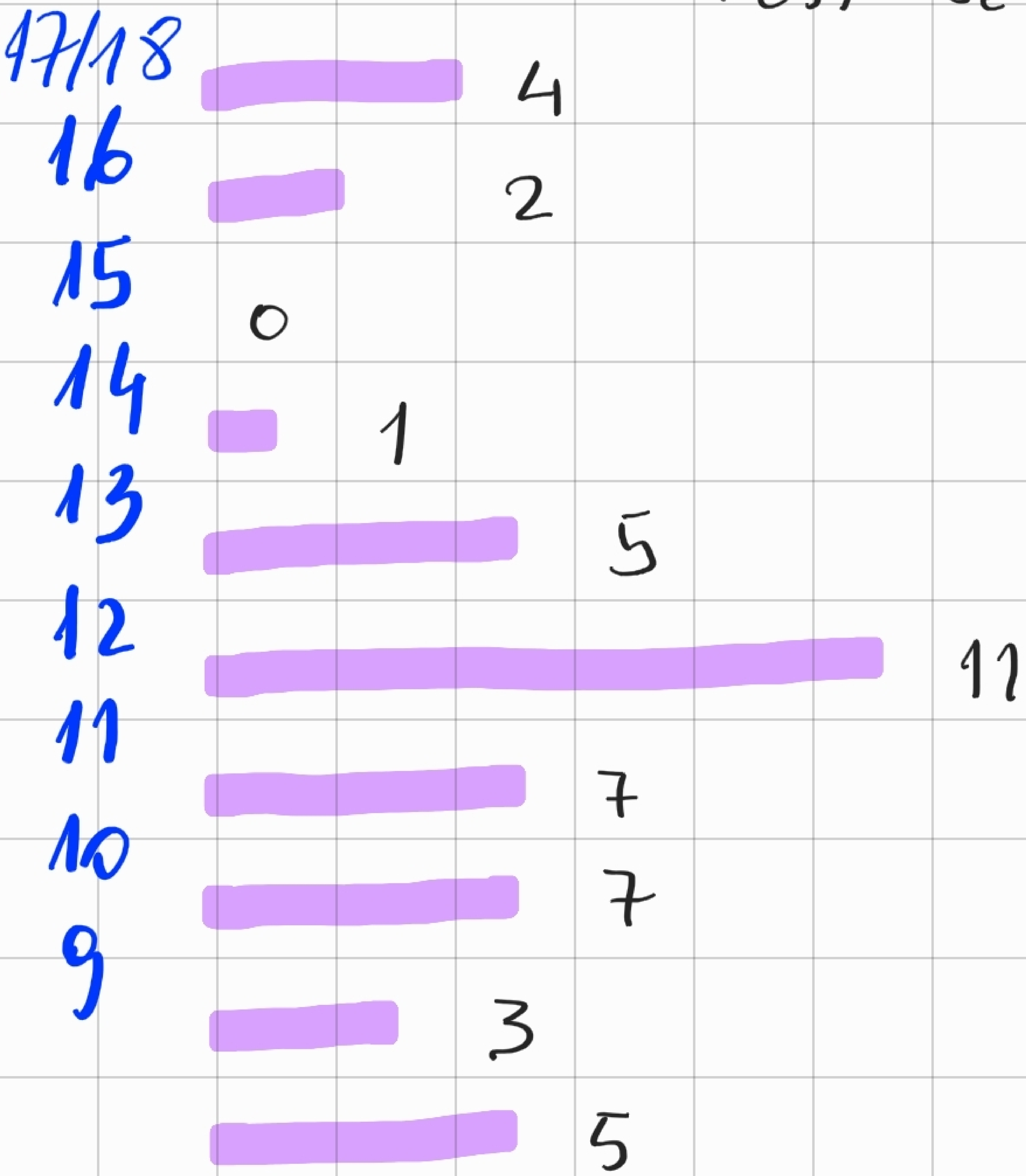
ANALISI MATEMATICA B
LEZIONE 29 - 30.11.2020

