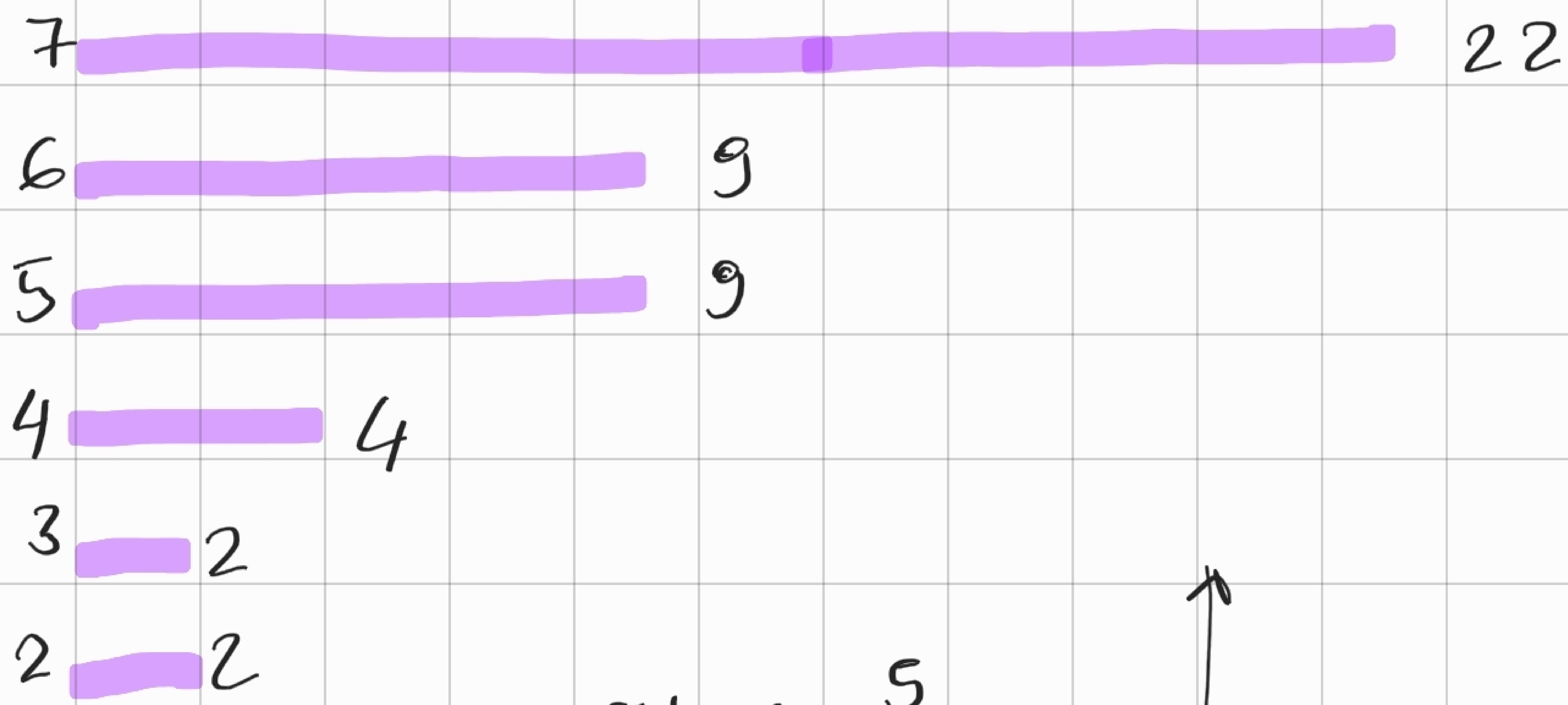
ANALISI MATEMATICA B
LEZIONE 41 - 18.1.2021

