

In order to prepare strong maximum principle we shall start with Hopf's Lemma. For this we shall assume the boundary of the domain Ω satisfies the ball condition, i.e.

$$\text{(H1)} \qquad \begin{cases} \text{for any } x_0 \in \partial\Omega \text{ there exists a ball } B \subset \Omega \\ \text{so that } x_0 \in \partial B \end{cases}$$

Lemma 2.3.1. *(Hopf's Lemma). Assume* (H1)*, $u \in C^2(\Omega)) \cap C^1(\overline{\Omega})$, and*

$$Lu = f \leq 0 \quad in\,\Omega.$$

If $x_0 \in \partial\Omega$ is such that

$$u(x_0) > u(x) \quad \textit{for all } x \in \Omega \tag{2.3.2}$$

then

$$\partial_\nu u(x_0) > 0,$$

where $\nu(x)$ is the exterior unit normal at $x \in \partial\Omega$.

Proof. Without loss of generality we can assume the ball in the assumption (H1) is $B(0,r)$, so that $|x_0| = r$ and

$$\nu(x_0) = x_0/r.$$

Then we take

$$w_\varepsilon(x) = u(x) + \varepsilon z(x),$$

where

$$z(x) = e^{-Rx^2} - e^{-Rr^2}$$

Then $z(x_0) = 0$ and

$$L(z)(x) = e^{-Rx^2}\left(-4R^2x^2 + 2nR - \sum_{j=1}^{n} 2V_j x_j R\right) < 0$$

for R big enough and $x \in B(0,r) \setminus B(0,r/2)$. In view of (2.3.2) we can find $\varepsilon > 0$ so that

$$w_\varepsilon(x_0) = u(x_0) > u(x) + \varepsilon z(x) = w_\varepsilon(x) \;\; |x| = r/2.$$

We have also

$$w_\varepsilon(x_0) = u(x_0) \geq u(x) + \varepsilon z(x) = w_\varepsilon(x), \;\; |x| = r.$$

Applyi ng the weak maximum principle for $w_\varepsilon(x) - w_\varepsilon(b)$ and the domain $B(0,r)\setminus \overline{B(0,r/2)}$ we find

$$\max_{\overline{B(0,r)}\setminus B(0,r/2)} w_\varepsilon - w_\varepsilon(b) \le 0$$

and hence

$$\partial_\nu w_\varepsilon(x_0) \ge 0.$$

In this way we find

$$\partial_\nu u(x_0) \ge -\varepsilon \partial_\nu z(x_0) = 2\varepsilon R r e^{-Rr^2} > 0.$$

□

Lemma 2.3.2. *If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of (2.1.1), then we have the following properties*

a) *(STRONG MIN principle) if $f(x)$ is continuous NON-NEGATIVE function ($f(x) \ge 0$ for any $x \in \Omega$) and f has minimum in internal point $x_0 \in \Omega$, then it is a constant;*

b) *(STRONG MAX principle) if $f(x)$ is continuous NON-POSITIVE function ($f(x) \le 0$ for any $x \in \Omega$)and f has maximum in internal point $x_0 \in \Omega$, then it is a constant.*

Proof. We shall prove only b). Set

$$M := \max_{\overline{\Omega}} u$$

and assuming u is not a constant we can decompose Ω as

$$\Omega = C \cup V,$$

where

$$C := \{x \in \Omega \mid u(x) = M\},$$

$$V := \{x \in \Omega \mid u(x) < M\}.$$

This means we can find two points $P \in C$ and $Q \in V$ and connect them with an arc in Ω. Let R be the closest point on the arc to Q such that $u(R) = M$ and $u(x) < M$ for all x on the arc between Q and R. Since R is in Ω, we can find y sufficiently close to R on the arc QR so that

$$y \in V,\ d(y,R) < d(y,\partial\Omega).$$

Then we can define the largest open ball $B(y,r) \subset V$ so that $\partial B(y,r)$ has a point $x_* \in C$. In this way we can apply for $B(y,r)$ the Hopf lemma and deduce $\partial_\nu u(x_*) > 0$ and this is a contradiction with the fact u has a maximum in x_*. The contradiction shows that V is empty. □

2.4 Maximum principle for Laplace operator with Coulomb potential

A natural question is to ask if the linear operator

$$P_\omega = -\Delta - \frac{1}{|x|} + \omega,$$

satisfies the weak maximum principle in the sense that

$$u \in H^2,\ P_\omega(u) = g \geq 0, \Longrightarrow u \geq 0. \tag{2.4.3}$$

The above maximum principle is incomplete, since additional behavior of u and g at infinity has to be imposed, namely, we shall suppose that

$$(1+|x|)^{-M} e^{\sqrt{\omega}|x|} u \in H^2, \quad (1+|x|)^{-M} e^{\sqrt{\omega}|x|} g \in H^2, \tag{2.4.4}$$

for some real number $M > 0$.

Note, that the energy levels of the hydrogen atom are described by the eigenvalues $\omega_k > 0$ of the eigenvalue problem

$$\Delta e_k(x) + \frac{e_k(x)}{|x|} = \omega_k e_k(x), \ \ e_k(x) \in H^2.$$

One has

$$\omega_k = \frac{1}{4(k+1)^2}, \quad k = 0, 1, \ldots$$

and $e_0(x) = ce^{-|x|/2}, c > 0$. The first observation is that all eigenfunctions $e_k(x)$, $k \geq 1$, are expressed in terms of Laguerre polynomials of $|x|$, having exactly k roots. This fact guarantees that the maximum principle is not valid for $\omega = \omega_k$. More precisely, we can show the following.

Lemma 2.4.1. *The weak maximum principle 2.4.3 is valid if an only if*

$$\omega \geq \frac{1}{4}.$$

2.5 Appendix: Maximum principle for subharmonic functions.

2.5.1 Mean value theorem. Harmonic functions

The Gauss Green identity

$$\int_\Omega (\Delta u v - u \Delta v)\, dy = \int_{\partial\Omega} (\partial_N u v - u \partial_N v)\, dS_y,$$

enables one to take $v(y) = 1/|x-y|$, where $x \in \Omega$ and modify the domain Ω as follows

$$\Omega_\delta = \{y : y \in \Omega, |y-x| \geq \delta\}$$

provided $\delta > 0$ is small.

Taking the limit as $\delta \to 0$ we get

Lemma 2.5.1. *(integral representation) If $u \in C^2(\mathbb{R}^n)$ then*

$$u(x) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{\partial_N u}{|x-y|} - u\partial_N\left(\frac{1}{|x-y|}\right)\right) dS_y$$
$$-\frac{1}{4\pi}\int_{\Omega}\frac{\Delta u}{|x-y|}dy.$$

In the particular case $\Omega = \{|x-x_0| < R\}$ we get

Lemma 2.5.2. *(Gauss mean value theorem) If $u \in C^2$ is a harmonic in $\{|x-x_0| < R\}$, then*

$$u(x_0) = \frac{1}{4\pi R^2}\int_{|x-x_0|=R} u(y)dS_y.$$

Problem 2.5.1. *(Strong Maximum principle) Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of*

$$\Delta u = 0, x \in \Omega. \tag{2.5.5}$$

If

$$u(c) = \max_{\overline{\Omega}} u(x)$$

for some $c \in \Omega$, then $u(x)$ is a constant.

2.5.2 Maximum principle for subharmonic functions

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$. Set

$$M(u)(x,R) = \frac{1}{\mu(|y| \leq R)}\int_{|y|\leq R} u(x+y)dy \tag{2.5.6}$$

and

$$M_S(u)(x,R) = \frac{1}{\mu(\mathbf{S}^{n-1})}\int_{|\omega|=1} u(x+R\omega)d\omega \tag{2.5.7}$$

Since $\mu(|y| \leq R) = \mu(\mathbf{S}^{n-1})R^n/n$, we have the relation

$$\int_0^R M_S(u)(x,r)r^{n-1}dr = \frac{R^n}{n}M(u)(x,R).$$

Definition 2.5.1. *A function $u \in L^1_{loc}$ is called subharmonic if $u(x) \le M(u)(x,R)$ for any $R > 0$ such that $\{y + x; |y| \le R\} \subset \Omega$ and for a.e. in $x \in \Omega$.*

Problem 2.5.2. *If $u \in C^\infty(\Omega)$ then u is subharmonic if and only if $\Delta u \ge 0$.*

Hint. If $\Delta u \ge 0$, then Gauss-Green gives

$$0 \le \int |y| \le R \Delta u(x+y)dy = cR^{n-1}\frac{d}{dR}M_S(u)(x,R),$$

so $M_S(u)(x,R) \ge M_S(u)(x,0) = u(x)$ and

$$M(u)(x,R) = nR^{-n}\int_0^R M_S(u)(x,r)r^{n-1}dr \ge u(x)nR^{-n}\int_0^R r^{n-1}dr = u(x).$$

Problem 2.5.3. *If $u \in L^1_{loc}(\Omega)$ then u is subharmonic if and only if $\Delta u \ge 0$.*

Problem 2.5.4. *If $u \in C(\overline{\Omega})$ is subharmonic, then the condition $u(c) = \max_{\overline{\Omega}} u(x)$ for some $c \in \Omega$ implies $u = cost$.*

The function

$$M(u)(x) = \sup_{R>0} M(u)(x,R) = \sup_{R>0} \frac{1}{\mu(|y| \le R)} \int_{|y|\le R} u(x+y)dy \tag{2.5.8}$$

is called Hardy - Littlewood MAXIMAL function.

Problem 2.5.5. *Show that there exists a constant $C > 0$ so that for any $u \in C_0^\infty(\mathbf{R}^n)$*

$$\|M(u)\|_{L^2} \le C\|u\|_{L^2}.$$

Chapter 3

Fundamental solution of Laplace operator in $\mathbb{R}^n$ and applications

3.1 Laplace equation in $\mathbb{R}^n$

Among the most important of all partial differential equations are undoubtedly Laplace's equation

$$\Delta u = 0 \tag{3.1.1}$$

and Poisson's equation

$$-\Delta u = f. \tag{3.1.2}$$

In both (3.1.1) and (3.1.2) , $x \in \Omega$ and the unknown is $u : \bar{\Omega} \to \mathbb{R}$, $u = u(x)$ where $\Omega \subset \mathbb{R}^n$ is a given open set. In (3.1.2) the function $f : U \to \mathbb{R}$ is also given. Remember that the Laplacian of u is

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}.$$

Definition 3.1.1. *A C^2 function u satisfying* (3.1.1) *is called a harmonic function.*

3.2 Decomposition of the Laplace operatore into radial and angular part

Start with the relation

$$|x|^2 \Delta = L^2 + (n-2)L + \Delta_{S^{n-1}}, \tag{3.2.3}$$

where

$$\Delta_{S^{n-1}} = \sum_{j,k=1}^{n} \Omega_{jk}^2, \Omega_{jk} = x_j \partial_k - x_k \partial_j \tag{3.2.4}$$

and

$$L = r\partial_r = \sum_{j=1}^{n} x_j \partial_j.$$

Per $n = 2$ we have

$$\Delta_{S^1} = x_1 \partial_2 - x_2 \partial_1.$$

Introduce polar coordinates

$$x_1 = r\cos\phi, x_2 = r\sin\phi.$$

Then

$$\partial_1 = \frac{x_1}{r}\partial_r - \frac{x_2}{x_1^2 + x_2^2}\partial_\phi = \cos\phi\partial_r - \frac{\sin\phi}{r}\partial_\phi$$

and

$$\partial_2 = \frac{x_2}{r}\partial_r + \frac{x_1}{x_1^2 + x_2^2}\partial_\phi = \sin\phi\partial_r + \frac{\cos\phi}{r}\partial_\phi.$$

So we get

$$\Omega_{12} = \partial_\phi$$

and we find

$$\Delta_{S^1} = \partial_\phi^2.$$

For $n = 3$ we have

$$x_1 = r\cos\phi\sin\theta, x_2 = r\sin\phi\sin\theta, x_3 = \cos\theta.$$

Then

$$\partial_1 = \frac{x_1}{r}\partial_r - \frac{x_2}{x_1^2 + x_2^2}\partial_\phi - \frac{x_3 x_1}{r^2(x_1^2 + x_2^2)^{1/2}}\partial_\theta =$$

$$= \cos\phi\sin\theta\partial_r - \frac{\sin\phi}{r\sin\theta}\partial_\phi - \frac{\cos\theta}{r}\partial_\theta.$$

$$\partial_2 = \frac{x_2}{r}\partial_r + \frac{x_1}{x_1^2 + x_2^2}\partial_\phi - \frac{x_3 x_2}{r^2(x_1^2 + x_2^2)^{1/2}}\partial_\theta$$

and

$$\partial_3 = \frac{x_3}{r}\partial_r + \frac{(x_1^2 + x_2^2)^{1/2}}{r^2}\partial_\theta$$

Hence,

$$\Omega_{12} = \partial_\phi, \ \ \Omega_{13} = \sin\phi\cot\theta\partial_\phi + \cos\phi\partial_\theta, \ \ \Omega_{13} = -\cos\phi\cot\theta\partial_\phi + \sin\phi\partial_\theta.$$

Taking the sum of squares, we get

$$\Delta_{S^2} = \partial_\theta^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2.$$

We have the following commutator properties

Problem 3.2.1. *If* $[A,B] = AB - BA$ *is the commutator of two operators then we have*

$$[\partial_j, x_k] = \delta jk,$$

$$[\Delta, \partial_j] = 0,$$

$$[\Delta, x_j] = 2\partial_j,$$

$$[\Delta, \Omega_{jk}] = 0,$$

$$[\Delta, L] = 2\Delta.$$

3.2.1 Spherical harmonics

If $Y(\omega)$ is an eigenfunction of $\Delta_{S^{n-1}}$, one can introduce

$$u(x) = |x|^M Y\left(\frac{x}{|x|}\right)$$

and see that $u(x)$ is harmonic, i.e. $\Delta u(x) = 0$ if and only if

$$\Delta_{S^{n-1}} Y = -M(M+n-2)Y.$$

One can take $u(x)$ to be a harmonic homogeneous POLYNOMIAL of order M (M is a non negative integer) and see that $Y(x/|x|) = |x|^{-M}u(x)$ is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $M(M+n-2)$. The application of Liuoville theorem shows that M is only integer and $u(x)$ has to be a homogeneous polynomial.