TEST SETTIMANALE



ES 18

a termini positivi

$$\sum_n x^{2n}\left(\sqrt{n+1}-\sqrt{n}\right) \qquad x \neq 0$$

Criterio del rapporto

$$a_n = x^{2n}\left(\sqrt{n+1}-\sqrt{n}\right)$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2(n+1)}\left(\sqrt{n+2}-\sqrt{n+1}\right)}{x^{2n}\left(\sqrt{n+1}-\sqrt{n}\right)}$$

$$= x^2\,\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}} = \boxed{A-B=\frac{A^2-B^2}{A+B}}$$

$$= x^2\,\frac{(\cancel{n}+2)-(\cancel{n}+1)}{\sqrt{n+2}+\sqrt{n+1}}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{(\cancel{n}+1)-\cancel{n}}$$

$$= x^2\,\frac{\cancel{\sqrt{n}}\left[\sqrt{1+\cancel{\tfrac{1}{n}}^{\,\to 0}}+1\right]}{\cancel{\sqrt{n}}\left[\sqrt{1+\underbrace{\cancel{\tfrac{2}{n}}}_{\to 0}}+\sqrt{1+\underbrace{\cancel{\tfrac{1}{n}}}_{\to 0}}\right]} \longrightarrow x^2\,\frac{\left[\sqrt{1+0}+1\right]}{\left[\sqrt{1+0}+\sqrt{1+0}\right]} = x^2.$$

Se $x^2 < 1$ la serie converge

se $x=0$ la serie converge.

se $x^2 > 1$ la serie diverge

se $x^2 = 1$? (il rapporto non ci dice niente.

se $x^2 = 1$ $(x = \pm 1)$

$$\sum_{k=0}^{n} 1 \cdot \left(\sqrt{k+1} - \sqrt{k}\right) = \sqrt{n+1} - \sqrt{0}$$

telescopica

per $n \to +\infty$ $\quad\quad +\infty$

Converge sse $|x| < 1$.

Esercizio 5

$$\sum_n \frac{n!}{n^n} \text{ converge.}$$

$n! << n^n$

$\frac{n!}{n^n} \to 0$

condizione necessaria

$\sum \frac{1}{2^n}$ converge

$$\frac{n!}{n^n} << \frac{1}{2^n} \left(<< \frac{1}{n^2} \right)$$

$$\frac{2^n \cdot n!}{n^n} \to 0 \quad \text{SI}$$

rapporto:

$$\frac{2^{\cancel{n+1}}\,\cancel{(n+1)!}}{\cancel{(n+1)}(n+1)^n}\cdot\frac{n^n}{\cancel{2^n}\,\cancel{n!}}$$

$$=\frac{2}{\left(\frac{n+1}{n}\right)^n}=\frac{2}{\left(1+\frac{1}{n}\right)^n}\longrightarrow\frac{2}{e}<1$$

Criterio di Dirichlet

segni alterni

Se $\quad a_n = (-1)^n \cdot b_n$, $\quad \underline{b_n \geq 0}$, $\quad (b_n = |a_n|)$

Se b_n è decrescente (monotona) e $\underline{b_n \to 0}$ (infinitesima)

