ANALISI MATEMATICA B
LEZIONE 35 – 14.12.2020

Test settimanale

1)

$$\lim_{n\to+\infty} \frac{\prod_{k=1}^{n} \sqrt[n]{2k+1}}{n}$$

$$\sqrt[n]{\prod_{k=1}^{n}(2k+1)} = \sqrt[n]{\prod_{k=1}^{n} 2k \cdot \prod_{k=1}^{n} 1}$$

$n=3$ $\quad \sqrt[3]{3\cdot 5\cdot 7} = \sqrt[3]{2\cdot 4\cdot 6\cdot 1\cdot 1\cdot 1}$

$$(a+b)(c+d) \neq ac\cdot b\cdot d$$

$$\prod_{k=1}^{n} 2k = 2^n \prod_{k=1}^{n} k$$

$$\frac{\sqrt[n]{2^n n!}}{n} = 2 \, \frac{\sqrt[n]{n!}}{n}$$

$$= 2 \sqrt[n]{\frac{n!}{n^n}}$$

$$\boxed{\begin{array}{l} \text{criterio del rapporto / radice} \\ \dfrac{a_{n+1}}{a_n} \to \ell \implies \sqrt[n]{a_n} \to \ell \end{array}}$$

$$\lim_{n\to+\infty} \frac{\prod_{k=1}^{n} \sqrt[n]{2k+1}}{n}$$

$$\frac{\sqrt[n]{2n+1}}{n} = \sqrt[n]{\frac{2n+1}{n^n}} \longrightarrow 0 \qquad \|$$

$$a_n = \frac{\prod_{k=1}^{n} \sqrt[n]{2k+1}}{n}$$

$$a_1 = \frac{3}{1} \qquad a_2 = \frac{\sqrt[2]{3\cdot 5}}{2}$$

$$a_3 = \frac{\sqrt[3]{3\cdot 5\cdot 7}}{3} \quad \ldots\ldots$$

$$\prod_{k=1}^{n}(2k+1) \neq (2n+1)!$$ No

$$1 \cdot 2 \cdot 3 \cdot 5 \cdot \ldots \cdot 13$$

$$3 \cdot 5 \cdot 7 \cdot \ldots \cdot 13$$

$$\prod_{k=1}^{n}(2k+1) = \frac{(2n+1)!}{\prod_{k=1}^{n} 2k}$$

$$= \frac{(2n+1)!}{2^n \cdot n!}$$

$$\frac{\prod_{k=1}^{n} \sqrt[n]{2k+1}}{n} = \sqrt[n]{\frac{\prod_{k=1}^{n}(2k+1)}{n^n}} = \sqrt[n]{a_n}$$

criterio $\frac{a_{n+1}}{a_n} \to l$ allora $\sqrt[n]{a_n} \to l$.

$$a_n = \frac{\prod_{k=1}^{n}(2k+1)}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\prod_{k=1}^{n+1}(2k+1)}{(n+1)^{n+1}} \frac{n^n}{\prod_{k=1}^{n}(2k+1)} = (*)$$

$$\left(\prod_{k=1}^{n+1}(2k+1) = \prod_{k=1}^{n}(2k+1) \cdot (2(n+1)+1) \right)$$

$$(*) = \underbrace{\frac{(2n+3)}{(n+1)}}_{\to 2} \cdot \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to \frac{1}{e}} \longrightarrow \frac{2}{e}$$

$$\sqrt[n]{3 \cdot 5 \cdot \ldots \cdot (2n+1)} =$$

$$= \sqrt[n]{3}\,\sqrt[n]{5}\,\ldots\,\sqrt[n]{2n+1}$$

$$\downarrow \quad \downarrow \quad \quad \downarrow$$

$$1 \quad 1 \quad \quad 1$$

NON POSSO DEDURRE CHE IL PRODOTTO TENDE A 1.