3.3 Laplace equation in $\mathbb{R}^n$ and its fundamental solution

One good strategy for investigating any partial differential equation is first to identify some explicit solutions and then, provided the PDE is linear, to assemble more complicated solutions out of the specific ones previously noted. Furthermore, in looking for explicit solutions it is often wise to restrict attention to classes of functions with certain symmetry properties. since Laplace's equation is invariant under rotations, it consequently seems advisable to search first for radial solutions, that is, functions of $r = |x|$ Let us therefore attempt to find a solution u of Laplace's equation in $\Omega = \mathbb{R}^n$, having the form

$$u(x) = v(r)$$

where $r = |x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$ and v is to be selected (if possible) so that $\Delta u = 0$ holds. First note for $i = 1, \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \quad (x \neq 0)$$

We thus have

$$u_{x_i} = v'(r)\frac{x_i}{r}, u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

for $i = 1, \ldots, n$, and so

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r} v' = 0. \tag{3.3.5}$$

If $v' \neq 0$, we deduce

$$\log\left(v'\right)' = \frac{v''}{v'} = \frac{1-n}{r}$$

and hence $v'(r) = \frac{a}{r^{n-1}}$ for some constant a. Consequently if $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases}$$

where b and c are constants. These considerations motivate the following.

Definition 3.3.1. *The function*

$$E(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)|B|}\frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{3.3.6}$$

defined for $x \in \mathbb{R}^n, x \neq 0$, is the fundamental solution of Laplace's equation.

Remark 3.3.1. *Recall that the volume of the ball $B(0,R)$ in $\mathbb{R}^n$ is*

$$|B(0,R)| = \int_0^R \rho^{n-1} d\rho \mu(S^{n-1}) = \frac{R^n \mu(\mathbb{S}^{n-1})}{n}$$

where $\mu(\mathbb{S}^{n-1})$ is the surface area of the unit sphere in $\mathbb{R}^n$.

The reason for the particular choices of the constants in (3.3.6) will be apparent in a moment.

We will sometimes slightly abuse notation and write $\Phi(x) = \Phi(|x|)$ to emphasize that the fundamental solution is radial. Observe also that we have the estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \left|D^2\Phi(x)\right| \leq \frac{C}{|x|^n} \quad (x \neq 0) \tag{3.3.7}$$

for some constant $C > 0$

3.4 Poisson equation

By construction the function $x \mapsto E(x)$ is harmonic for $x \neq 0$. If we shift the origin to a new point y, the PDE (1) is unchanged; and so $x \mapsto E(x-y)$ is also harmonic as a function of $x, x \neq y$. Let us now take $f : \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $x \mapsto E(x-y)f(y)(x \neq y)$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points y. This reasoning might suggest that the convolution

$$\begin{aligned} u(x) &= \textstyle\int_{\mathbb{R}^n} E(x-y)f(y)dy \\ &= \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log(|x-y|)f(y)dy & (n=2) \\ \frac{1}{n(n-2)|B|}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy & (n \geq 3) \end{cases} \end{aligned} \tag{3.4.8}$$

will solve Laplace equation (3.1.1). However, this is wrong: we cannot just compute

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x E(x-y)f(y)dy = 0$$

Indeed, as intimated by estimate (3.3.7) $D^2\Phi(x-y)$ is not summable near the singularity at $y=x$, and so the differentiation under the integral sign above is unjustified (and incorrect). We must proceed more carefully in calculating Δu Let us for simplicity now assume $f \in C_c^2(\mathbb{R}^n)$; that is, f is twice continuously differentiable, with compact support.

Theorem 3.4.1. *(Solving Poisson's equation). Define u by* (3.4.8) *. Then*

(i) $u \in C^2(\mathbb{R}^n)$

and

(ii) $-\Delta u = f \quad$ *in* $\mathbb{R}^n$ *We consequently see that* (3.4.8) *provides us with a formula for a solution of Poisson's equation* (3.1.2) *in* $\mathbb{R}^n$.

Proof. We have

$$u(x) = \int_{\mathbb{R}^n} E(x-y) f(y) dy = \int_{\mathbb{R}^n} E(y) f(x-y) dy$$

hence

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} E(y) \left[\frac{f\left(x+he_i-y\right)-f(x-y)}{h} \right] dy$$

where $h \neq 0$ and $e_i = (0,\ldots,1,\ldots,0)$, the 1 in the i^{th} -slot. But

$$\frac{f\left(x+he_i-y\right)-f(x-y)}{h} \to \frac{\partial f}{\partial x_i}(x-y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$, and thus

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} E(y) \frac{\partial f}{\partial x_i}(x-y) dy \quad (i=1,\ldots,n)$$

Similarly

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbf{K}^n} E(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) dy \quad (i,j=1,\ldots,n) \tag{3.4.9}$$

As the expression on the right hand side of eq.PE10 is continuous in the variable x, we see $u \in C^2(\mathbb{R}^n)$

Since E blows up at 0, we will need for subsequent calculations to isolate this singularity inside a small ball. So fix $\varepsilon > 0$. Then

$$\begin{aligned} \Delta u(x) &= \int_{B(0,z)} E(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^m - B(0,t)} E(y) \Delta_x f(x-y) dy \\ &=: I_\varepsilon + J_\varepsilon \end{aligned}$$

Now

$$|I_\varepsilon| \le C \left\| D^2 f \right\|_{L^\infty(\mathbb{R}^-)} \int_{B(0,\varepsilon)} |E(y)|dy \le \begin{cases} C\varepsilon^2|\log\epsilon| & (n=2) \\ C\varepsilon^2 & (n \ge 3) \end{cases} \tag{3.4.10}$$

An integration by parts yields

$$\begin{aligned} J_c &= \int_{\mathbb{R}^n - B(0,c)} E(y)\Delta_y f(x-y)dy \\ &= -\int_{\mathbb{R}^- - B(0,z)} DE(y)\cdot D_y f(x-y)dy \\ &\quad + \int_{\partial B(0,\varepsilon)} E(y)\frac{\partial f}{\partial \nu}(x-y)dS(y) \\ &=: K_\varepsilon + L_\varepsilon \end{aligned} \tag{3.4.11}$$

ν denoting the inward pointing unit normal along $\partial B(0,\varepsilon)$. We readily check

$$|L_\varepsilon| \le \|Df\|_{L^\infty(\mathbf{K}^n)} \int_{\partial B(0,\varepsilon)} |E(y)|dS(y) \le \begin{cases} C\varepsilon|\log\varepsilon| & (n=2) \\ C\varepsilon & (n \ge 3) \end{cases} \tag{3.4.12}$$

We continue by integrating by parts once again in the term K_ε, to discover

$$\begin{aligned} K_\varepsilon &= \int_{\mathbb{R}^- - B(0,\varepsilon)} \Delta E(y) f(x-y)dy - \int_{\partial B(0,\varepsilon)} \frac{\partial E}{\partial \nu}(y) f(x-y)dS(y) \\ &= -\int_{\partial B(0,\varepsilon)} \frac{\partial E}{\partial \nu}(y) f(x-y)dS(y) \end{aligned}$$

since E is harmonic away from the origin. Now $DE(y) = \frac{-1}{n|B|}\frac{y}{y'}(y \neq 0)$ on $\partial B(0,\varepsilon)$. since $n|B|\varepsilon^{n-1}$ is the surface area of the sphere $\partial B(0,\varepsilon)$, we have

$$\begin{aligned} K_\varepsilon &= -\frac{1}{n|B|\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)dS(y) \\ &= -f_{\partial B(z,t)} f(y)dS(y) \to -f(x) \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{3.4.13}$$

(Remember that a slash through an integral denotes an average.) Combining now (3.4.10)-(3.4.13)and letting $\varepsilon \to 0$, we find $-\Delta u(x) = f(x)$, as asserted. □

3.5 Weak solutions of Poisson equation

Typical example of application of the notion of distribution is the definition of a weak solution of the Laplace equation.

Definition 3.5.1. *If u, f are distributions in $\mathbb{R}^n$, then u is a weak solution of the equation*

$$-\Delta u = f, \;\; \Delta = \partial_1^2 + \cdots \partial_n^2, \tag{3.5.14}$$

if for any test function $\varphi(x)$ we have

$$< u, \Delta\varphi >= - < f, \varphi > .$$

3.5.1 Case $n = 3$

One can verify that

$$\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta$$

in the sense of distributions in $\mathbb{R}^3$. Indeed taking any test function φ we apply Gauss - Green formula for the domain $\{|x| \geq \varepsilon\}$ and using the fact that

$$\Delta\left(\frac{1}{|x|}\right) = 0 \quad |x| \neq 0,$$

we find

$$\int_{|x|>\varepsilon}\left(\Delta\left(\frac{1}{|x|}\right)\right)\varphi(x)dx - \int_{|x|>\varepsilon}\left(\frac{1}{|x|}\right)\Delta\varphi(x)dx =$$
$$-\int_{|x|=\varepsilon}\partial_r\left(\frac{1}{|x|}\right)\varphi(x)dS_x + \int_{|x|=\varepsilon}\left(\frac{1}{|x|}\right)\partial_r\varphi(x)dS_x,$$

where here and below

$$\partial_r = \sum_{j=1}^n \frac{x_j}{|x|}\partial_j.$$

Taking into account the fact that

$$\partial_r\left(\frac{1}{|x|}\right) = -\frac{1}{|x|^2}$$

and introducing spherical coordinate $x = \varepsilon\omega, |\omega| = 1$, we find

$$\int_{|x|=\varepsilon}\partial_r\left(\frac{1}{|x|}\right)\varphi(x)dS_x = -\int_{|\omega|=1}\varphi(\varepsilon\omega)dS_\omega,$$

$$\int_{|x|=\varepsilon}\left(\frac{1}{|x|}\right)\partial_r\varphi(x)dS_x = \varepsilon\int_{|\omega|=1}\partial_r\varphi(\varepsilon\omega)d\omega$$

so taking the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \partial_r \left(\frac{1}{|x|}\right) \varphi(x)dS_x = -4\pi\varphi(0),$$

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \left(\frac{1}{|x|}\right) \partial_r\varphi(x)dS_x = 0,$$

so we arrive at

$$-\int_{\mathbb{R}^3} \left(\frac{1}{|x|}\right) \Delta\varphi(x)dx = 4\pi\varphi(0)$$

and the identity

$$-\frac{1}{4\pi}\,\Delta\left(\frac{1}{|x|}\right) = \delta. \tag{3.5.15}$$

3.5.2 Case $n \geq 3$.

The function

$$E(x) \in C^\infty(\mathbb{R}^n \setminus 0)$$

satisfying

$$-\Delta E = \delta$$

in the sense of distributions is called fundamental solutions of the Laplace operator and they enable one to represent the solution of the Poisson equation

$$-\Delta u = f, \;\; f \in C_0^\infty,$$

as follows

$$u(x) = \int_{\mathbb{R}^n} E(x-y)f(y)dy.$$

To verify that

$$u(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}dy$$

is a weak solution of

$$\Delta u = f,$$

we consider

$$u_\varepsilon(x) = c\int_{|x-y|\geq\varepsilon} \frac{f(y)}{|x-y|^{n-2}}dy = c\int_{|y|\geq\varepsilon} \frac{f(x-y)}{|y|^{n-2}}dy$$

$$= c\int_\varepsilon^\infty \int_{|\omega|=1} \frac{f(x-r\omega)}{r^{n-2}}r^{n-1}dr.$$

