ANALISI MATEMATICA B
LEZIONE 23 – 16.11.2020

NUMERO e

$$\boxed{e = \lim_{\substack{n\to+\infty \\ n\in\mathbb{N}}} \left(1+\frac{1}{n}\right)^n}$$

$e\in\mathbb{R} \qquad 2\le e\le 4.$

Limiti notevoli

1. $$\boxed{\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e}$$

2. $$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

3. $$\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$$

① $\lim\limits_{\substack{x\to+\infty \\ x\in\mathbb{R},\, x>0}} \left(1+\frac{1}{x}\right)^x \overset{?}{=} e$

$\lfloor x\rfloor \le x \le \lfloor x\rfloor+1$

dim

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}$$

$n=n(x)=\lfloor x\rfloor+1$ $\quad \mathbb{R}\to\mathbb{N}$

(*) $n=\lfloor x\rfloor$

$$\left(1+\frac{1}{n}\right)^{n-1} = \frac{\left(1+\frac{1}{n}\right)^n}{1+\frac{1}{n}} \longrightarrow \frac{e}{1}=e$$

(*) $$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{n}\right)$$

$$\to e\cdot 1 = e \qquad \square$$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} \overset{?}{=} e \qquad y=-x$$

$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^{-x}$$

$$\left(1-\frac{1}{x}\right)^{-x} = \left(\frac{x-1}{x}\right)^{-x} = \left(\frac{x-1+1}{x-1}\right)^{x}$$

$$= \left(1+\frac{1}{x-1}\right)^{x} = \underbrace{\left(1+\frac{1}{x-1}\right)^{x-1}}_{\downarrow\ e}\underbrace{\left(1+\frac{1}{x-1}\right)}_{\downarrow\ 1}$$

per $x \to +\infty$

$\longrightarrow e$ $\square$

$$\lim_{x\to0^+}(1+x)^{\frac{1}{x}} = e$$

$x = \frac{1}{y} \qquad y \to +\infty$

$$\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} = e \qquad \begin{cases} x = \frac{1}{y} \\ y \to -\infty \end{cases}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$x \to 0$ $\quad \log_e(1+x)^{1/x} \longrightarrow \log_e e = 1$

$$\|$$

$$\frac{1}{x}\log_e(1+x) = \frac{\log_e(1+x)}{x}$$

③

$\boxed{1+x=e^y}$ se $y \to 0$

$\boxed{x = e^y - 1} \to 0$, $e^y \to 1$

$$\frac{\log_e(1+x)}{x} = \frac{\log_e(e^y)}{e^y - 1} = \frac{y}{e^y - 1}$$

$$\lim_{y \to 0} \frac{y}{e^y - 1} = 1$$

(2) $$\lim_{y \to 0} \frac{e^y - 1}{y} = \frac{1}{1} = 1.$$

$$\boxed{\begin{array}{c} \left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1} \\ \uparrow \quad \forall n \in \mathbb{N} \quad \uparrow \\ n \neq 0 \end{array}}$$

ORDINI DI INFINITO

(o DI INFINITESIMO)

Per $x \to x_0 \in \bar{\mathbb{R}}$ diremo che

"f è molto più piccola di g"

$$f(x) << g(x) \qquad \text{per } \underline{\underline{x \to +\infty}}$$

se

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$$

$$\boxed{\sqrt{x} << x << x^2}$$

ES per $\underline{x \to +\infty}$

$$\sqrt{x} << x \qquad \frac{\sqrt{x}}{x} = \sqrt{\frac{\cancel{x}}{x^{\cancel{2}}}} = \frac{1}{\sqrt{x}} \to 0$$

$x << x^2$ $\qquad \frac{x}{x^2} = \frac{1}{x} \to 0$

per $x \to +\infty$

$x^\alpha << x^\beta$ se $\alpha < \beta$

Es per $x \to 0^+$ $\quad \sqrt{x} >> x >> x^2$

per $x \to 0^+$ $\quad x^\alpha >> x^\beta$ se $\alpha < \beta$

$$\frac{x^\beta}{x^\alpha} = x^{\beta-\alpha} \to 0 \qquad \square$$

$\beta - \alpha > 0$

Es $\quad a > b > 1$ $\quad a^x >> b^x$

per $x \to +\infty$

$$\to \frac{b^x}{a^x} = \left(\frac{b}{a}\right)^x \to 0 \Rightarrow b^x << a^x$$