$$\begin{cases} a_0 = 0 \\ a_{n+1} = a_n + \frac{1}{n^2} \end{cases}$$

$a_0 = 0 \qquad a_1 = 0 + \frac{1}{1^2} \qquad a_2 = \frac{1}{1^2} + \frac{1}{2^2}$

$a_3 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \qquad \ldots \; a_n = \sum_{k=1}^{n} \frac{1}{k^2}$

a_n è convergente

$$\begin{cases} a_0 = 1 \\ a_{n+1} = a_n + \frac{1}{a_n \cdot n^2} \end{cases}$$

<u>IDEA</u> $a_n \geq 0$, $\boxed{a_n \text{ crescente}}$

$a_n \geq 1$ $\qquad \boxed{a_n \leq \sum_{k=1}^{n} \frac{1}{k^2}}$

① $a_n > 0 \ \forall n$

per induzione $\quad a_0 = 1 > 0$

$$a_{n+1} = a_n + \frac{1}{a_n \cdot n^2}$$

$$> \frac{1}{a_n \cdot n^2} > 0 \qquad \checkmark$$

② $a_{n+1} \geq a_n \qquad \forall n$

$$\boxed{a_{n+1} = a_n + \frac{1}{a_n \cdot n^2}} \geq a_n \qquad \checkmark$$

③ $a_0 = 1 \implies a_n \geq a_0 = 1 \ \forall n.$

$a_n \geq 1 \quad \forall n.$

a_n crescente $\implies a_n \to \ell$

$$a_{n+1} = a_n + \frac{1}{a_n \cdot n^2}$$

$$\leq a_n + \frac{1}{n^2}$$

$a_n \geq 1$

$$a_1 \leq a_0 + 1 = 1 + 1$$

$$a_2 \leq a_1 + \frac{1}{1} = 1 + \frac{1}{1^2} + \frac{1}{2^2}$$

$$a_3 \leq a_2 + \frac{1}{3^2} = 1 + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}$$

$+ \ldots$

$$a_n \overset{?}{\leq} 1 + \sum_{k=1}^{n} \frac{1}{k^2}$$

per induzione:

$n=0$ $\qquad a_0 = 1$ ✓

$$a_{n+1} = a_n + \frac{1}{a_n (n+1)^2}$$

ip. indutt.

$$\leq a_n + \frac{1}{(n+1)^2} \leq \left(1 + \sum_{k=1}^{n} \frac{1}{k^2}\right)$$

Problema col testo

$$\begin{cases} a_0 = 1 \\ \uparrow \\ a_{n+1} = a_n + \dfrac{1}{a_n \cdot n^2} \end{cases}$$

$$\boxed{a_1 = a_0 + \frac{1}{a_0 \cdot 0}}$$

NON HA SENSO.

TESTO SENSATO

$$\boxed{\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + \dfrac{1}{a_n \cdot n} \end{cases}}$$

$$a_1 = 1$$

$$a_2 = 1 + \frac{1}{a_1 \cdot 1^2} \leq 1 + \frac{1}{1^2}$$

$$a_3 = a_2 + \frac{1}{a_2 \cdot 2^2} \leq 1 + \frac{1}{1^2} + \frac{1}{2^2}$$

$$\vdots$$

$$a_n \overset{?}{\leq} 1 + \sum_{k=1}^{n-1} \frac{1}{k^2}$$

$n=1$ $\quad a_1 = 1 \leq 1$ ✓

$$a_{n+1} = a_n + \frac{1}{a_n \cdot n^2}$$

$$\leq \left(1 + \sum_{k=1}^{n-1} \frac{1}{k^2}\right) + \frac{1}{n^2}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k^2}$$ ✓