One can easily derive that if $f \in C^2$ and has a compact support, then $u_\varepsilon \in C^2$ and

$$\Delta u_\varepsilon(x) = c\int_{|y|\geq\varepsilon} \frac{\Delta_y f(x-y)}{|y|^{n-2}}dy$$

Applying the Gauss - Green formula, we find

$$\Delta u_\varepsilon(x) = c\int_{|y|\geq\varepsilon} f(x-y)\Delta\left(\frac{1}{|y|^{n-2}}\right)dy+$$

$$+c\int_{|y|=\varepsilon}\frac{\partial_N f(x-y)}{|y|^{n-2}}dy - c\int_{|y|=\varepsilon} f(x-y)\partial_N\left(\frac{1}{|y|^{n-2}}\right)dy,$$

where $\partial_N = -\sum_j y_j/|y|\partial_j = -\partial_r$, so we get

$$\Delta u_\varepsilon(x) = c\int_{|\omega|=1}\frac{\partial_r f(x-\varepsilon\omega)}{\varepsilon^{n-2}}\varepsilon^{n-1}d\omega - (n-2)c\int_{|\omega|=1} f(x-\varepsilon\omega)d\omega$$

so taking the limit as $\varepsilon \to 0$, we obtain

$$\Delta u(x) = -(n-2)c\mu(S^{n-1})f(x).$$

So

$$c = -\frac{1}{(n-2)\mu(S^{n-1})}.$$

3.6 Mean-value formulas.

Consider now an open set $\Omega \subset \mathbb{R}^n$ and suppose u is a harmonic function within Ω. We next derive the important mean-value formulas, which declare that $u(x)$ equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$, provided $B(x,r) \subset \Omega$. These implicit formulas involving u generate a remarkable number of consequences, as we will momentarily see.

Theorem 3.6.1. *(Mean-value formulas for Laplace's equation). If $u \in C^2(\Omega)$ is harmonic, then*

(3.6.16)
$$u(x) = \fint_{\partial B(x,r)} u\,dS = \fint_{B(x,r)} u\,dy$$

for each ball $B(x,r) \subset \Omega$.

Proof. Set

$$\phi(r) := \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{\partial B(0,1)} u(x+rz)dS(z)$$

Then

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z dS(z)$$

and consequently, using Green's formulas we compute

$$\begin{aligned}\phi'(r) &= \fint_{\partial B(z,r)} Du(y)\cdot \frac{y-x}{r} dS(y)\\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(y)\\ &= \frac{r}{n}\fint_{B(x,r)} \Delta u(y) dy = 0\end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} u(y) dS(y) = u(x)$$

Observe next that our employing polar coordinates, gives

$$\begin{aligned}\int_{B(x,r)} u dy &= \int_0^r \left(\int_{\partial B(x,s)} u dS\right) ds\\ &= u(x)\int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n u(x)\end{aligned}$$

□

Theorem 3.6.2. *THEOREM 3 (Converse to mean-value property). If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint u dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. Proof. If $\Delta u \neq 0$, there exists some ball $B(x,r) \subset \Omega$ such that, say, $\Delta u > 0$ within $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n}\fint_{B(x,r)} \Delta u(y) dy > 0$$

a contradiction. □

3.7 Properties of harmonic functions.

We now present a sequence of interesting deductions about harmonic functions, all based upon the mean-value formulas. Assume for the following that $\Omega \subset \mathbb{R}^n$ is open and bounded.

3.7.1 Strong maximum principle, uniqueness.

Theorem 3.7.1. *(Strong maximum principle). Suppose* $u \in C^2(\Omega) \cap C(O)$ *is harmonic within* Ω.

(i) Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(ii) Furthermore, if Ω *is connected and there exists a point* $x_0 \in \Omega$ *such that*

$$u(x_0) = \max_{\overline{\Omega}} u$$

then u is constant within Ω. *Assertion (i) is the marimum principle for Laplace's equation and (ii) is the strong maximum principle. Replacing u by* $-u$, *we recover also similar assertions with "min" replacing "max".*

Proof. Suppose there exists a point $x_0 \in \Omega$ with $u(x_0) = M := \max_\Omega u$ Then for $0 < r < \mathrm{dist}(x_0, \partial\Omega)$, the mean-value property asserts

$$M = u(x_0) = \fint_{B(x_0,r)} u\,dy \leq M$$

As equality holds only if $u \equiv M$ within $B(x_0, r)$, we see $u(y) = M$ for all $y \in B(x, r)$. Hence the set $\{x \in \Omega \mid u(x) = M\}$ is both open and relatively closed in Ω, and thus equals Ω if Ω is connected. This proves assertion (ii), from which (i) follows. □

Remark 3.7.1. *The strong maximum principle asserts in particular that if* Ω *is connected and* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfies*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \quad u = g \text{ on } \partial\Omega \end{cases}$$

where $g \geq 0$, *then u is positive everywhere in* Ω *if g is positive somewhere on* $\partial\Omega$.

An important application of the maximum principle is establishing the uniqueness of solutions to certain boundary-value problems for Poisson's equation.

Theorem 3.7.2. *(Uniqueness). Let* $g \in C(\partial\Omega)$, $f \in C(\Omega)$. *Then there exists at most one solution* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *of the boundary-value problem*

$$\begin{cases} -\Delta u = f \text{ in } U \\ \quad u = g \text{ on } \partial U \end{cases} \tag{3.7.17}$$

Proof. If u and $\tilde{u}$ both satisfy (3.7.17) one can apply apply Theorem 3.7.1 to the harmonic functions $w := \pm(u - \tilde{u})$ □

3.7.2 Regularity

Now we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus harmonic functions are automatically infinitely differentiable. This sort of assertion is called a regularity theorem. The interesting point is that the algebraic structure of Laplace's equation $\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$ leads to the analytic deduction that all the partial derivatives of u exist, even those which do not appear in the PDE.

Theorem 3.7.3. *THEOREM 6 (Smoothness). If u $E\,C(U)$ satisfies the mean-tualue property (16) for each ball $B(x,r) \subset U$, then*

$$u \in C^\infty(U)$$

Note carefully that u may not be smooth, or even continuous, up to ∂U.

Proof. Proof. Let η be a standard mollifier, as described in \$C.4, and recall that η is a radial function. Set $u^\epsilon := \eta_c * u$ in $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$ As shown in \$C.4, $u^r \in C^\infty(U_e)$ We will prove u is smooth by demonstrating that in fact $u \equiv u^f$ on U_ε Indeed if $x \in U_{e,}$ then Thus $u^\epsilon \equiv u$ in U_ε, and so $u \in C^\infty(U_e)$ for each $\epsilon > 0$. ☐

3.8 Liouville's Theorem.

Next we see that there are no nontrivial bounded harmonic functions on all of R^n

THEOREM 8 (Liouville's Theorem). Suppose u : R" → R is harmonic and bounded. Then u is constant. Proof. Fix $x_0 \in \mathbb{R}^n, r > 0$, and apply Theorem 7 on $B(x_0, r)$:

$$\begin{aligned} |Du(x_0)| &\leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(z_0,r))} \\ &\leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0 \end{aligned}$$

as $r \to \infty$. Thus $Du \equiv 0$, and so u is constant. THEOREM 9 (Representation formula). Let $f \in C_c^2(\mathbb{R}^n), n \geq 3$. Then any bounded solution of

$$-\Delta u = f \quad \text{in } \mathrm{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^0} \Phi(x-y) f(y) dy + C \quad \left(x \in R^n\right)$$

for some constant C.

3.9 Liouville theorem for harmonic functions in $\mathbb{R}^n$, $n \geq 3$.

The Laplace equation

$$\Delta u = f, \;\; f \in C_0^\infty,$$

has a solution

$$u(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

provided $n \geq 3$ and

$$c = -\frac{1}{(n-2)\mu(S^{n-1})}.$$

We shall use this fundamental solution to derive the Liouville theorem. For simplicity we consider only the case $n = 3$.

Lemma 3.9.1. *(Liouville) If $u \in C^2(\mathbb{R}^n)$ is a harmonic function, i.e. $\Delta u = 0$ in $\mathbb{R}^n$, then the condition*

$$\max_{|x|=R} |u(x)| \leq C$$

implies $u = cost$.

Idea of Proof: Take cut off function $\varphi(s)$ such that $\varphi(s) = 1$ for $|s| \leq 1$ and $\varphi(s) = 0$ for $|s| \geq 2$. Then setting $\varphi_R(x) = \varphi(|x|/R)$ we have

$$\Delta(\varphi_R u) = 2\nabla(u\nabla\varphi_R) - u\Delta\varphi_R,$$

since u is harmonic. Applying the formula for the fundamental solution, we get

$$\varphi_R(x)u(x) = -\frac{1}{4\pi}\int_{R^3} \frac{2\nabla(u(y)\nabla\varphi_R(y)) - u(y)\Delta\varphi_R(y)}{|x-y|} dy =$$

$$= -\frac{1}{4\pi}\int_{R^3} \frac{2(u(y)(x-y)\nabla\varphi_R(y))}{|x-y|^3} dy + \frac{1}{4\pi}\int_{R^3} \frac{u(y)\Delta\varphi_R(y)}{|x-y|} dy.$$

Take $|x| \sim R/2$. Then $|y| \sim R$ on the support of $\nabla\varphi_R(y)$. So one can show that

$$|\nabla u(x)| \leq \frac{C}{|x|}$$

provided $|x| \sim R/2$. Since $\nabla u(x)$ is also harmonic, applying the maximum principle of Lemma 3.9.1 we complete the proof that $\nabla u = 0$.

Problem 3.9.1. *(Liouville) If $u \in C^2(\mathbb{R}^n)$ is a harmonic function, i.e. $\Delta u = 0$ in $\mathbb{R}^n$, then the condition*

$$\max_{|x|=R} |u(x)| \leq C(1+|x|)^M$$

implies $u(x)$ is a polynomial.

Chapter 4

Harmonic functions in domains (ball and semispace) and conformal transform.

First step in this section is to make change of variables in the Laplace operator. Indeed, let us take

$$y \to x = F(y)$$

invertible change of variables in $\mathbb{R}^n$.

Our goal is the compute

$$\Delta_y u(F(y)).$$

For the purpose we start with the representation

$$u(F(y)) = \frac{1}{(2\pi)^n} \int e^{i<F(y).\xi>} \hat{u}(\xi) d\xi.$$

Since

$$\Delta e^{i<F(y).\xi>} = \left(-\sum_k < \partial_k F(y), \xi >< \partial_k F(y), \xi > + i < \Delta F(y), \xi > \right) e^{i<F(y).\xi>},$$

we can set

$$g^{jm}(y) = \sum_k \partial_k F_j(y) \partial_k F_m(y)$$

and obtain

$$\Delta e^{i<F(y).\xi>} = \left(-\sum_{jm} g^{jm}(y) \xi_j \xi_m + i < \Delta F(y), \xi > \right) e^{i<F(y).\xi>}.$$

Using $i\xi_k \hat{u}(\xi) = \widehat{\partial_k u}(\xi)$, we

$$\Delta_y u(F(y)) = \sum_{jm} g^{jm}(y)(\partial_j \partial_m u)(F(y)) + < \Delta F(y), (\nabla u)(F(y)) > .$$

In the particular case

$$F(y) = \frac{y}{|y|^2}$$

one can verify that we have

$$\left(g^{jm}\right)_{j,m=1}^{n} = \frac{I}{|y|^4}, \quad \Delta F(y) = \frac{2y}{|y|^4}.$$

Given any $u(x) \in C_0^\infty$ we define

$$u^*(y) = |y|^{-1} u(y/|y|^2) \tag{4.0.1}$$

Then we can assume $n = 3$ and derive

$$\Delta_y u^*(y) = |y|^{-5} \Delta_x u(y/|y|^2). \tag{4.0.2}$$

Idea of Proof: Introduce polar coordinates $R = |x|$, $r = |y|$ and $\omega = x/|x| = y/|y|$.Then the transform is given by

$$R = \frac{1}{r}.$$

We have

$$\Delta_x = \partial_R^2 + \frac{2}{R}\partial_R + \frac{1}{R^2}\Delta_\omega$$

$$\Delta_y = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_\omega$$

Note that

$$R\partial_R = -r\partial_r, \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\frac{f}{r} = \frac{1}{r}\partial_r^2 f,$$

so we obtain

$$\left(\partial_r^2 + \frac{2}{r}\partial_r\right)\frac{f}{r} = R(R^2\partial_R)^2 f = R^5\left(\partial_R^2 + \frac{2}{R}\partial_R\right)f.$$

Problem 4.0.1. *How can be generalized the conformal low (4.0.2) for dimensions $n \geq 3$.?*

Turning to the Dirichlet problem, we study the boundary value problem

$$\Delta u = 0, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

For the purpose we look for

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - h(x,y)\right)$$

so that $h(x,y)$ is harmonic with respect to y and on the boundary $\partial\Omega$ we have

$$h(x,y) = \frac{1}{|x-y|}, y \in \partial\Omega.$$

The Gauss Green identity

$$\int_\Omega (\Delta u v - u \Delta v)\, dy = \int_{\partial\Omega} (\partial_N u v - u \partial_N v)\, dS_y,$$

enables one to take $v(y) = 1/|x-y|$, where $x \in \Omega$ and modify the domain Ω as follows

$$\Omega_\delta = \{y : y \in \Omega, |y-x| \geq \delta\}$$

provided $\delta > 0$ is small.