$(b_n \to \ell > 0)$

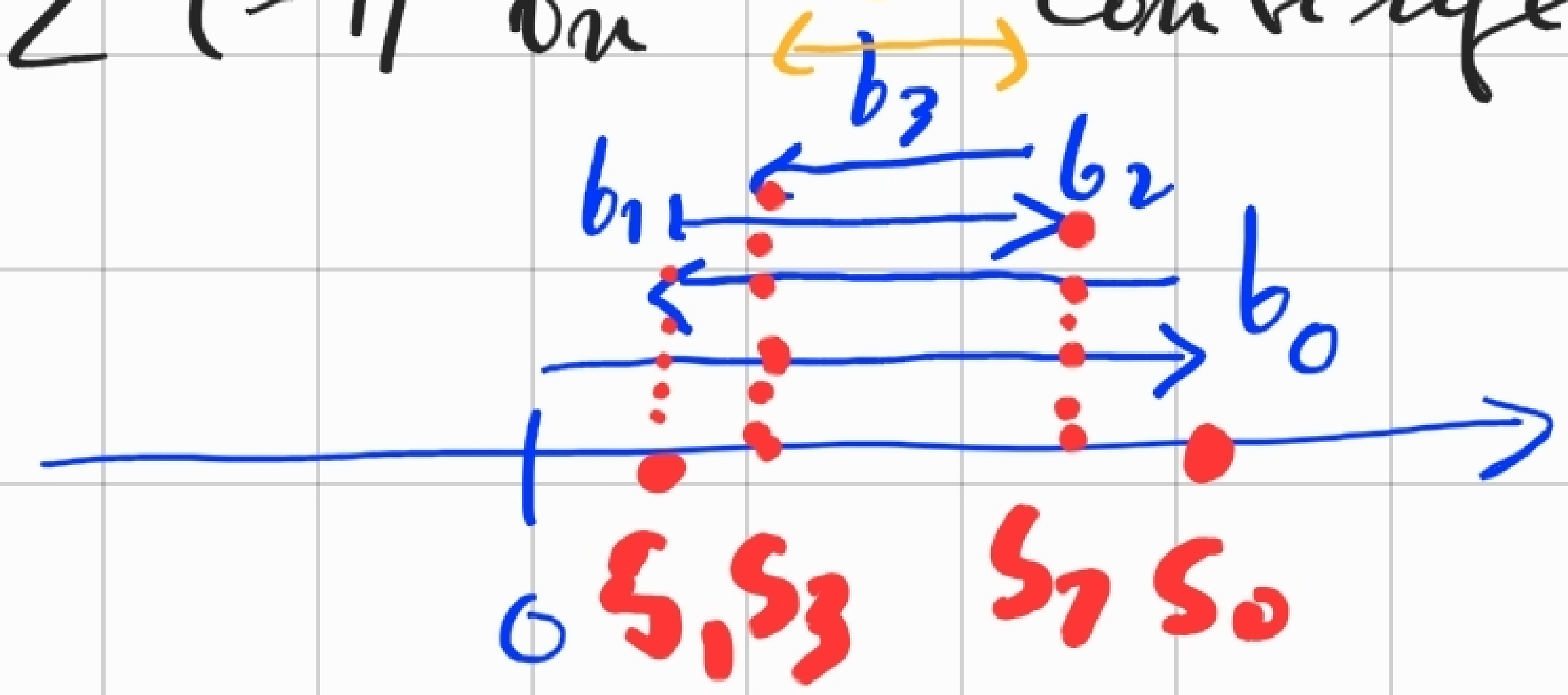
$\Downarrow$

la serie è indeterminata

Allora

$$\sum (-1)^n b_n \text{ è converge.}$$

dim



$S_0 = b_0 \qquad S_1 = b_0 - b_1$

$S_2 = b_0 - b_1 + b_2$

idea
$$\begin{cases} S_{2n} \text{ decrescente} \checkmark \\ S_{2n+1} \text{ crescente} \checkmark \\ S_{2n} - S_{2n+1} \to 0 \end{cases}$$

$$S_{2n+1} = S_{2n} + a_{2n+1} = S_{2n} - b_{2n+1}$$

$$S_{2n} - S_{2n+1} = b_{2n+1} \to 0$$

$$S_{2n+2} = S_{2n} + a_{2n+1} + a_{2n+2}$$
$$= S_{2n} - b_{2n+1} + b_{2n+2}$$
$$\leq S_{2n}$$

$\left(b_{2n+2} \leq b_{2n+1} \right)$
$\Uparrow$
b_n decrescente

S_{2n} decrescente.

$$S_{2n+3} = S_{2n+1} + a_{2n+2} + a_{2n+3}$$
$$= S_{2n+1} + b_{2n+2} - b_{2n+3}$$
$$\geq S_{2n+1}$$

$b_{2n+3} \leq b_{2n+2}$

S_{2n+1} crescente.

$S_{2n} \longrightarrow S$

(le successioni monotone sono regolari)

$$S_{2n+1} \to S'$$

$$S_{2n} = S_{2n-1} + a_{2n}$$

$$= S_{2n-1} + b_{2n} \geq S_{2n-1}$$

$$(b_{2n} \geq 0)$$

$$S_{2n} \geq S_{2n-1} \geq S_1 = b_0 - b_1 \geq 0$$

S_{2n} è decrescente e inferiormente limitata

$$0 \leq S_1 \leq S_{2n} \leq S_0 = b_0 < +\infty$$

$S_{2n} \to S$ finito $S \in \mathbb{R}$.
(converge)

$$S_{2n+1} - S_{2n} \to 0$$

$$S_{2n+1} \longrightarrow S' + 0 = S'.$$

$$S_n \longrightarrow S'$$

$$\| $$

$$\sum_{k=0}^{n} (-1)^k b_k \quad \text{converge!} \quad \square$$

<u>Oss</u> $b_n \geq 0$, monotona, infinitesima ($b_n \to 0$)

Allora b_n decrescente

<u>Es</u>

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{n \cdot \ln n} \qquad b_n = \frac{1}{n \ln n}$$

$n \ln n$ crescente

b_n decrescente

$b_n > 0$

$n \ln n \to +\infty \qquad b_n \to 0$

la serie converge.

OSS non è assolutamente convergente

$$\sum_n \frac{1}{n \ln n} = +\infty$$

(criterio di condensazione)

ES

$$b_n = \begin{cases} n & \text{se } n \text{ pari} \\ 1 & \text{se } n \text{ dispari} \end{cases}$$

$$\sum (-1)^n b_n$$

(b_n non è infinitesima)

$$0 - 1 + 1 - 1 + 2 - 1 + 3 - 1 + 4 - 1 \dots$$

$$= +\infty$$

ES

$$+\infty = \sum_{n=1}^{+\infty} \frac{1}{n} = \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \right) = \sum_{n=1}^{+\infty} a_n$$

$$a_n = \begin{cases} \frac{1}{n} + \frac{1}{\sqrt{n}} & \text{se } n \text{ pari} \\ -\frac{1}{\sqrt{n}} & \text{se } n \text{ dispari} \end{cases}$$

$a_n \to 0$, a_n ha segni alterni

ma $\sum a_n = +\infty$.