---

a_n crescente

$$a_n \leq 1 + \sum_{k=1}^{n-1} \frac{1}{k^2}$$

$$\downarrow \infty$$

$$1 + \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

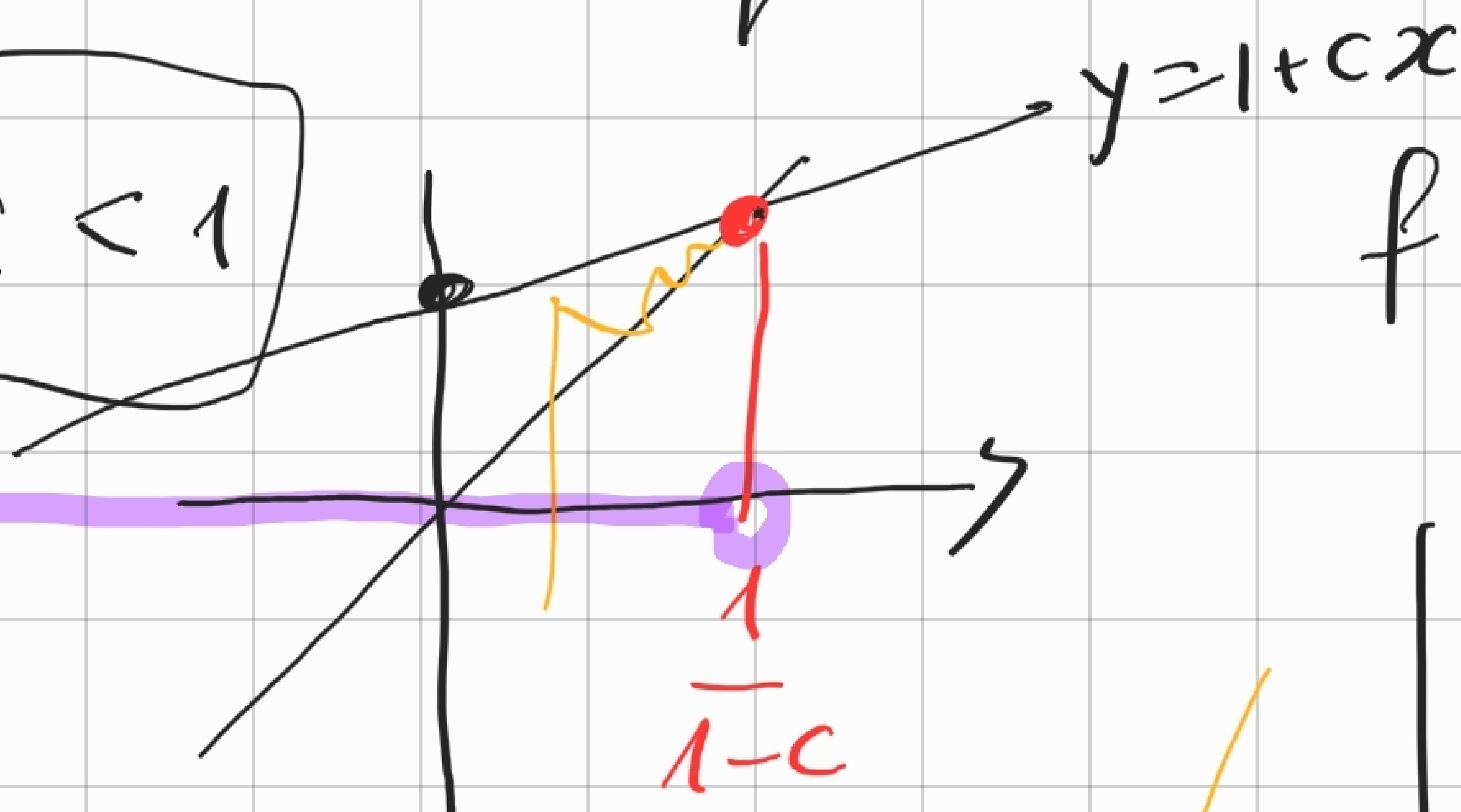
$a_n \longrightarrow \ell < +\infty$.

