ANALISI MATEMATICA (B)

LEZIONE 70

23.3.2020

Es 4. $\displaystyle\int_2^{+\infty} \frac{1}{x^p (\ln x)^q}\,dx \sim \sum \frac{1}{k^p \ln^q k}$

$p=1$: $\displaystyle\int_2^{+\infty} \frac{1}{x \ln^q x}\,dx = \int_{\ln 2}^{+\infty} \frac{dy}{y^q}$ converge $\Leftrightarrow q>1$

$y = \ln x$
$\frac{dx}{x} = dy$

se $p>1$ per $x \to +\infty$ $\displaystyle\frac{1}{x^p(\ln x)^q} \ll \frac{1}{x^{p-\varepsilon}}$ $\quad \displaystyle\int_2^{+\infty} \frac{1}{x^{p-\varepsilon}} < +\infty$ se $p-\varepsilon>1$

$\displaystyle \gg \frac{1}{x^{p+\varepsilon}}$ $\quad \displaystyle\int \frac{1}{x^{p+\varepsilon}} = +\infty$ se $p+\varepsilon<1$.

se $p<1$

Es 5. $\displaystyle\int_0^{1/2} \frac{1}{x^p |\ln x|^q}\,dx$

converge se $p<1$
diverge se $p>1$

come prima
ma per $x \to 0$

se $p=1$ $\displaystyle\int_0^{1/2} \frac{dx}{x(-\ln x)^q} = -\int_{+\infty}^{\ln 2} \frac{dy}{y^q} = \int_{\ln 2}^{+\infty} \frac{dy}{y^q}$ converge $\Leftrightarrow q>1$.

$y = -\ln x$
$dy = -\frac{dx}{x}$

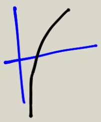
$y = \frac{1}{x^p |\ln x|^q}$

Es 4.

Es 5.

Es 6

$$\int_{\frac{1}{2}}^{2} \frac{1}{x^p |\ln x|^q} dx$$

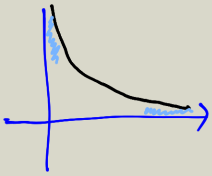


per $x \to 1$

$$\frac{1}{x^p |\ln x|^q} \sim \frac{1}{|\ln x|^q} \sim \frac{1}{|x-1|^q}$$

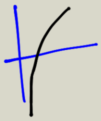
$\ln x = \ln(1 + (x-1)) \sim x - 1$

$$\int_{1-\varepsilon}^{1+\varepsilon} \frac{1}{|x-1|^q}$$ converge $\Leftrightarrow$ $q < 1$



Es 1

$$\int_0^{+\infty} \frac{1}{(1+x)^\alpha \cdot x^\beta}\, dx$$



per $x \to +\infty$ $\quad \frac{1}{(1+x)^\alpha x^\beta} \sim \frac{1}{x^{\alpha+\beta}}$ se $x \to +\infty$

per $x \to 0^+$ $\quad \frac{1}{(1+x)^\alpha x^\beta} \sim \frac{1}{x^\beta}$

$$\left.\begin{array}{l} \alpha+\beta>1 \\ \beta<1 \end{array}\right\} \text{converge.}$$

<u>Esercizio per casa</u> trovare $f:[0,+\infty) \to \mathbb{R}$, continua, positiva, $\int_0^{+\infty} f(x)\,dx < +\infty$

ma non $\lim_{x\to+\infty} f(x) = 0$.

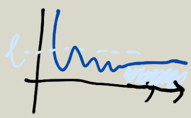
GIULIO $\int \frac{1}{x^2 \sin^2 x}$

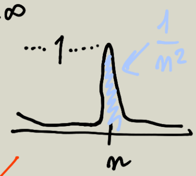
ALESSIO $\int_1^{+\infty} \sin^2\left(\frac{1}{x}+1\right)$

Oss

se $\lim_{x\to+\infty} f(x) = \ell > 0$ per $x\to+\infty$ $f(x) \sim \ell$

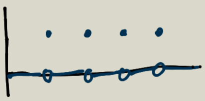
allora $\int^{+\infty} f(x) = +\infty$ $\int \ell = +\infty$

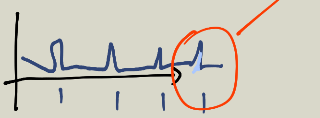




$\sin(e^{-x})$

Giulio





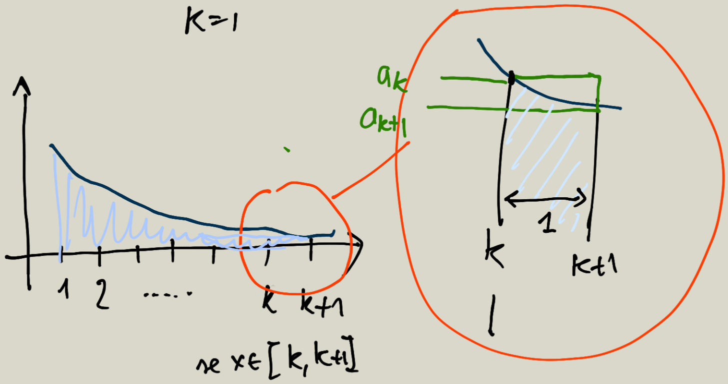
Teorema (collegamento tra serie ed integrali impropri)

Sia $f:[1,+\infty) \to \mathbb{R}$ una funzione non negativa e decrescente

e sia $a_k = f(k)$, $k \in \mathbb{N}$.