As in Lemma 2.5.1 taking the limit as $\delta \to 0$ we get

$$u(x) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{\partial_N u}{|x-y|} - u\partial_N\left(\frac{1}{|x-y|}\right)\right) dS_y$$
$$-\frac{1}{4\pi}\int_\Omega \frac{\Delta u}{|x-y|} dy.$$

If we take $v(y) = h(x,y)$ we can use the fact that $h(x,y)$ is more regular than $1/|x-y|$ and find

$$0 = \frac{1}{4\pi}\int_{\partial\Omega}\big(h(x,y)\partial_N u - u\partial_N h\big)\, dS_y$$
$$-\frac{1}{4\pi}\int_\Omega h(x,y)\Delta u dy.$$

The above relations imply

$$u(x) = \int_{\partial\Omega}\big(G(x,y)\partial_N u - u\partial_N G(x,y)\big)\, dS_y$$
$$-\int_\Omega G(x,y)\Delta u dy.$$

Since $G(x,y) = 0$ on the boundary and since u is harmonic, we get

Lemma 4.0.1. *(Green function) Let $u \in C^2(\Omega)$ be a harmonic function in the domain Ω, i.e. $\Delta u = 0$ in Ω. If*

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - h(x,y)\right)$$

where $h(x,y)$ is harmonic with respect to y and on the boundary $\partial\Omega$ we have

$$h(x,y) = \frac{1}{|x-y|}, y \in \partial\Omega,$$

then the solution to the problem

$$\Delta u = 0, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

is given by

$$u(x) = -\int_{\partial\Omega} \left(f(y)\partial_N G(x,y)\right) dS_y$$

Example 4.0.1. *Let $\Omega = \{x, |x| < 1\}$. To construct $h(x,y)$ we consider the exterior of the domain $\Omega_{ext} = \{x, |x| > 1\}$ and use the conformal map*

$$y \in \Omega_{ext} \to \frac{y}{|y|^2} \in \Omega.$$

Given any x, $|x| < 1$ the function

$$\frac{1}{|x-y|}$$

is harmonic in Ω_{ext} so following (4.0.1) we see that

$$h(x,y) = \frac{1}{|y||x - y/|y|^2|}$$

is harmonic in Ω. Introduce polar coordinates $y = r\omega$. Then Lemma 4.0.1 implies that

$$u(x) = -\int_{|\omega|=1} \left(f(\omega)\partial_r G(x,r\omega)\right) d\omega$$

Since

$$4\pi G(x,r\omega) = \frac{1}{|x-r\omega|} - \frac{1}{|rx-\omega|}$$

and since

$$\partial_r \frac{1}{|x-r\omega|} = -\frac{< r\omega - x, \omega >}{|x-r\omega|^3}$$

$$\partial_r \frac{1}{|rx-\omega|} = -\frac{< rx-\omega, x >}{|rx-\omega|^3}$$

we obtain taking $r = 1$

$$4\pi \partial_r G(x, r\omega) = -\frac{1-|x|^2}{|x-\omega|^3}.$$

Finally, we can write

$$u(x) = \frac{1-|x|^2}{4\pi} \int_{|\omega|=1} \left(f(\omega) \frac{d\omega}{|x-\omega|^3} \right).$$

and this is the unique solution to the Dirichlet problem

$$\Delta u = 0, x \in \Omega$$
$$\lim_{x\to\infty} u(x) = 0,$$
$$u(x) = f(x), x \in \partial\Omega.$$

To show that this is a solution really, we take into account the relation

$$(2L_x + 1)\left(\frac{1}{|x-y|}\right) = \left(\frac{1-|x|^2}{|x-y|^3}\right), \; |y| = 1, \; |x| \neq 1, \; L_x = x.\nabla_x$$

as well as the commutator relations

$$[\Delta, L] = 2\Delta.$$

So from

$$\Delta_x \left(\frac{1}{|x-y|}\right) = 0$$

we can derive

$$\Delta_x \left((2L_x + 1)\left(\frac{1}{|x-y|}\right)\right) = \Delta_x \left(\frac{1-|x|^2}{|x-y|^3}\right) = 0$$

so

$$u(x) = \frac{1-|x|^2}{4\pi} \int_{|\omega|=1} \left(f(\omega) \frac{d\omega}{|x-\omega|^3} \right)$$

is a harmonic function!!!

Example 4.0.2. *Let* $\Omega = \{x, |x| > 1\}$. *To construct* $h(x, y)$ *we consider the interior of the domain* $\Omega_{int} = \{x, |x| < 1\}$ *and use the conformal map*

$$y \in \Omega_{ext} \to \frac{y}{|y|^2} \in \Omega.$$

Given any x, $|x|>1$ *the function*

$$\frac{1}{|x-y|}$$

is harmonic in Ω_{int} *so following (4.0.1) we see that*

$$h(x,y)=\frac{1}{|y||x-y/|y|^2|}$$

is harmonic with respect to y *in* Ω. *Introduce polar coordinates* $y=r\omega$. *Then Lemma 4.0.1 implies that*

$$u(x)=-\int_{|\omega|=1}\left(f(\omega)\partial_r G(x,r\omega)\right)d\omega$$

Since

$$4\pi G(x,r\omega)=\frac{1}{|x-r\omega|}-\frac{1}{|rx-\omega|}$$

and since

$$\partial_r\frac{1}{|x-r\omega|}=-\frac{<r\omega-x,\omega>}{|x-r\omega|^3}$$
$$\partial_r\frac{1}{|rx-\omega|}=-\frac{<rx-\omega,x>}{|rx-\omega|^3}$$

we obtain taking $r=1$

$$4\pi\partial_r G(x,r\omega)=-\frac{1-|x|^2}{|x-\omega|^3}.$$

Finally, we can write

$$u(x)=\frac{1-|x|^2}{4\pi}\int_{|\omega|=1}\left(f(\omega)\frac{d\omega}{|x-\omega|^3}\right).$$

and this is the unique solution to the Dirichlet problem

$$\Delta u=0, x\in\Omega$$
$$u(x)=f(x), x\in\partial\Omega.$$

Problem 4.0.2. *Let* $u(x)$ *be a solution to*

$$\Delta u=0, |x|>1$$
$$\lim_{x\to\infty}u(x)=0,$$
$$u(x)=f(x), |x|=1.$$

Show that

$$|u(x)|\leq\frac{C}{1+|x|}\max_{|x|=1}|f(x)|.$$

Problem 4.0.3. *Construct the Green function for the domains:*
a)$\{x, |x| < R\}$,
b)$\{x, |x| > R\}$,

Answer: a)

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{|y|=R} \left(f(y) \frac{dS_y}{|x-y|^3} \right).$$

Problem 4.0.4. *Let* $u(x)$ *be a solution to*

$$\Delta u = 0, |x| > R$$
$$u(x) = f(x), |x| = R.$$

Show that

$$|u(x)| \leq \frac{C}{1+|x|} \max_{|x|=R} |f(x)|.$$

Problem 4.0.5. *Construct the Green function for the domain* $\{x \in \mathbb{R}^3, x_3 > 0\}$.

Answer:

$$u(x) = \frac{x_3}{2\pi} \int_{\mathbb{R}^2} \left(f(y') \frac{dy'}{(|x'-y'|^2 + x_3^2)^{3/2}} \right), \;\; x' = (x_1, x_2).$$

Chapter 5

Applications: a priori estimates

5.0.1 Laplace equation in the space $\mathbb{R}^n$.

The equation

$$\Delta u(x) = f(x), x \in \mathbb{R}^n$$

has a unique solution provided

$$\lim_{x\to\infty} u(x) = 0$$

Taking $f(x) \in C(\mathbb{R}^n)$ with compact support one can represent the unique solution as follows (for simplicity we take $n = 3$.)

$$u(x) = -\frac{1}{4\pi}\int_K \frac{f(y)}{|x-y|}dy,$$

where here and below K denotes the support of f.

Problem 5.0.1. *(smoothing property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 5.0.2. *(smoothing property) If $f(x) \in L^\infty(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 5.0.3. *(smoothing property) If $f(x) \in C^k(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^{k+1}(\mathbb{R}^n)$*

Problem 5.0.4. *(convergence) If $f_k(x)$ have a fixed compact support K and tend uniformly in $C^k(K)$, then*

$$u_k(x) = -\frac{1}{4\pi}\int_K \frac{f_k(y)}{|x-y|}dy,$$

converge uniformly in $C^{k+1}(\mathbb{R}^n)$.

Problem 5.0.5. *(smoothing and decay property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$ and*

$$|u(x)| \leq \frac{C\|f\|_{C(K)}}{1+|x|}.$$

Problem 5.0.6. *(decay property for f without compact support) If $f(x) \in C(\mathbb{R}^n)$ has the property*

$$\|(1+|x|)^{3+\varepsilon} f\|_{L^\infty(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3} |(1+|x|)^{3+\varepsilon} f(x)| < \infty,$$

then $u(x) \in C^1(\mathbb{R}^n)$ and

$$|u(x)| \leq \frac{C\|(1+|x|)^{3+\varepsilon} f\|_{L^\infty(\mathbb{R}^3)}}{1+|x|}.$$

Problem 5.0.7. *(Coulomb behaviour property) If $f(x) \in C(\mathbb{R}^n)$ is non-negative, has a compact support and $f(x_0) > 0$ for some point $x_0 \in \mathbb{R}^n$, then there is a positive constant C_0 so that*

$$|u(x)| \geq \frac{C_0}{1+|x|}.$$

Hence u is not square integrable. More precisely, we have

$$\int_{|x|\leq R} |u(x)|^2 dx \geq CR$$

for R sufficiently large.

5.0.2 Laplace equation in bounded domain with Dirichlet boundary condition.

The general problem

$$\Delta u = F, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

can be solved extending F in $\mathbb{R}^n$ and representing u as follows

$$u = u_0 + w,$$

where

$$u_0(x) = -\frac{1}{4\pi}\int_\Omega \frac{F(y)}{|x-y|} dy.$$

Then w solves

$$\Delta w = 0, x \in \Omega$$
$$w(x) = g(x), x \in \partial\Omega.$$

where

$$g = f - u_0.$$

Applying the maximum principle we get

$$\|w\|_{L^\infty(\Omega)} \le C\|g\|_{C(\partial\Omega)} \le C\left(\|f\|_{C(\partial\Omega)} + \|F\|_{L^\infty(\Omega)}\right).$$

This estimate guarantees the uniqueness of the solution. The existence is delicate: needs or sub and supersolutions (Peron method) or methods from Functional analysis (Friedrich's extentions). For the concrete domains: interior or exterior ball, half space we have concrete solutions.