Esercizio 4

$$\begin{cases} a_0 = 0 \\ a_{n+1} = 1 + c \cdot a_n \end{cases}$$

a_n converge $\iff$ $c \in (-1, 1)$

$\boxed{0 \leq c < 1}$



$\underline{\underline{f(x) = 1 + cx}}$

$\boxed{b_n = a_n - \frac{1}{1-c}}$

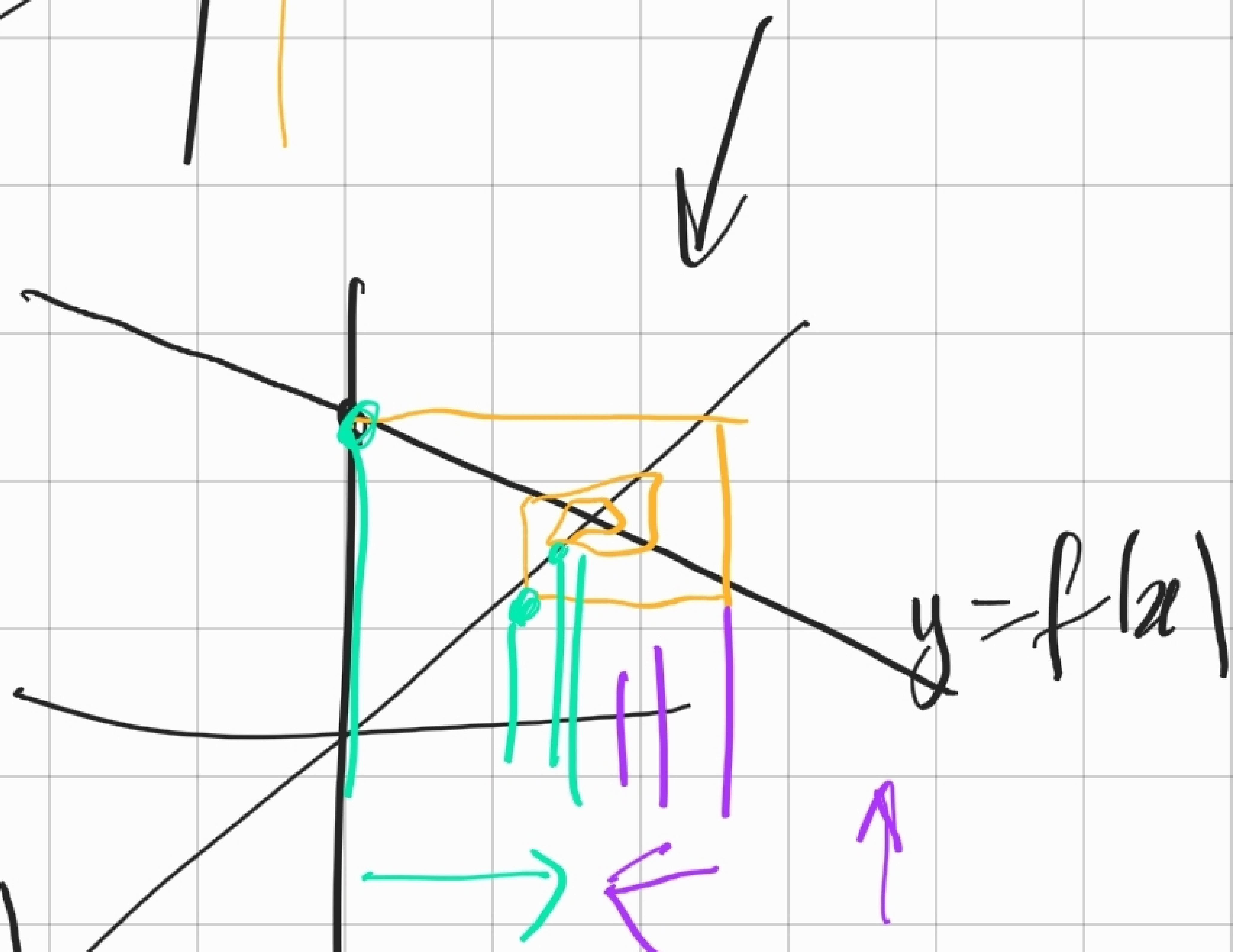
$\boxed{b_n = b_0 \cdot c^n}$

$\boxed{c = 1}$

$f(x) = x \iff x = \frac{1}{1-c}$