Allora la serie $\sum_{k=1}^{+\infty} a_k$ è convergente $\iff \int_1^{+\infty} f(x)\,dx$ è convergente.

dim



se $x \in [k, k+1]$

$$a_{k+1} \leq f(x) \leq a_k$$

$$1 \cdot a_{k+1} \leq \int_k^{k+1} f(x)\,dx \leq a_k \cdot 1$$

$$\int_1^n f(x)\,dx$$

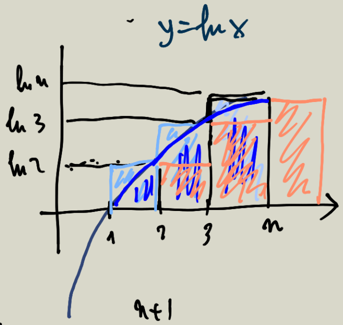
$$\left(\sum_{k=1}^{n} a_k\right) - a_1 = \sum_{k=2}^{n} a_k = \sum_{k=1}^{n-1} a_{k+1} \leq \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx \leq \sum_{k=1}^{n-1} a_k$$

$\lim n \to +\infty$

$$\left| \int_1^{+\infty} f(x)\,dx - \sum_{k=1}^{+\infty} a_k \right| \leq a_1.$$

Esempio approssimare $n!$

$$\ln(n!) = \ln \prod_{k=1}^{n} k = \sum_{k=1}^{n} \ln k$$



$$\int_1^n \ln x \, dx \le \sum_{k=1}^{n} \ln k \le \int_2^{n+1} \ln x \, dx$$

$$\int_1^n \ln x \, dx = \left[x\ln x - x\right]_1^n = n\ln n - n + 1 \sim n \ln n$$

$$\int_2^{n+1} \ln x \, dx = \left[x \ln x - x\right]_2^{n+1} = (n+1)\ln(n+1) - (n+1) - 2\ln 2 + 2 \sim n\ln n$$

$$\ln(n!) \sim n \ln n \qquad \square$$

CONVERGENZA ASSOLUTA

$\int_a^b f(x)\,dx$ integrale improprio $\quad f$ cambia segno.

<u>Teo</u> <u>Th.</u> $\int_a^b |f(x)|\,dx$ convergente $\Longrightarrow \int_a^b f(x)\,dx$ convergente.

<u>Hyp</u> f localmente R-integrabile su $[a,b]$. $(\Longrightarrow |f|$ loc. R-integrabile$)$

<u>Def</u> diremo che $\int_a^b f(x)\,dx$ converge assolutamente

se $\int_a^b |f(x)|\,dx$ converge.

inoltre

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|$$

<u>dim</u>

$$f = f^+ - f^- \qquad f^+(x) = \begin{cases} f(x) & \text{se } f(x) \geq 0 \\ 0 & \text{se } f(x) < 0 \end{cases}$$

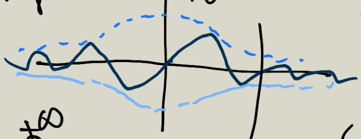
$$|f| = f^+ + f^- \qquad f^-(x) = \begin{cases} -f(x) & \text{se } f(x) \leq 0 \\ 0 & \text{se } f(x) > 0 \end{cases}$$

$$0 \leq f^+ \leq |f| \qquad \int f^+ \leq \int |f| < +\infty$$

$$0 \leq f^- \leq |f| \qquad \int f^- \leq \int |f| < +\infty$$

$$\int f = \int (f^+ - f^-) = \int f^+ - \int f^- \quad \text{è finito} \quad \square$$

Es $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{1+x^2}$ convergente $\left(\left|\int\right| \le \pi \right)$.



$$\int_{-\infty}^{+\infty} \left|\frac{\sin x}{1+x^2}\right| dx \le \int_{-\infty}^{+\infty} \frac{1}{1+x^2} \text{ converge. } (=\pi)$$

$$\frac{1}{1+x^2} \sim \frac{1}{x^2} \text{ per } x \to \pm\infty$$

Es $\displaystyle\int_1^{+\infty} \frac{\sin x}{x}\, dx$? esiste? finito? infinito?

$$\sum \frac{(-1)^k}{k}$$

$$\int_1^{+\infty} \frac{|\sin x|}{x} = ? \quad (+\infty)$$

$$\le \int_1^{+\infty} \frac{1}{x}\, dx = +\infty$$

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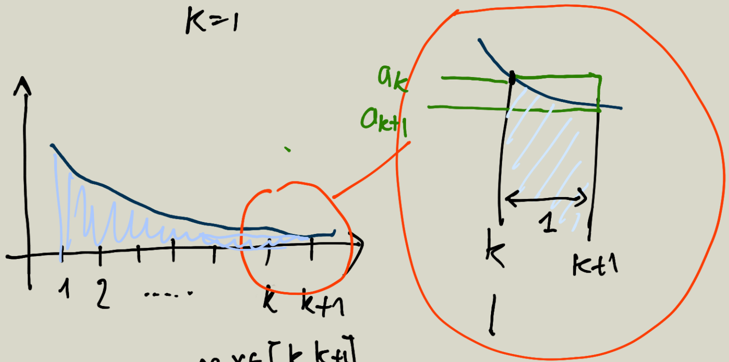

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per $n\to+\infty$

$$\left|\int_1^{+\infty} f(x)\,dx - \sum_{k=1}^{+\infty} a_k\right| \le a_1$$