$b_n = |a_n|$ non è monotona.

SISTEMI DINAMICI DISCRETI

(Successioni definite per ricorrenza)
(ricorsive)

$$\boxed{\begin{cases} a_0 = \alpha \\ a_{n+1} = f(a_n) \end{cases}}$$

f

<u>Esempio</u> (Algoritmo di Erone)

Voglio calcolare $\sqrt{p}$. $(p > 1)$

$$\rightarrow \begin{cases} a_0 = p \\ a_{n+1} = \dfrac{a_n + \frac{p}{a_n}}{2} \end{cases}$$

$$\boxed{f(x) = \frac{x + \frac{p}{x}}{2}}$$

dimostriamo che $a_n \to \sqrt{p}$.

<u>Es</u> $p = 2$

$a_0 = \underline{2}$, $a_1 = \dfrac{2 + \frac{2}{2}}{2} = \dfrac{3}{2} = \underline{1{,}5}$

$$a_2 = \frac{a_1 + \frac{p}{a_1}}{2} = \frac{\frac{3}{2} + 2 \cdot \frac{2}{3}}{2} = \frac{3}{4} + \frac{2}{3}$$

$$= \frac{9+8}{12} = \frac{17}{12} = \underline{1{,}41\bar{6}}$$

$$a_3 = \frac{17}{24} + \frac{12}{17} = \frac{17^2 + 24 \cdot 12}{24 \cdot 17}$$

$$= \underline{1{,}414213}56 \ldots$$

$$\left[\sqrt{2} = 1.4142135 \ldots \right]$$

<u>FATTO BANALE</u> (MA IMPORTANTE)

se <u>a_n converge</u> allora

$a_n \to \sqrt{p}.$ $\quad (\text{se } \ell > 0)$

$$a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2} \to \frac{\ell + \frac{p}{\ell}}{2}$$

ℓ

(se $a_n \to \ell$ anche $a_{n+1} \to \ell$)

$$\ell = \frac{\ell + \frac{p}{\ell}}{2}$$

$$2\ell = \ell + \frac{p}{\ell} \qquad \ell = \frac{p}{\ell}$$

$$\ell^2 = p. \quad \Rightarrow \quad \ell = \sqrt{p}$$

$(\ell = -\sqrt{p})$

1. $a_n > 0$ $\forall n$ (per induzione)

(i) $a_0 = p > 1$

(ii) $a_{n+1} = \dfrac{a_n + \frac{p}{a_n}}{2} > 0$

perché $a_n > 0$.

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② a_n decrescente ? SI

$$a_{n+1} \le a_n$$

$$\| $$

$$\frac{a_n + \frac{p}{a_n}}{2} \Leftarrow \frac{a_n + \frac{p}{a_n}}{2} \le a_n \qquad (a_n > 0)$$

$$\Updownarrow$$

$$a_n^2 + p \le 2 \cdot a_n^2$$

$\to \boxed{a_n^2 \ge p}$ se $a_n \ge \sqrt{p}$

a_n decrescente se $a_n \ge \sqrt{p}$

(3) $a_n \geq \sqrt{p}$ $\forall$

$\boxed{a_n \geq 0}$

$$a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2} \geq \sqrt{p}$$

$$\Uparrow$$

$$a_n^2 + p \geq 2\sqrt{p}\, a_n$$

$$\Uparrow$$

$$a_n^2 - 2\sqrt{p}\, a_n + p \geq 0$$

$$\Uparrow$$

$$(a_n - \sqrt{p})^2 \geq 0 \quad \text{ok!}$$

$a_n \geq \sqrt{p}$

a_n decrescente

$a_n \to \ell \geq \sqrt{p}$ è finito

$$a_{n+1} = \frac{a_n + \frac{p}{a_n}}{2} \to \frac{l + \frac{p}{l}}{2}$$

$$\underbrace{l = \frac{l + \frac{p}{l}}{2} \qquad l > 0}$$

$$l = \sqrt{p} \qquad \square$$

$$a_n \to \sqrt{p}$$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{m\to+\infty} a_m = \lim_{n\to+\infty} a_n.$$

$$m = n+1$$

a_n: a_0, a_1, a_2, a_3, ...

a_{n+1}: a_1, a_2, a_3, ...