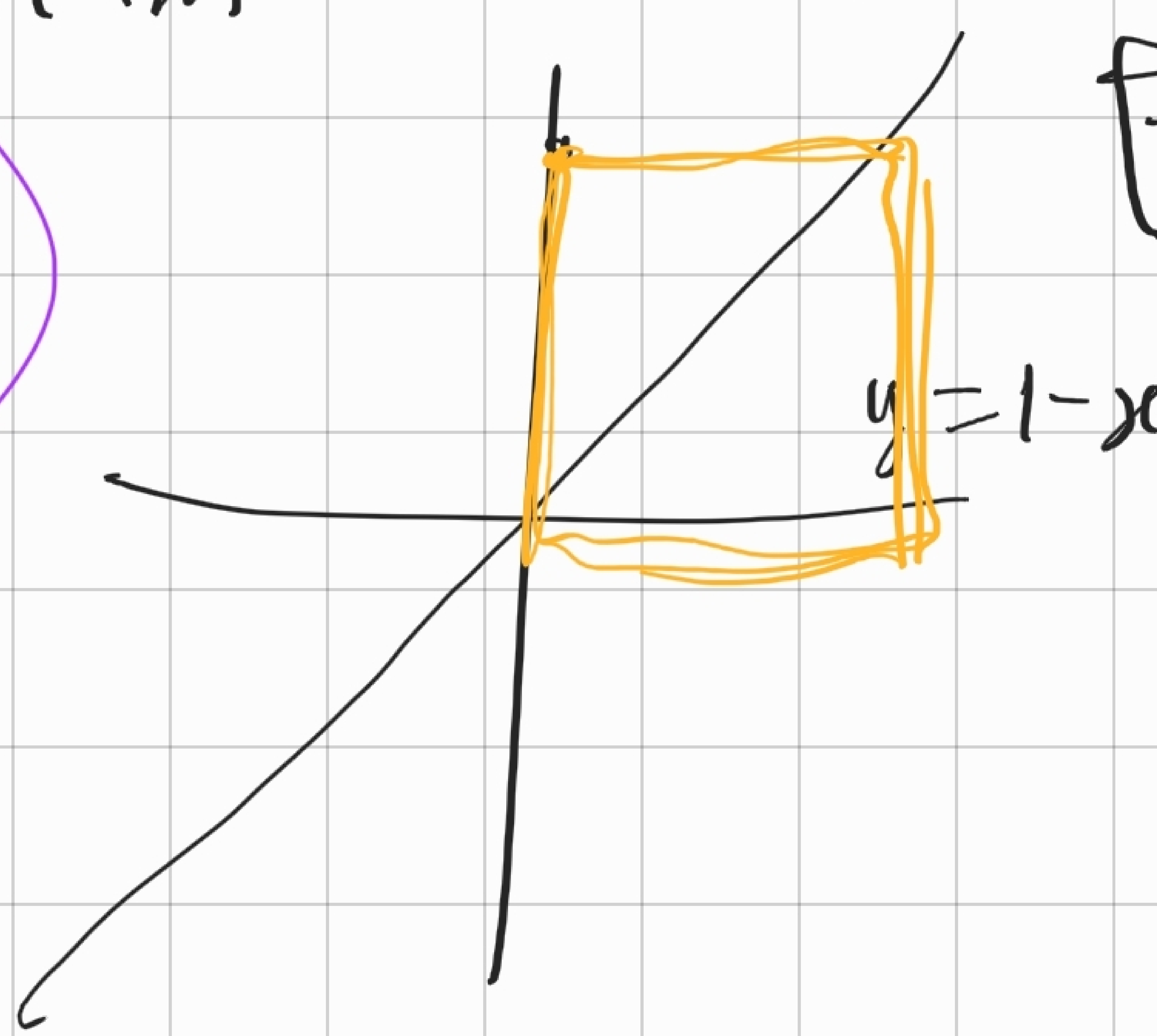
$c > 1$

$-1 < c < 0$

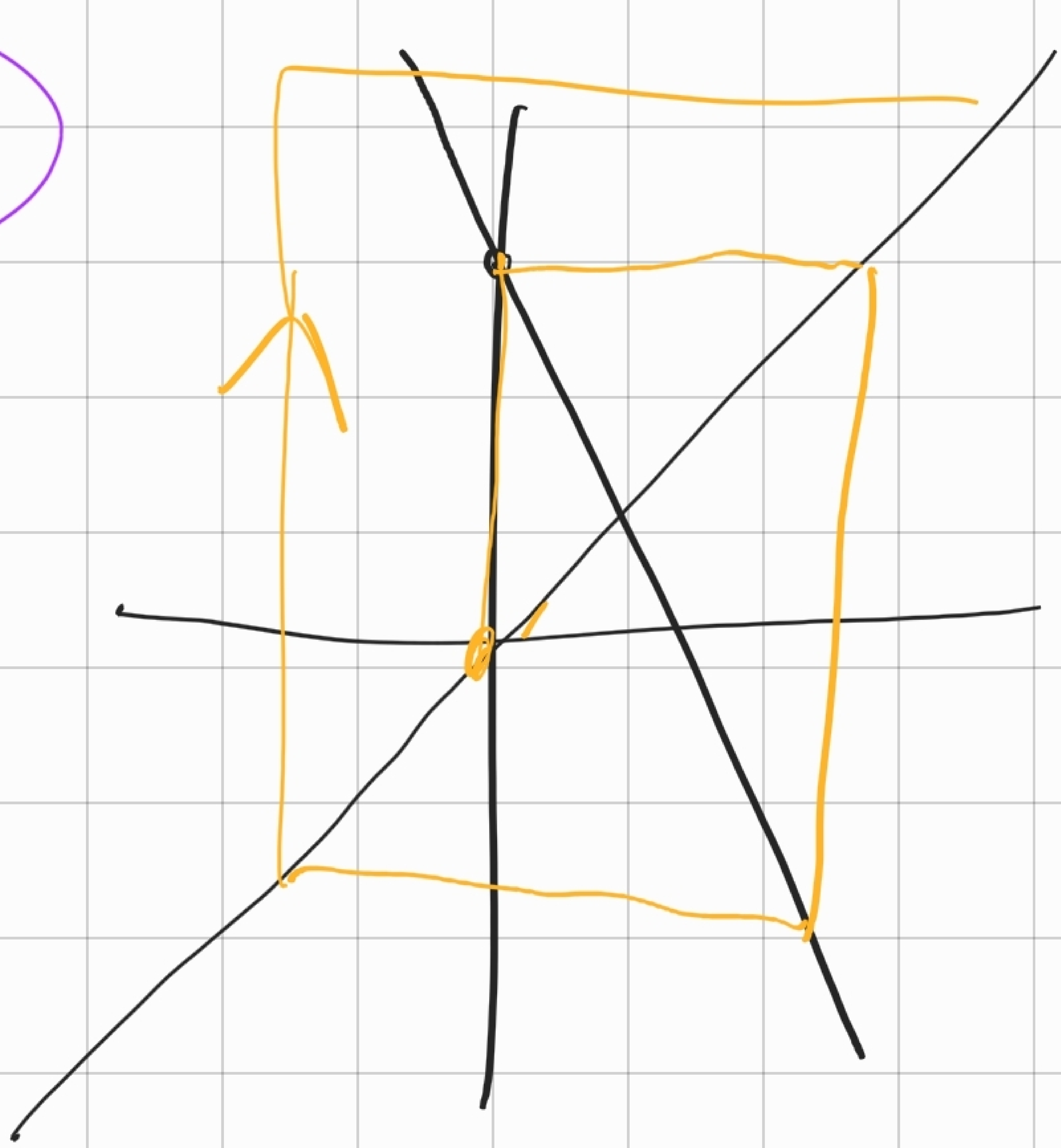


$$a_{n+2} = f(f(a_n)) = 1 + c f(a_n) = 1 + c(1 + c a_n) = \boxed{1 + c + c^2 a_n}$$