$\frac{b}{a} < 1$

CORRETTO!

per $x \to -\infty$

se $a > b > 1$ $\qquad a^x << b^x$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \longrightarrow 0 \quad \text{per } x \to -\infty$$

$\frac{a}{b} > 1$

Teorema 1 per $n \to +\infty$, $n \in \mathbb{N}$, $a > 1$, $\alpha > 0$

$$\underbrace{n^\alpha << a^n} << n! << n^n$$

Teorema 2 per $x \to +\infty$

$$\log_a x << \underbrace{x^\alpha << a^x}$$

LIMITI NOTEVOLI

ORDINI DI INFINITO

Teorema (criterio del rapporto)

Se a_n è una successione, $a_n > 0$, $\forall n$

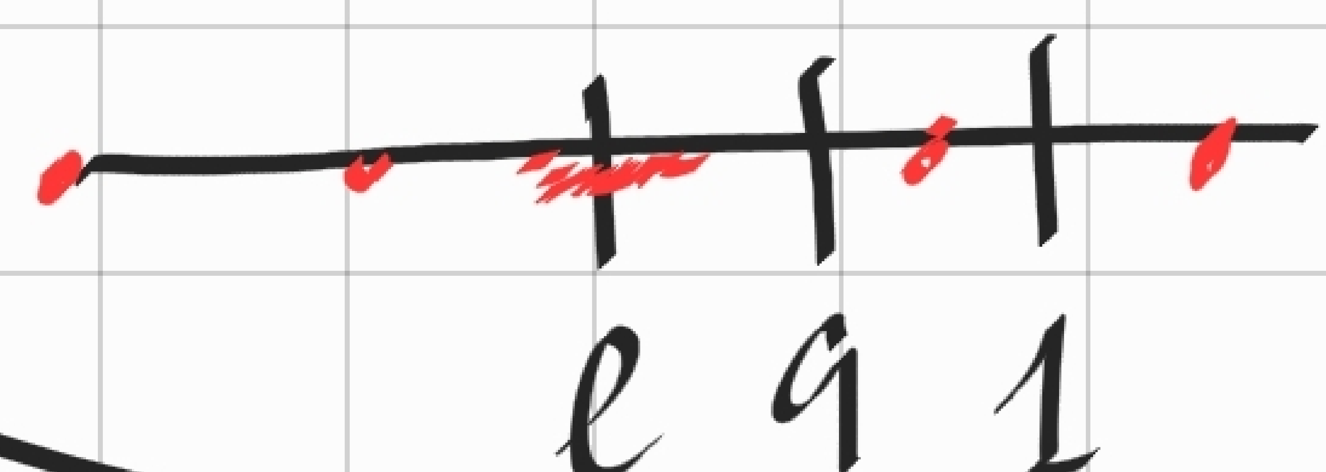
e se $$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l < 1$$

allora $$\lim_{n\to+\infty} a_n = 0.$$

dim $\frac{a_{n+1}}{a_n} \to l<1$ significa:

Scelgo q: $l<q<1$

$$\exists \alpha : n>\alpha \quad \frac{a_{n+1}}{a_n} \le q$$



$$\left(q - \frac{a_{n+1}}{a_n} \to q - l > 0\right)$$

permanenza del segno

Prendo $N\in\mathbb{N}$, $N>\alpha$

$$\forall n \ge N \quad a_{n+1} \le q\cdot a_n$$

$$\frac{a_{N+1}}{a_N} \leq q \qquad a_{N+1} \leq q \cdot a_N$$

$$\frac{a_{N+2}}{a_{N+1}} \leq q \qquad a_{N+2} \leq q \cdot a_{N+1} \leq q^2 a_N$$