5.0.3 Idea of Peron method

4.8. Metodo di Perron per il problema di Dirichlet su domini generici Ω. Una volta risolto il problema di Dirichlet su $B_R(0)$ si puo' risolvere il problema di Dirichlet su una famiglia molto ampia di domini Ω senza bisogno di costruire la funzione di Green associata (come fatto su $B_R(0)$). Introduciamo le funzioni subarmoniche

Definition 5.0.1. *Una funzione $u : \Omega \to \mathbb{R}$ si dice subarmonica e diremo*

$$u \in \mathrm{SUB}(\Omega)$$

se $u \in C(\Omega)$ (ossia e' continua) ed inoltre

$$\forall x \in \Omega \quad \exists r(x) > 0 \text{ t.c. } u(x) \le \fint_{S_r(x)} u d\sigma, \quad \forall r \in (0, r(x))$$

Proposition 5.0.1. *Per ogni funzione $u \in \mathrm{SUB}(\Omega)$ si ha* $\max_{\bar\Omega} u = \max_{\partial\Omega} u$

Proof. Segue dallo stesso argomento usato per provare il principio del massimo per funzioni armoniche (usando il teorema della media). □

Proposition 5.0.2. *Siano $u, v \in \mathrm{SUB}(\Omega)$ allora* $\max\{u, v\} \in \mathrm{SUB}(\Omega)$

Proof. E ben noto che $\max\{u, v\}$ e'continua se u e v sono continue. Abbiamo inoltre per ipotesi

$$u(x) \le \fint_{S_r(x)} u d\sigma \le \fint_{S_r(x)} \max\{u, v\} d\sigma$$

e $v(x) \leq ⨍_{S_r(x)} u d\sigma \leq ⨍_{S_r(x)} \max\{u, v\} d\sigma$ quindi

$$\max\{u, v\}(x) \leq ⨍_{S_r(x)} \max\{u, v\} d\sigma$$

□

Proposition 5.0.3. *Se u e' armonica su Ω allora $u \in \mathrm{SUB}(\Omega)$.*

Proof. Abbiamo provato che per ogni funzione armonica vale l'identita' della media. □

Proposition 5.0.4. *Sia u_n una successione di funzioni armoniche su B_R che siano monotone crescenti ed uniformemente limitate. Allora la funzione $u(x)$, definita come il limite puntuale di $u_n(x)$ e' armonica, ossia $u \in C^2(\Omega)$ e $\Delta u = 0$.*

Proof. Basta provare che $u(x)$e$'$ continua e $u(x) = ⨍_{S_R(x)} u d\sigma$. A tal fine osserviamo che per il principio della media (sia su B_R che su S_R) si ha

$$u_n(x) = ⨍_{S_R(x)} u_n(y) d\sigma \text{ e } u_n(x) = ⨍_{B_R(x)} u_n(y) dy$$

Passando al limite abbiamo che $u(x) = ⨍_{B_R(x)} u(y) dy$ e da questa rappresentazione e facile dedurre che $u(x)$ e' continua (basta applicare il teorema di Lebesgue alla successione di funzioni $u(y)\chi_{B_R}(x_n)$ dove $x_n \overset{n\to\infty}{\to} x$). Passando sempre al limite (nell'integrale di superficie) si ha $u(x) = ⨍_{S_R(x)} u(y) d\sigma$ □

Proposition 5.0.5. *Proposizione* 4.10. *Sia $B_R(x_0) \subset \Omega$ e $u \in \mathrm{SUB}(\Omega)$. Sia inoltre v tale che $\Delta v = 0$ su $B_R(x_0)$ e $v = u$ su $\partial B_R(x_0)$. Allora si ha $w \in \mathrm{SUB}(\Omega)$ dove*

$$w(x) = \begin{cases} v(x), & x \in B_R(x_0) \\ u(x), & x \in \Omega \backslash B_R(x_0) \end{cases}$$

Proof. Ovviamente w e' continua. Per provare che e' subarmonica consideriamo tre casi. Se $x \in \Omega \backslash B_R(x_0)$ allora essendo u subarmonica si ha

$$w(x) = u(x) \leq ⨍_{S_r(x)} u d\sigma = ⨍_{S_r(x)} w d\sigma$$

per r abbastanza piccolo e per $x \in \Omega \backslash B_R(x_0)$.

Se $x \in B_R(x_0)$ invece essendo v armonica su $B_R(x_0)$ concludiamo

$$w(x) = v(x) = ⨍_{S_r(x)} v d\sigma = ⨍_{S_r(x)} w d\sigma$$

per r abbastanza piccolo.

Resta il caso in cui $x \in \partial B_R(x_0)$. Allora osserviamo che $u - v \in \mathrm{SUB}(B_R(x_0))$ ed inoltre

$u - v = 0$ su $S_R(x_0)$. Quindi per la Proposizione 5.0.1 si ha $u - v \le 0$ su $B_R(x_0)$. Da cio' deduciamo

$$\begin{aligned}\fint_{S_r(x)} w d\sigma &= \fint_{S_r(x)\cap B_R(x_0)} v d\sigma + \fint_{S_r(x)\cap B_R^c(x_0)} u d\sigma \\ &\ge \fint_{S_r(x)\cap B_R(x_0)} u d\sigma + \fint_{S_r(x)\cap B_R^s(x_0)} u d\sigma \ge u(x) = w(x)\end{aligned}$$

dove abbiamo usato il fatto che $u(x) = w(x)$ su $\partial B_R(x_0)$. □

Possiamo ora introdurre la candidata soluzione al problema (24)

$$\begin{cases} \Delta u = 0, & y \in \Omega \\ u(y) = g(y), & y \in \partial\Omega \end{cases} \tag{5.0.1}$$

dove $g(y) \in \mathscr{C}(\partial\Omega)$. Nel seguito assumeremo $\bar{\Omega}$ compatto. Definiamo (25)

$$v(x) = \sup\{w(x) \mid w \in \mathrm{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega}), w(y) \le g(y) \quad \forall y \in \partial\Omega\}. \tag{5.0.2}$$

Osserviamo che l'insieme di funzioni $w \in \mathscr{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ tale che $w(y) \le g(y) \quad \forall y \in \partial\Omega$ e' non vuoto. Infatti basta osservare che la funzione costante $w(x) = \inf_{\partial\Omega} g$ soddisfa questa proprieta'. Osserviamo anche che necessariamente $v(x) < \infty$. A tal fine osserviamo che per la Proposizione 4.6 necessariamente $w(x) \le \sup_{y\in\partial\Omega} g(y)$ per ogni $w \in \mathscr{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ tale che $w(y) \le g(y) \quad \forall y \in \partial\Omega$. Come primo passo provi- amo che $\Delta v = 0$ in Ω. Il secondo passo consistera' nell'individuare delle condizioni su Ω che garantiscano $v \in \mathscr{C}(\bar{\Omega})$, $v(y) = g(y)$ per $y \in \partial\Omega$.

Theorem 5.0.1. *Teorema 4.11. La funzione $v(x)$ definita in* (5.0.2) *é armonica in Ω.*

Proof. sia $x_0 \in \Omega$ e sia $B_R(x_0) \subset \Omega$. Selezioniamo quindi $w_{k,x_0}(x)$ tali che $w_{k,x_0} \in \mathscr{SUB}(\Omega) \cap \mathscr{C}(\bar{\Omega})$ e $w_{k,x_0}(y) \le g(y) \quad \forall y \in \partial\Omega$ ed inoltre $w_{k,x_0}(x_0) \overset{k\to\infty}{\to} v(x_0)$. Allora os- serviamo che siccome $v(x)$ e'definito come sup possiamo assumere le funzioni $w_n(x)$ crescenti (se non lo fossero basta lavorare con

$$\sup_{i=1,\dots,n} \left\{w_{1,x_0}(x), \dots, w_{n,x_0}(x)\right\}.$$

Inoltre possiamo anche assumere che $w_{k,x_0}(x)$ sia armonica su $B_R(x_0)$. Infatti se cosi' non fosse basterebbe considerare

$$\tilde{w}_{k,x_0}(x) = \begin{cases} w_{k,x_0}(x), & x \in \Omega \backslash B_R(x_0) \\ v_k(x), & x \in B_R(x_0) \end{cases}$$

dove $\Delta v_k = 0$ e $v_k(y) = w_{k,x_0}(y)$ per $y \in \partial B_R(x_0)$. Allora dalle proprieta' delle funzioni subarmoniche si vede che $\tilde{w}_{k,x_0}(x) \in \mathcal{SUB}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, inoltre $\tilde{w}_{k,x_0}(y) \leq g(y) \quad \forall y \in \partial\Omega$ ed anche $\tilde{w}_{k,x_0}(x) \geq w_{k,x_0}(x)$. Questa disuguaglianza e ovvia in $\Omega \backslash B_R(0)$ ed e' anche vera su $B_R(x_0)$ poiche' $-\bar{w}_{k,x_0}(x) + w_{k,x_0}(x) = -v_k(x) + w_{k,x_0}(x)$ e' subarmonica su $B_R(x_0)$ ed e' nulla su $\partial B_R(x_0)$. Quindi per Proposizione 4.6 si ha $\tilde{w}_{k,x_0}(x) \geq w_{k,x_0}(x)$. Riassumendo abbiamo $\tilde{w}_{k,x_0}(x)$ armoniche su $\overline{B_R}(x_0)$ ed inoltre $\tilde{w}_{k,x_0}(x)$ crescenti rispetto a k. Come conseguenza del Teorema della media e del Teorema di passaggio al limite sotto il segno di integrale si puo' provare che il limite puntuale di

funzioni armoniche, crescenti e uniformemente limitate e' armonica. Pertanto abbiamo che detta $\bar{w}_{x_0}(x) = \lim_{k\to\infty} \bar{w}_{k,x_0}(x)$ si ha che $\bar{w}_{x_0}(x)$ e' armonica su $B_R(x_0)$ ed inoltre $\bar{w}_{x_0}(x_0) = v(x_0)$ Dico che necessariamente (26)

$$v(x) = \bar{w}_{x_0}(x), \quad \forall x \in B_R(x_0)$$

e questo ci permetterebbe di conclude che $v(x)$ e' armonica su $B_R(x_0)$ e quindi su Ω data l'arbitrarieta' della palla $B_R(x_0)$. Se (26) non fosse vera avremmo $v(y) > \bar{w}_{x_0}(y)$ per qualche $y \in B_R(x_0)$ e ragionando come sopra potremmo considerare una successione $w_{k,y}$ tali che $w_{k,y}(y) \overset{k\to\infty}{\to} v(y)$. A meno di considerare $\max\{w_{k,y}, \tilde{w}_{k,x_0}\}$ possiamo assumere che $w_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$. Inoltre ragionando come sopra possiamo costruire una ulteriore nuova successione

$$\tilde{w}_{k,y}(x) = \begin{cases} w_{k,y}(x), & x \in \Omega \backslash B_R(x_0) \\ u_k(x), & x \in B_R(x_0) \end{cases}$$

dove $\Delta u_k = 0$ su $B_R(x_0)$ e $u_k = w_{k,y}$ su $\partial B_R(x_0)$ Usando il principio del massimo ed il fatto che $w_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ per ogni $x \in \Omega$ e quindi anche per $x \in \partial B_R(x_0)$ avremo che $\bar{w}_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ per ogni $x \in \Omega$ Definendo quindi $\bar{w}_y(x) = \lim_{k\to\infty} \bar{w}_{k,y}(x)$ avremo che $\bar{w}_y$ e' armonica su $B_R(x_0)$ ed inoltre (27)

$$v(x_0) = \bar{w}_y(x_0) \geq \bar{w}_{x_0}(x_0) = v(x_0)$$

La disuguaglianza in (27) segue dal fatto che $\tilde{w}_{k,y}(x) \geq \tilde{w}_{k,x_0}(x)$, per ogni $x \in \Omega$ invece la prima uguaglianza in (27) segue da

$$v(x_0) = \lim_{k\to\infty} \tilde{w}_{k,x_0}(x_0) \leq \lim_{k\to\infty} \tilde{w}_{k,y}(x_0) = v(x_0)$$

dove nell'ultima uguaglianza abbiamo usato che (data la definizione di $v(x_0)$ come sup) $v(x_0) \geq \lim_{k\to\infty} \tilde{w}_{k,y}(x_0) \geq \lim_{k\to\infty} \tilde{w}_{k,x_0}(x_0) = v(x_0)$. Osserviamo inoltre che sic- come $\bar{w}_{k,y}(x) \geq \bar{w}_{k,x_0}(x)$ su $B_R(x_0)$ e siccome entrambe sono armoniche e monotone crescenti e' facile dedurre che i loro limiti puntuali saranno armonici su $B_R(x_0)$ ed inoltre $\bar{w}_y(x) \geq \bar{w}_{x_0}(x)$ su $B_R(x_0)$. Siccome da

(27) segue che $\bar{w}_y(x_0) = \bar{w}_{x_0}(x_0)$ ne de- duciamo per il principio del massimo che $\bar{w}_y = \vec{w}_{x_0}$ su $B_R(x_0)$ e quindi $v(y) = \bar{w}_y(y) = \bar{w}_{x_0}(y)$, e quindi data l'arbitrarieta' di y abbiamo

Resta solo da capire se $v \in \mathscr{C}(\bar{\Omega})$ e se $v(y) = g(y)$ su $\partial\Omega$. Senza ipotesi ulteriori su Ω questa proprieta' e' falsa ma e' vera per una ampia classe di aperti come vedremo di seguito.