$C = -1$



$c < -1$



$a_n = \ldots\ldots\ldots$

$-1 < c < 0$

$$\begin{cases} a_0 = 0 \\ a_{n+1} = 1 + c\, a_n \end{cases}$$

$a_0 = 0$

$a_1 = 1 + c \cdot a_0 = 1$

$$a_2 = 1 + c \cdot 1 = 1 + c$$

$$a_3 = 1 + c(1+c) = 1 + c + c^2$$

$\vdots$

$$a_n \overset{?}{=} 1 + c + c^2 + \dots + c^{n-1}$$

lo dimostro per induzione

$$a_n \overset{?}{=} \sum_{k=0}^{n-1} c^k$$

$n=0$ $\quad a_0 = 0 \quad \sum_{k=0}^{-1} c^k = 0$

$n=1$ $\quad a_1 = 1 \quad \sum_{k=0}^{0} c^k = c^0 = 1$

$$a_{n+1} = 1 + c\,a_n \underset{\text{ip. induttiva}}{=} 1 + c \sum_{k=0}^{n-1} c^k$$

$$= 1 + \sum_{k=0}^{n-1} c^{k+1}$$

$$= \underset{\substack{\| \\ c^0}}{1} + \sum_{k=1}^{n} c^k = \sum_{k=0}^{n} c^k \quad \checkmark$$

$$a_n = \sum_{k=0}^{n-1} c^k \longrightarrow \sum_{k=0}^{+\infty} c^k$$

Se $c < 0$ la successione non è monotona.

$f(x) = 1 + cx$ è decrescente

$\mathbb{R}$ è invariante

a_{2n} e a_{2n+1} sono

monotone.

$a_{2n} \to \ell \qquad a_{2n+1} \to \ell'$

$$a_{2n+1} = f(a_{2n}) = 1 + c\, a_{2n}$$

$$a_{2n+1} \downarrow \ell' \qquad \to 1 + c\,\ell$$

$$a_{2n+2} = f(a_{2n+1}) = 1 + c\, a_{2n+1}$$

$$a_{2n+2} \downarrow \ell \qquad 1 + c\, a_{2n+1} \downarrow 1 + c\,\ell'$$

$$\boxed{\begin{cases} \ell' = 1 + c\ell \\ \ell = 1 + c\ell' \end{cases}}$$

Se $l = +\infty$ $\quad l' = -\infty$

in quanto $c < 0$

$cl \to -\infty \qquad 1 + cl \to -\infty$

Se $l = -\infty \quad l' = +\infty$

$a_0 = 0 \qquad a_1 = 1$

$a_2 = 1 + c \qquad a_3 = 1 + c(1+c)$

$= 1 + c + c^2$

$\underbrace{\text{Se } c > -1}$

$a_2 \geq a_0 \implies a_3 \leq a_1$

f decrescente

a_{2n} è crescente $\qquad a_{2n+1}$ è decrescente.

$a_{2n} \geq a_0 = 0$ $\qquad$ $a_{2n+1} \leq a_1 = 1$

$[0,1]$ é invariante

$0 \leq x \leq 1$ $\qquad$ $f(x) = 1 + cx$

$$\underset{\underset{1}{\parallel}}{f(0)} \geq f(x) \geq \underset{\underset{1+c}{\parallel}}{f(1)}$$