$\vdots$

PER INDUZIONE $\rightarrow$ $a_{N+k} \leq q^k \cdot a_N$ $\quad (*)$

$\forall n \geq N \quad 0 < a_n \leq q^{n-N} \cdot a_N$

$n = N+k$

$= q^n \cdot \frac{a_N}{q^N}$ costante

$a_n \downarrow 0 \qquad q^n \downarrow 0 \quad q<1$

per i 2 carabinieri $\quad a_n \to 0$ $\square$

$(*)$ Sapendo che se $n \geq N$

$$a_{n+1} \overset{(\#)}{\leq} q\, a_n$$

dimostrare per induzione

che $\boxed{a_{N+k} \leq q^k \cdot a_N}$ $\underline{\underline{P(k)}}$

dim per induzione su $k \in \mathbb{N}$

(i) $k=0$: $a_N \overset{?}{\leq} q^0 \cdot a_N$

(ii) $P(k) \overset{?}{\Rightarrow} P(k+1)$

$$a_{N+k+1} \overset{(\#)}{\leq} q \cdot a_{N+k} \overset{P(k)}{\leq}$$

$$q \cdot q^k \cdot a_N = q^{k+1} \cdot a_N \qquad \square$$

dim (Teorema 1) $(\alpha > 0,\ a > 1)$

- $\underbrace{n^\alpha \overset{?}{<<} a^n}$ a_n

$$\underbrace{\lim_{n \to +\infty} \frac{n^\alpha}{a^n} = 0}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{1}{a} < 1$$

$$a_{n+1} = \frac{(n+1)^\alpha}{a^{n+1}} \qquad a_n = \frac{n^\alpha}{a^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha}{a^{n+1}} \cdot \frac{a^n}{n^\alpha} = \left(\frac{n+1}{n}\right)^\alpha \cdot \frac{\cancel{a^n}}{a\cdot\cancel{a^n}}$$

$$= \left(1+\underset{\downarrow\,0}{\cancel{\frac{1}{n}}}\right)^\alpha \cdot \frac{1}{a} \longrightarrow 1\cdot\frac{1}{a} = \frac{1}{a} < 1$$

se $x\to 1$: $\quad x^\alpha \to 1^\alpha \qquad\qquad x\to x_0 \Rightarrow f(x)\to f(x_0)$

per continua

- $a^n << n!$ per casa
- $n! << n^n$ per casa

Teorema 2

Sapendo che $n^\alpha << a^n$ per $n \to +\infty$ $n \in \mathbb{N}$

Mostrare che $\boxed{x^\alpha << a^x}$ per $x \to +\infty$ $x \in \mathbb{R}$

$$0 \leq \frac{x^\alpha}{a^x} \leq \frac{(\lfloor x \rfloor + 1)^\alpha}{a^{\lfloor x \rfloor}}$$

$\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1$

$n = \lfloor x \rfloor + 1$

$$= \frac{n^\alpha}{a^{n-1}} = \frac{n^\alpha}{a^n} \cdot a \to 0 \cdot a = 0.$$

- $\log_a x \overset{?}{<<} x^\alpha$

$y = \log_a x$

$$\frac{\log_a x}{x^\alpha} = \frac{y}{(a^y)^\alpha} =$$

$$= \frac{y}{a^{\alpha y}} = \frac{y^1}{(a^{\alpha})^y} \to 0 \quad \square$$

$a>1,\ \alpha>0 \qquad a^{\alpha}>1$

$\log_e n << n$

Esercizio $\displaystyle\lim_{n\to+\infty} \sqrt[n]{n} = 1$

$\dfrac{\log_e n}{n} \to 0$

$$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{1}{n}\log_e n} \to e^0 = 1.$$

$$e^{\log_e n} = n$$

Esercizio $\displaystyle\lim_{n\to+\infty} \frac{2^{\sqrt{n}} - \sqrt{2n}}{n! + 3^n - 4}$

$$2^{\sqrt{n}} - 2^{n/2} = 2^{\sqrt{n}}\left(1 - 2^{n/2 - \sqrt{n}}\right)$$

$\square$