5.0.4 Eigenvalues of Laplace equation in bounded domain with Dirichlet boundary condition.

The general eigenvalue problem

$$\begin{aligned} \Delta u &= \lambda u, x \in \Omega \\ u(x) &= 0, x \in \partial\Omega. \end{aligned}$$

for the case of bounded domain Ω can have only negative eigenvalues. Indeed, multiplying by u and integrating into Ω we find

$$-\int_\Omega |\nabla u|^2 = \lambda \int_\Omega |u|^2$$

so if $\lambda \geq 0$ we get $u = cost = 0$. The fact that λ is real follows from this relation too.

Any two eigenfunctions $(j = 1,2)$

$$\begin{aligned} \Delta u_j &= \lambda_j u_j, x \in \Omega \\ u_j(x) &= 0, x \in \partial\Omega. \end{aligned}$$

we have the orthogonality relation

$$\int_\Omega u_1(x) u_2(x) dx = 0.$$

The set of all eigenfunctions is complete, i.e. if $f(x)$ is orthogonal to all eigenfunctions, then $f = 0$. This property needs more details from functional analysis and we shall stop here this argument.

5.0.5 Application: Nonlinear problem for Laplace equation with Dirichlet data

Let $\Omega = \{|x| < R\}$ be a bounded domain with smooth boundary $\partial\Omega = \{|x| = R\}$. Consider the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned} \tag{5.0.3}$$

where $F(u)$ is a C^1 function, such that

$$F(u) = O(|u|^p)$$

near $u = 0$.

Lemma 5.0.1. *(small data solutions) Let*

$$p > 1.$$

There exists $\varepsilon > 0$ such that for any $f(x) \in C(\partial\Omega)$ with

$$\max_{\partial\Omega} |f(x)| \le \varepsilon,$$

there exists a unique solution $u(x) \in C(\overline{\Omega}) \cap C^1(\Omega)$ to the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Idea of the proof. Let u_k be the sequence defined as follows $u_0 = 0$, and

$$\begin{aligned} \Delta u_{k+1} &= F(u_k), x \in \Omega \\ u_{k+1}(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Set

$$X_k = \sup_{|x| \le R} |u_k(x)|.$$

The the apriori estimates imply

$$X_k \le C\varepsilon + C X_{k+1}^p.$$

Show that this inequality implies that there exists a constant $C_0 > C$ so that

$$X_k \le C_0 \varepsilon.$$

Apply the contraction principle for the sequence u_k showing that

$$Y_k = \max_{|x| \le R} |u_k(x) - u_{k+1}(x)|$$

satisfies

$$Y_k \le q Y_{k-1}$$

with some $q \in (0,1)$. So $u_k(x)$ tends uniformly to a function $u(x)$ Using the Poisson formula one can show that $\partial_j u_k(x)$ tends uniformly to a function $\partial_j u(x)$ for $x \in K$, where K is any compact set in Ω.

5.0.6 Laplace equation in exterior domain with Dirichlet boundary condition.

Let $\Omega \subset \mathbb{R}^3$ be the exterior of a compact K with smooth boundary $\partial\Omega$. The general problem

$$\Delta u = F, x \in \Omega$$
$$u(x) = f(x), x \in \partial\Omega.$$

has to be considered together with a condition that guarantees the uniqueness of the solution, i.e.

$$u(x) = o(1)$$

as $x \to \infty$. As before the problem can be solved extending F in $\mathbb{R}^3$ and representing u as follows

$$u = u_0 + w,$$

where

$$u_0(x) = -\frac{1}{4\pi}\int_\Omega \frac{F(y)}{|x-y|}dy.$$

Then w solves

$$\Delta w = 0, x \in \Omega$$
$$w(x) = g(x), x \in \partial\Omega.$$

where

$$g = f - u_0.$$

Applying the estimates of the previous sections, we get

$$\|(1+|x|)w\|_{L^\infty(\Omega)} \leq C\|g\|_{C(\partial\Omega)} \leq C\left(\|f\|_{C(\partial\Omega)} + \|(1+|x|)u_0\|_{L^\infty(\Omega)}\right) \leq$$

$$\leq C\left(\|f\|_{C(\partial\Omega)} + \|(1+|x|)^{3+\varepsilon}u_0\|_{L^\infty(\Omega)}\right).$$

Problem 5.0.8. *Generalize the above argument for $n \geq 3$.*

This estimate guarantees the uniqueness of the solution. The existence is delicate: needs or sub and supersolutions (Peron method) or methods from Functional analysis (Friedrich's extentions). For the concrete domains: interior or exterior ball, half space we have concrete solutions.

5.0.7 Exterior nonlinear problem for Laplace equation with Dirichlet data

Let $\Omega = \{|x| > R\}$ be an exterior domain with smooth boundary $\partial\Omega = \{|x| = R\}$. Consider the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned} \tag{5.0.4}$$

where $F(u)$ is a C^1 function, such that

$$F(u) = O(|u|^p)$$

near $u = 0$.

Lemma 5.0.2. *(small data solutions) Let*

$$p > 3.$$

There exists $\varepsilon > 0$ such that for any $f(x) \in C(\partial\Omega)$ with

$$\max_{\partial\Omega} |f(x)| \le \varepsilon,$$

there exists a unique solution $u(x) \in C(\overline{\Omega}) \cap C^1(\Omega)$ to the problem

$$\begin{aligned} \Delta u &= F(u), x \in \Omega \\ u(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Idea of the proof. Let u_k be the sequence defined as follows $u_0 = 0$, and

$$\begin{aligned} \Delta u_{k+1} &= F(u_k), x \in \Omega \\ u_{k+1}(x) &= f(x), x \in \partial\Omega. \end{aligned}$$

Set

$$X_k = \sup_{|x| \ge R} |(1 + |x|) u_k(x)|.$$

The the apriori estimates and the assumption $p > 3$ imply

$$X_k \le C\varepsilon + C X_{k+1}^p.$$

Show that this inequality implies that there exists a constant $C_0 > C$ so that

$$X_k \le C_0 \varepsilon.$$

Apply the contraction principle for the sequence u_k showing that

$$Y_k = \max_{|x|\le R} |u_k(x) - u_{k+1}(x)|$$

satisfies

$$Y_k \le q Y_{k-1}$$

with some $q \in (0,1)$. So $u_k(x)$ tends uniformly to a function $u(x)$ Using the Poisson formula one can show that $\partial_j u_k(x)$ tends uniformly to a function $\partial_j u(x)$ for $x \in K$, where K is any compact set in Ω. Hence $F(u_k)$ tends uniformly in $C^1(K)$ and we conclude that $u \in C^2(\Omega)$.

Problem 5.0.9. *Generalize the above argument for $n \ge 3$.*

5.1 Bessel functions and Stoke's phenomena

The standard Bessel functions $J_\nu(z)$ are solutions to the differential equation

$$z^2 \frac{d^2}{dz^2} w(z) + z \frac{d}{dz} w(z) + (z^2 - \nu^2) w(z) = 0 \tag{5.1.5}$$

More precisely, we have the definition of $J_\nu(z)$ given by

$$J_\nu(z) = z^\nu \left(\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m m! \Gamma(m+1+\nu)} \right), \tag{5.1.6}$$

where $z^\nu = e^{\nu \log z}$ with $\log z = \log|z| + i \arg z$ being the branch of the logarithm defined via the choice of $\arg z$. For example the principle branch is with $\arg z \in (-\pi, \pi)$. Therefore, $J_\nu(z)$ is well defined and analytic for $|\arg z| < \pi$ and has singularity at $z = 0$ when ν is not an integer. The equation (5.1.5) has two linearly independent solutions $J_\nu(z)$ and $J_{-\nu}(z)$.

Near the origin we have the asymptotic expansion

$$J_\nu(z) = z^\nu \left(1 + O(|z|^2)\right), \quad z \to 0. \tag{5.1.7}$$

The fact that a solution of an ODE, near an irregular singularity, in different sectors of the complex plane in general shows different asymptotic behavior was observed and studied by Stokes and is, therefore, named Stokes' phenomenon.

More precisely, the asymptotic expansion for the $J_\nu(x)$ can be found from section 7.2 in [28]

$$\begin{aligned} J_\nu(z) = {} & c_1 \left(\frac{2}{\pi}\right)^{1/2} e^{-(\log|z| + \mathrm{i}\arg z)/2} e^{i(z - \pi\nu/2 - \pi/4)} \left(1 + O\left(|z|^{-1}\right)\right) + \\ & + c_2 \left(\frac{2}{\pi}\right)^{1/2} e^{-(\log|z| + \mathrm{i}\arg z)/2} e^{-i(z - \pi\nu/2 - \pi/4)} \left(1 + O\left(|z|^{-1}\right)\right), \end{aligned} \tag{5.1.8}$$

where the constants c_1, c_2 depend on the choice of the sector in the complex plane.

Indeed, for any integer p we have

$$c_1 = c_2 = \frac{1}{2} e^{\mathrm{i}p(2\nu+1)\pi} \text{ if } \arg z \in ((2p-1)\pi, (2p+1)\pi) \tag{5.1.9}$$

and

$$c_1 = \frac{1}{2} e^{\mathrm{i}(p+1)(2\nu+1)\pi}, c_2 = \frac{1}{2} e^{\mathrm{i}p(2\nu+1)\pi} \text{ if } \arg z \in (2p\pi, (2p+2)\pi) \tag{5.1.10}$$

due to the Stokes phenomenon (see section 7.2 in [28]).

The proof of these asymptotic expansions can be deduced via the relation

$$J_\nu(ze^{\mathrm{i}m\pi}) = e^{\mathrm{i}m\pi\nu} J_\nu(z) \tag{5.1.11}$$

valid for any integer m. The Neumann function is defined as a linear combination of $J_\pm$

$$Y_\nu(z) = \frac{1}{\sin(\nu\pi)} \left(\cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)\right). \tag{5.1.12}$$

The Hankel functions are given by

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + \mathrm{i}Y_\nu(z), \\ H_\nu^{(2)}(z) &= J_\nu(z) - \mathrm{i}Y_\nu(z). \end{aligned} \tag{5.1.13}$$

Their integral representation (see 7.3.6 (28) in [**?**]) provided $\arg z \in (0, \pi)$

$$\begin{aligned} \pi H_\nu^{(1)}(z) &= -\mathrm{i}e^{-\mathrm{i}\nu\pi/2} \int_{-\infty}^{\infty} e^{\mathrm{i}z\cosh t} e^{-\nu t} dt = \\ &= -\mathrm{i}e^{-\mathrm{i}\nu\pi/2} \int_0^\infty e^{\mathrm{i}z(s+1/s)/2} s^{-\nu-1} ds \end{aligned} \tag{5.1.14}$$

Some partial values of $\nu = \pm 1/2$ give the following result

$$H_{-1/2}(w) = iH_{1/2}(w)$$

$$H_{1/2}(w) = -i\left(\frac{2}{\pi w}\right)^{1/2} e^{iw}.$$

In particular we shall need for $w = iy$ with $y > 0$ the following

$$H_{1/2}(iy) = -ie^{-i\pi/4} \left(\frac{2}{\pi y}\right)^{1/2} e^{-y}. \tag{5.1.15}$$

$$H_{-1/2}(iy) = e^{-i\pi/4} \left(\frac{2}{\pi y}\right)^{1/2} e^{-y}. \tag{5.1.16}$$

The modified Bessel functions are combinations of $J_\nu(iz), J_{-\nu}(\mathrm{i}z)$ and therefore they are solutions to the differential equation

$$z^2\frac{d^2}{dz^2}w(z)+z\frac{d}{dz}w(z)-(z^2+\nu^2)w(z)=0. \tag{5.1.17}$$

The modified Bessel function $I_\nu(z)$ is defined by

$$I_\nu(z)=e^{-\mathrm{i}\nu\pi/2}J_\nu(\mathrm{i}z). \tag{5.1.18}$$

For $\nu=0$ we have the expansion near the origin

$$I_0(z)=1+O(|z|^2). \tag{5.1.19}$$

The relation (5.1.11) implies

$$I_\nu(ze^{\mathrm{i}m\pi})=e^{\mathrm{i}m\pi\nu}I_\nu(z) \tag{5.1.20}$$

valid for any integer m.