$c < 0$

$$f([0,1]) \subseteq [1+c, 1]$$

$$\subseteq [0,1]$$

se $1+c \geq 0 \iff c \geq -1$

$\ell, \ell' \in [0,1]$

a_{2n}, a_{2n+1} convergono

$$\begin{cases} l' = 1 + c\,l \\ l = 1 + c\,l' \end{cases}$$

$$\begin{cases} l = 1 + c(1 + c\,l) = f(f(l)) \\ l' = 1 + c(1 + c\,l') \end{cases}$$

$$l = 1 + c + c^2 l$$

$$(1 - c^2)\, l = 1 + c$$

$$l = \frac{1+c}{1-c^2} = \frac{\cancel{1+c}}{\cancel{(1+c)}(1-c)}$$

$$= \frac{1}{1-c}$$

$$l' = \dots = \frac{1}{1-c}$$

$$\ell = \ell' = \frac{1}{1-c}$$

$-1 < c < 0$

la successione converge.

---

se $c < -1$

$a_0 = 0$

$a_2 = 1 + c$

a_{2n} decrescente, $a_2 < a_0$

a_{2n+1} crescente

$a_{2n} \leq a_0 \leq 0$

$a_{2n+1} \geq a_1 = 1$

l'unico punto fisso di $f \circ f$

è $\frac{1}{1-c} \in [0,1]$

$l, l' \nexists$ punto fisso

$l = -\infty \qquad l' = +\infty$

Posso applicare radice/rapporto?

Solo se voglio dimostrare che

$a_n \to 0 \quad$ o $\quad a_n \to +\infty$

Se voglio mostrare che $a_n \to l \in \mathbb{R}$

$l \neq 0$, posso provare a

considerare $\underbrace{b_n} = \underbrace{a_n - l}$

e dimostro che $b_n \to 0$

Ad esempio $\begin{cases} a_0 = 0 \\ a_{n+1} = 1 + c\, a_n \end{cases}$

Forse $a_n \to \ell = \frac{1}{1-c}$

$c \neq 1$

$$\boxed{b_n = a_n - \frac{1}{1-c}}$$

$a_n = b_n + \frac{1}{1-c}$

$$\begin{cases} b_0 = a_0 - \frac{1}{1-c} = \frac{1}{c-1} \\ b_{n+1} = a_{n+1} - \frac{1}{1-c} = 1 + c\, a_n - \frac{1}{1-c} \end{cases}$$

$$= 1 + c\left(b_n + \frac{1}{1-c}\right) - \frac{1}{1-c}$$

$$= 1 + \frac{c}{1-c} - \frac{1}{1-c} + c\, b_n$$

$$= 1 + \frac{c-1}{1-c} + c\, b_n$$

$$= 1 - 1 + c\, b_n = c\, b_n$$

$$\begin{cases} b_0 = \frac{1}{c-1} \\ b_{n+1} = c\, b_n . \end{cases}$$

$$\frac{|b_{n+1}|}{|b_n|} = \frac{|c\, b_n|}{|b_n|} = |c|$$

se $|c| < 1$

$|b_n| \to 0$ $\quad$ $\left| a_n - \frac{1}{1-c} \right| \to 0$

$$a_n \to \frac{1}{1-c} .$$

Se $|c| > 1$ $\quad$ $|b_n| \to +\infty$

$$\left| a_n - \frac{1}{1-c} \right| \to +\infty$$

$$|a_n| \to +\infty$$

a_n non converge.

Se $f: A \to A$ A invariante

f decrescente

allora $\quad a_{2n}, a_{2n+1}$ sono monotone

$$a_{2n} \to l$$

$$a_{2n+1} \to l'$$

se l è finito e f continua in l

$$l = f(f(l))$$

$$a_{2n+2} = f(a_{2n+1}) = f(f(a_{2n}))$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$l \qquad = \qquad f(f(l))$$

$\ell = |(f \circ f)(\ell)|$

$$\frac{1}{z} = \frac{\bar{z}}{z^2} \qquad (\text{se } z \neq 0)$$

$$z \cdot \frac{\bar{z}}{z^2} = \frac{z \cdot \bar{z}}{z^2} = \frac{z^2}{z^2} = 1$$

$$\frac{z}{w} = z \cdot \frac{\bar{w}}{w^2}$$