Then two linearly independent solutions to (5.1.17) are $I_\nu(z)$ and $I_{-\nu}(z)$, for $\nu\neq 0$.

The asymptotics expansion of $I_\nu(z)$ for $|z|\to\infty$ depends where is $\mathrm{Arg}z$, therefore we have to be careful for Stokes phenomena. If $\arg z\in(-\pi/2,3\pi/2)$, then we have the asymptotics

$$\begin{aligned}I_\nu(z)=&\frac{e^z}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}\left(1+O(|z|^{-1})\right)+\\&+\frac{e^{-z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right).\end{aligned} \tag{5.1.21}$$

From this we take $\arg z\in(-\pi/2+m\pi,3\pi/2+m\pi)$ we get

$$\begin{aligned}I_\nu(z)=&\frac{e^{(-1)^m z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}m(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right)+\\&+\frac{e^{-(-1)^m z}}{\sqrt{2\pi}}e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{\mathrm{i}(m+1)(\nu+1/2)\pi}\left(1+O(|z|^{-1})\right).\end{aligned} \tag{5.1.22}$$

In this way we conclude

$$\begin{aligned}&\arg z\in(-\pi/2+m\pi,3\pi/2+m\pi)\implies I_\nu(z)=\\&=\frac{e^{-\mathrm{i}\arg z)/2}}{\sqrt{2\pi|z|}}\left[e^{(-1)^m z}e^{\mathrm{i}m(\nu+1/2)\pi}+e^{-(-1)^m z}e^{\mathrm{i}(m+1)(\nu+1/2)\pi}\right]\left(1+O(|z|^{-1})\right).\end{aligned} \tag{5.1.23}$$

If $\arg z \in (-3\pi/2+2p\pi, \pi/2+2p\pi)$, p integer, then we take $m = 2p-1$ and then we get

$$I_\nu(z) = \frac{e^{-\mathrm{i}\arg z)/2}}{\sqrt{2\pi|z|}}\left[e^{-z}e^{\mathrm{i}(2p-1)(\nu+1/2)\pi} + e^{z}e^{\mathrm{i}2p(\nu+1/2)\pi}\right]\left(1+O(|z|^{-1})\right). \tag{5.1.24}$$

The modified Bessel function $K_\nu(z)$ is defined by

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)}\left(I_{-\nu}(z) - I_\nu(z)\right) = \tag{5.1.25}$$

$$= \frac{\pi}{2\sin(\nu\pi)}\left(e^{\mathrm{i}\nu\pi/2}J_{-\nu}(\mathrm{i}z) - e^{-\mathrm{i}\nu\pi/2}J_\nu(\mathrm{i}z)\right).$$

Obviously, we have

$$K_\nu(z) = K_{-\nu}(z). \tag{5.1.26}$$

We have the following relation between $K_\nu(z)$ and Hankel functions (see 10.27.8 in [**?**]) with $-\pi \le \arg z \le \pi/2$

$$K_\nu(z) = \frac{\pi\mathrm{i}}{2}e^{\nu\mathrm{i}\pi/2}H_\nu^{(1)}(\mathrm{i}z). \tag{5.1.27}$$

The differential equation satisfied by $K_\nu(z)$ is again (5.1.17).
Similarly to (5.1.32) , we have the relations

$$K_\nu(ze^{\mathrm{i}m\pi}) = e^{-\mathrm{i}m\pi\nu}K_\nu(z) - i\pi\frac{\sin(m\nu\pi)}{\sin(\nu\pi)}I_\nu(z). \tag{5.1.28}$$

valid for any integer m.
Near the origin we have the expansion

$$K_0(z) = -I_0(z)\log\left(\frac{z}{2}\right) + O(1) = -\log(z) + O(1), \quad z \to 0. \tag{5.1.29}$$

We have the following asymptotic expansion valid if $|z| \to \infty$ and $\arg z \in (-3\pi/2, 3\pi/2)$ (see relation (20), section 7.23 in [28])

$$K_\nu(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{-(\log|z|+\mathrm{i}\arg z)/2}e^{-z}\left(1+O(|z|^{-1})\right). \tag{5.1.30}$$

For $\arg w \in (\pi/2, 5\pi/2)$ we can take $w = e^{\mathrm{i}2\pi}z$ so that

$$\arg w \in (\pi/2, 5\pi/2) \Longrightarrow \operatorname{Arg} z \in (-3\pi/2, 3\pi/2)$$

use (5.1.32), so we can write

$$K_\nu(w) = K_\nu(e^{2\mathrm{i}\pi}z) = \frac{\pi}{2\sin(\nu\pi)}\left(I_{-\nu}(e^{2\mathrm{i}\pi}z) - I_\nu(e^{2\mathrm{i}\pi}z)\right) =$$

$$= \frac{\pi}{2\sin(\nu\pi)}\left(e^{-2i\pi\nu}I_{-\nu}(z) - e^{2i\pi\nu}I_{\nu}(z)\right)$$

and using the asymptotic expansions (5.1.24) for z we arrive at

(5.1.31) $$K_\nu(w) = \sqrt{\frac{\pi}{2}}\frac{e^{-i(arg w)/2}}{\sqrt{|w|}}\left[2i\cos(\nu\pi)e^{w} + e^{-w}\right]\left(1 + O(|w|^{-1})\right)$$

(5.1.32) $$I_\nu(ze^{im\pi}) = e^{im\pi\nu}I_\nu(z)$$

For $\nu = 1/4$ we get

(5.1.33) $$K_{1/4}(z) = \frac{\pi}{\sqrt{2}}(I_{-1/4}(z) - I_{1/4}(z)) = \frac{\pi}{\sqrt{2}}\left(e^{i\pi/8}J_{-1/4}(iz) - e^{-i\pi/8}J_{1/4}(iz)\right).$$

The modified Bessel function $K_\nu(z)$ has the integral representation (see Eq. (16) in section 7.3.4 in [2])

(5.1.34) $$\Gamma(\nu+1/2)K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2}dt$$

provided $\mathrm{Re}(\nu) > -1/2$, $|\arg(z)| < \pi$. If $z > 0$, the a change of variable implies

(5.1.35) $$\Gamma(\nu+1/2)K_\nu(z) = \sqrt{\pi}\frac{e^{-z}}{2^\nu}z^\nu\int_0^\infty e^{-tz}t^{\nu-1/2}(2+t)^{\nu-1/2}dt.$$

In equivalent way, we can write

(5.1.36) $$\sqrt{\pi}e^{-z}\int_0^\infty e^{-tz}t^{\nu-1/2}(2+t)^{\nu-1/2}dt = z^{-\nu}2^\nu\Gamma(\nu+1/2)K_\nu(z).$$

5.2 Foundamental solution of the Helmholtz equation

The Helmholtz equation

$$(\lambda - \Delta)G = \delta$$

with $\lambda > -$ has a fundamental solution

$$G_{\lambda,n}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\xi}\frac{d\xi}{\lambda+|\xi|^2} = (2\pi)^{-n}\mathrm{Re}\left(\int_{\mathbb{R}^n} e^{-ix\xi}\frac{d\xi}{\lambda+|\xi|^2}\right).$$

Here and below $\lambda > 0$. Setting

$$G_n(x) = G_{1,n}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} \frac{d\xi}{1+|\xi|^2},$$

we see that

$$G_{\lambda,n}(x) = \lambda^{(n-2)/2} G_n(\sqrt{\lambda}|x|). \tag{5.2.37}$$

The main result of the section is the following

Lemma 5.2.1. *We have the relation*

$$G_{\lambda,n}(x) = (2\pi)^{-n/2} \lambda^{(n-2)/4} \frac{K_{(n-2)/2}(\sqrt{\lambda}|x|)}{|x|^{(n-2)/2}}. \tag{5.2.38}$$

Proof. **Case:** $n = 1$

In this case, the right side of (5.2.38) is given by

$$(2\pi)^{-1/2}|x|^{1/2} K_{-1/2}(\sqrt{\lambda}|x|)$$

Since

$$K_{1/2}(|x|) = K_{-1/2}(|x|) = \sqrt{\frac{\pi}{2}} \frac{e^{-|x|}}{\sqrt{|x|}},$$

we see that (5.2.38) becomes

$$G_{\lambda,1}(x) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x|} \tag{5.2.39}$$

We have the relation

$$G_1(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-\mathrm{i}x\xi} \frac{d\xi}{1+\xi^2} =$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos(x\xi) \frac{d\xi}{1+\xi^2} = (2\pi)^{-1} \int_{\infty}^{\infty} e^{\mathrm{i}x\xi} \frac{d\xi}{1+\xi^2}.$$

The function $K_1(x)$ is even and hence it is sufficient to consider the case $x > 0$. A simple application of the Cauchy theorem implies that for any $x > 0$ we have the identities

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}xz}}{1+z^2}\, dz = \frac{\mathrm{i}^{-1}}{2} e^{-x} \tag{5.2.40}$$

and more generally for any $\lambda > 0$ we have

$$\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}xz}}{\lambda+z^2}\, dz = \int_{-\infty}^{\infty} \frac{\cos(xz)}{\lambda+z^2}\, dz = \frac{\pi}{\sqrt{\lambda}} e^{-\sqrt{\lambda}|x|} \tag{5.2.41}$$

so

$$G_1(x) = \frac{e^{-|x|}}{2} \tag{5.2.42}$$

and hence we have (5.2.39).

Case: $n = 2$**.**

We assume $x = (0, |x|)$ and then

$$G_2(x) = (2\pi)^{-2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{e^{-i|x|\xi_2} d\xi_2}{1+\xi_1^2+\xi_2^2} \right) d\xi_1.$$

So applying (5.2.41) with $\lambda = 1+\xi_1^2$, we find

$$G_2(x) = \int_0^{\infty} \frac{1}{2\pi\sqrt{1+\xi_1^2}} e^{-\sqrt{1+\xi_1^2}|x|} d\xi_1$$

Now we make change of variables $\xi_1 \in (0,\infty) \to t \in (0,\infty)$ defined by

$$1+\xi_1^2 = (1+t)^2, \implies \xi_1 = \sqrt{t(t+2)}, \;\; \xi_1 d\xi_1 = (t+1)dt$$

and get

$$G_2(x) = (2\pi)^{-1} e^{-|x|} \int_0^{\infty} e^{-t|x|} t^{-1/2} (2+t)^{-1/2} dt$$

Using (5.1.36), we find

$$G_2(x) = \frac{1}{2\pi^{3/2}} \Gamma(1/2) K_0(|x|).$$

Using the identity

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

we find

$$G_2(x) = (2\pi)^{-1} K_0(|x|). \tag{5.2.43}$$

Further, (5.2.37) implies

$$G_{\lambda,2}(x) = (2\pi)^{-1} K_0(\sqrt{\lambda}|x|). \tag{5.2.44}$$

Case: $n = 3$**.**

We have the relations

$$G_3(x) = (2\pi)^{-2} \int_0^{\infty} \left(\int_0^{\pi} e^{-i|x|\rho\cos\theta} \sin\theta d\theta \right) \frac{\rho^2 d\rho}{1+\rho^2} =$$

$$= \frac{1}{(2\pi)^2 \mathrm{i}|x|} \int_0^\infty \left(e^{i|x|\rho} - e^{-i|x|\rho} \right) \frac{\rho d\rho}{1+\rho^2} = \frac{1}{2\pi^2 |x|} \int_0^\infty \sin(|x|\rho) \frac{\rho d\rho}{1+\rho^2}.$$

A simple application of the Cauchy theorem implies that for any $R > 0$ we have the identities

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}Rz}}{1+z^2} z^k dz = \frac{\mathrm{i}^{k-1}}{2} e^{-R}. \tag{5.2.45}$$

Hence, we have the relations

$$\pi \mathrm{i}^k e^{-R} = \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}Rz}}{1+z^2} z^k dz =$$

$$= \int_0^\infty \frac{e^{\mathrm{i}Rz} + (-1)^{k+2} e^{-\mathrm{i}Rz}}{1+z^2} z^k dz.$$

Now we can use the fact that

$$e^{\mathrm{i}Rz} + (-1)^{k+2} e^{-\mathrm{i}Rz} = \begin{cases} 2\cos(Rz), & \text{if } k \text{ is even, } z > 0, R > 0; \\ 2\mathrm{i}\sin(Rz), & \text{if } k \text{ is odd, } z > 0, R > 0. \end{cases}$$

Thus for k even we have

$$\int_0^\infty \frac{2\cos(Rz)}{1+z^2} z^k dz = \pi \mathrm{i}^k e^{-R}$$

and we deduce

$$\int_0^\infty \frac{\cos(R\rho)}{1+\rho^2} \rho^k dz = \frac{\pi}{2} (-1)^{k/2} e^{-R}. \tag{5.2.46}$$

For k odd we have

$$\int_0^\infty \frac{\sin(R\rho)}{1+\rho^2} \rho^k dz = \frac{\pi}{2} (-1)^{(k-1)/2} e^{-R}. \tag{5.2.47}$$

Applying now (5.2.47) with $k = 1$, we find

$$G_3(x) = \frac{1}{4\pi|x|} e^{-|x|}. \tag{5.2.48}$$

Then (5.2.37) implies

$$G_{\lambda,3}(x) = \frac{1}{4\pi|x|} e^{-\sqrt{\lambda}|x|}. \tag{5.2.49}$$

General case $n \geq 3$

Since $G_n(x)$ is a radial function, we can take $x = (0, \cdots, 0, x_n) = (0, \cdots, 0, |x|)$. We have to compute

$$G_{1,n}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} \frac{d\xi}{1+|\xi|^2} =$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} e^{-\mathrm{i}|x|\xi_n} \frac{d\xi_n}{1+|\xi'|^2+\xi_n^2} \right) d\xi'.$$

Applying the identity (5.2.41), we find

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}|x|\xi_n} \frac{d\xi_n}{1+|\xi'|^2+\xi_n^2} = \frac{\pi}{\sqrt{1+|\xi'|^2}} e^{-\sqrt{1+|\xi'|^2}|x|}$$

and introducing polar coordinates $\rho = |\xi'|$ in R^{n-1} we get

$$G_{1,n}(x) = (2\pi)^{-n} \mu(S^{n-2}) \int_0^{\infty} \frac{\pi}{\sqrt{1+|\rho|^2}} e^{-\sqrt{1+|\rho|^2}|x|} \rho^{n-2} d\rho$$

Now we make change of variables $\rho \in (0,\infty) \to t \in (0,\infty)$ defined by

$$1+\rho^2 = (1+t)^2, \implies \rho = \sqrt{t(t+2)}, \ \ \rho d\rho = (t+1)dt$$

and get

$$G_{1,n}(x) = (2\pi)^{-n} \mu(S^{n-2}) \pi e^{-|x|} \int_0^{\infty} e^{-t|x|} t^{(n-2)/2-1/2} (2+t)^{(n-2)/2-1/2} dt$$

Using (5.1.36), we find

$$G_{1,n}(x) = (2\pi)^{-n} \mu(S^{n-2}) \sqrt{\pi} 2^{\nu} \Gamma(\nu+1/2) \frac{K_{\nu}(|x|)}{|x|^{\nu}}, \nu = \frac{n-2}{2} \tag{5.2.50}$$

Comparing the relation (5.2.48) with (5.2.50) and using the relations

$$K_{1/2}(|x|) = \sqrt{\frac{\pi}{2}} \frac{e^{-|x|}}{\sqrt{|x|}}, \ \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \ \ \mu(\mathbb{S}^1) = 2\pi,$$

we see that (5.2.50) with $n = 3$ coincides with (5.2.48).

Since $\mu(\mathbb{S}^{n-2}) = (2\pi)^{(n-1)/2} / \Gamma((n-1)/2)$, we find

$$G_n(x) = G_{1,n}(x) = (2\pi)^{-n/2} \frac{K_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}. \tag{5.2.51}$$

Further, (5.2.37) implies (5.2.38).

□

Now we turn to the case of Helmholtz equation

$$(-z^2 - \Delta)\mathfrak{G} = \delta$$

with $z = i\sqrt{\lambda}$. Its fundamental solution is

$$\mathfrak{G}(x) = \mathfrak{G}_{z,n}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi}\frac{d\xi}{-z^2+|\xi|^2} = G_{-z^2,n}(|x|).$$

First we consider the case $n = 1$. Then for any z with $\mathrm{Im} z > 0$ we have

$$\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}xw}}{-z^2+w^2}\,dw = \frac{\mathrm{i}\pi}{z}e^{\mathrm{i}z|x|} \tag{5.2.52}$$

and therefore

$$\mathfrak{G}_{z,1}(x) = \frac{\mathrm{i}}{2z}e^{\mathrm{i}z|x|} \tag{5.2.53}$$

From Lemma 5.2.1 we have the relation

$$G_{\lambda,n}(x) = (2\pi)^{-n/2}\lambda^{(n-2)/4}\frac{K_{(n-2)/2}(\sqrt{\lambda}|x|)}{|x|^{(n-2)/2}}$$

so

$$\mathfrak{G}_{z,n}(x) = G_{-z^2,n}(|x|) = (2\pi)^{-n/2}|z|^{(n-2)/2}\frac{K_{(n-2)/2}(-iz|x|)}{|x|^{(n-2)/2}}.$$

Now we use the relation (5.1.27) and find

$$K_{(n-2)/2}(-iz|x|) = \frac{\pi\mathrm{i}}{2}e^{(n-2)\mathrm{i}\pi/4}H^{(1)}_{(n-2)/2}(z|x|)$$

so

$$\mathfrak{G}_{z,n}(x) = (2\pi)^{-n/2}\frac{\pi\mathrm{i}}{2}|z|^{(n-2)/2}e^{(n-2)\mathrm{i}\pi/4}\frac{H^{(1)}_{(n-2)/2}(z|x|)}{|x|^{(n-2)/2}}. \tag{5.2.54}$$

For $n = 1$ we use

$$\mathfrak{G}_{z,1}(x) = (2\pi)^{-1/2}\frac{\pi\mathrm{i}}{2}|z|^{-1/2}e^{-\mathrm{i}\pi/4}|x|^{1/2}H^{(1)}_{-1/2}(z|x|)$$

and (5.1.16) and find

$$\mathfrak{G}_{z,1}(x) = (2\pi)^{-1/2}\frac{\pi\mathrm{i}}{2}|z|^{-1/2}e^{-\mathrm{i}\pi/4}|x|^{1/2}e^{-i\pi/4}\left(\frac{2}{\pi|z||x|}\right)^{1/2}e^{\mathrm{i}z|x|} =$$

$$= (2\pi)^{-1/2}\frac{\pi\mathrm{i}}{2}\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{\mathrm{i}z|x|}}{z} = \frac{\mathrm{i}}{2z}e^{\mathrm{i}z|x|}$$

and this relation is compatible with (5.2.53). For $n = 2$ we have

$$\mathfrak{G}_{z,2}(x) = (2\pi)^{-1}\frac{\pi\mathrm{i}}{2}H_0^{(1)}(z|x|) = \frac{i}{4}H_0^{(1)}(z|x|).$$

This result is compatible with (5.15) Chapter I.5 in [1].

5.2.1 Helmholz equation in the space $\mathbb{R}^3$.

The equation

$$\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3$$

is called Helmholz equation Taking $f(x) \in C(\mathbb{R}^3)$ with compact support one can represent the unique solution as follows

$$u(x) = -\frac{1}{4\pi}\int_K \frac{e^{\pm i\lambda|x-y|}f(y)}{|x-y|}dy,$$

where here and below K denotes the support of f.

One can verify that

$$(\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x-y|}}{|x|}\right) = -4\pi\delta$$

in the sense of distributions in $\mathbb{R}^3$. Indeed taking any test function φ we apply Gauss - Green formula for the domain $\{|x| \geq \varepsilon\}$ and using the fact that

$$(\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) = 0 \quad |x| \neq 0,$$

we find

$$\int_{|x|>\varepsilon}\left((\Delta + \lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\right)\varphi(x)dx - \int_{|x|>\varepsilon}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)(\Delta + \lambda^2)\varphi(x)dx =$$

$$-\int_{|x|=\varepsilon}\partial_r\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\varphi(x)dS_x + \int_{|x|=\varepsilon}\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right)\partial_r\varphi(x)dS_x,$$

where here and below

$$\partial_r = \sum_{j=1}^{n}\frac{x_j}{|x|}\partial_j.$$

Taking into account the fact that

$$\partial_r\left(\frac{1}{|x|}\right) = -\frac{1}{|x|^2}$$

and introducing spherical coordinate $x = \varepsilon\omega, |\omega| = 1$, we find

$$\int_{|x|=\varepsilon}\partial_r\left(\frac{1}{|x|}\right)\varphi(x)dS_x = -\int_{|\omega|=1}\varphi(\varepsilon\omega)dS_\omega,$$

$$\int_{|x|=\varepsilon}\left(\frac{1}{|x|}\right)\partial_r\varphi(x)dS_x = \varepsilon\int_{|\omega|=1}\partial_r\varphi(\varepsilon\omega)d\omega$$

so taking the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \partial_r \left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) \varphi(x)dS_x = -4\pi\varphi(0),$$

$$\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) \partial_r\varphi(x)dS_x = 0,$$

so we arrive at

$$-\int_{\mathbb{R}^3} \left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) (\Delta+\lambda^2)\varphi(x)dx = 4\pi\varphi(0)$$

and the identity

$$-\frac{1}{4\pi}\,(\Delta+\lambda^2)\left(\frac{e^{\pm i\lambda|x|}}{|x|}\right) = \delta.$$

The function

$$E_\pm(x) \in C^\infty(\mathbb{R}^3 \setminus 0)$$

satisfying

$$(\Delta+\lambda^2)E_\pm = \delta$$

in the sense of distributions is called fundamental solutions of the Helmholz operator and they enable one to represent the solution of the Laplace equation

$$\Delta u = f, \;\; f \in C_0^\infty,$$

as follows

$$u_\pm(x) = \int_{\mathbb{R}^n} E_\pm(x-y)f(y)dy.$$

The uniqueness of the solution is guaranteed by the radiation condition

$$u(x) = \frac{e^{\pm i\lambda|x|}}{|x|} a\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|^2}\right)$$

at infinity.

Problem 5.2.1. *(smoothing property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 5.2.2. *(smoothing property) If $f(x) \in L^\infty(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$*

Problem 5.2.3. *(smoothing property) If $f(x) \in C^k(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^{k+1}(\mathbb{R}^n)$*

Problem 5.2.4. *(smoothing and decay property) If $f(x) \in C(\mathbb{R}^n)$ has a compact support, then $u(x) \in C^1(\mathbb{R}^n)$ and*

$$|u(x)| \leq \frac{C\|f\|_{C(K)}}{1+|x|}.$$

Problem 5.2.5. *(representation) If $f(x) \in C(\mathbb{R}^3)$ satisfies the estimate*

$$|f(x)| \leq ce^{-|x|}$$

and $u(x) \in C^2(\mathbb{R}^3)$ is a bounded function that is a solution to

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

then

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\lambda|x-y|} f(y)}{|x-y|} dy.$$

Hint: Apply the max principle and derive the uniqueness.

Problem 5.2.6. *(a priori estimate) If $f(x) \in C(\mathbb{R}^3)$ satisfies the estimate*

$$|f(x)| \leq ce^{-A|x|}$$

and $u(x) \in C^2(\mathbb{R}^3)$ is a bounded function that is a solution to

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

with $0 < \lambda < A$ then

$$|u(x)| \leq \frac{C\, e^{-\lambda|x|}}{|x|}.$$

Problem 5.2.7. *(some integrals for radial functions)*

$$\int_{\mathbb{S}^2} F(|x + r\omega|) d\omega = \frac{c}{|x| r}\int_{||x|-r|}^{|x|+r} F(\lambda)\lambda d\lambda$$

Problem 5.2.8. *(representation for radial solutions) If $f(x) = f(|x|) \in C(\mathbb{R}^3)$ satisfies the estimate*

$$|f(x)| \leq ce^{-|x|}$$

and $u(x) = u(|x|) \in C^2(\mathbb{R}^3)$ is a bounded radial function that is a solution to

$$-\Delta u(x) + \lambda^2 u = f(x), x \in \mathbb{R}^3,$$

then

$$u(x) = -\frac{c}{|x|}\int_0^\infty e^{-r}\int_{||x|-r|}^{|x|+r} f(\lambda)\lambda d\lambda dr =$$

$$= \int_0^{|x|} e^{-|x|}\sinh(\lambda) f(\lambda)\lambda d\lambda + \int_{|x|}^\infty \sinh(|x|) e^{-\lambda} f(\lambda)\lambda d\lambda.$$

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