test settimanale



$|z^5| = |z|^5$

$z^5 = 1 \Rightarrow |z| = 1$

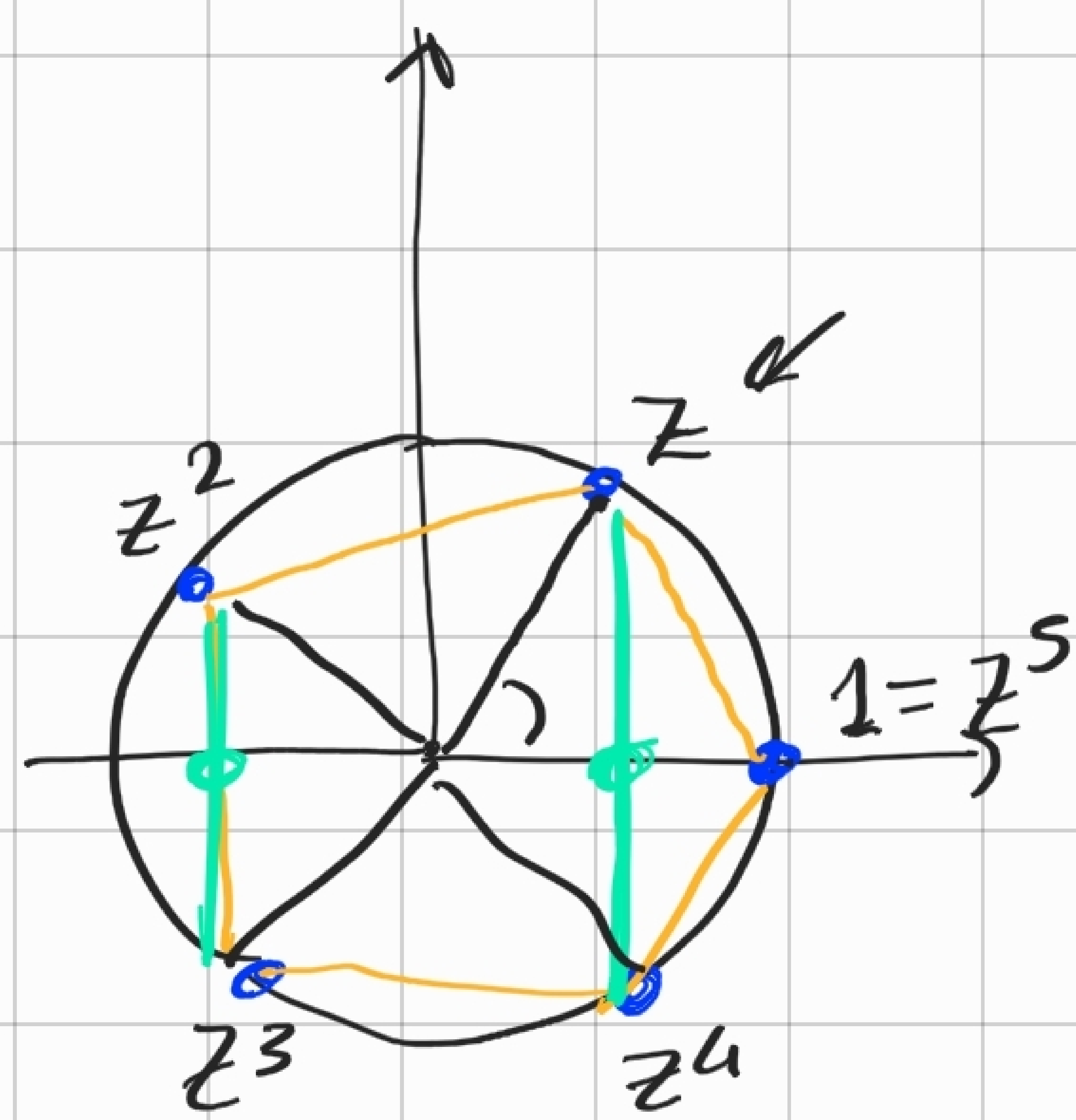
$\arg(z)$

$z^4 + z^3 + z^2 + z + 1 = 0$

$x = \operatorname{Re} z$

$4x^2 + 2x - 1 = 0$

$x_{1,2} = \dfrac{-1 \pm \sqrt{5}}{4}$



$\operatorname{Re} z = \dfrac{\sqrt{5} - 1}{4} = \cos \dfrac{2\pi}{5}$

$z = e^{i \frac{2\pi}{5}}$

ES5 se $\bar{z} = \frac{1}{z}$

$$(z+\bar{z})^2 = z^2 + 2\cancel{z\bar{z}} + \bar{z}^2$$

$$z^2 + \bar{z}^2 = (\underbrace{z+\bar{z}}_{})^2 - 2$$

$$x = \operatorname{Re} z = \frac{z+\bar{z}}{2}$$

$$z^2 + \bar{z}^2 = (2x)^2 - 2$$

$$= 4x^2 - 2$$

ES6 se $\bar{z} = \frac{1}{z}$

$$z^4 + z^3 + z^2 + z + 1 = 0$$

moltiplichiamo per $\bar{z}^2$

$$z^2 + z + 1 + \bar{z} + \bar{z}^2 = 0$$

$$(z^2 + \bar{z}^2) + (z + \bar{z}) + 1 = 0$$

$x = \operatorname{Re} z \qquad z + \bar{z} = 2x$

$z^2 + \bar{z}^2 = 4x^2 - 2$

$4x^2 - 2 + 2x + 1 = 0$

$4x^2 + 2x - 1 = 0 \qquad \square$

$\rightarrow \cos(60^\circ) = \frac{1}{2} \qquad \sin(60^\circ) = \frac{\sqrt{3}}{2}$

$\rightarrow \cos(30^\circ) = \frac{\sqrt{3}}{2} \qquad \sin(30^\circ) = \frac{1}{2}$

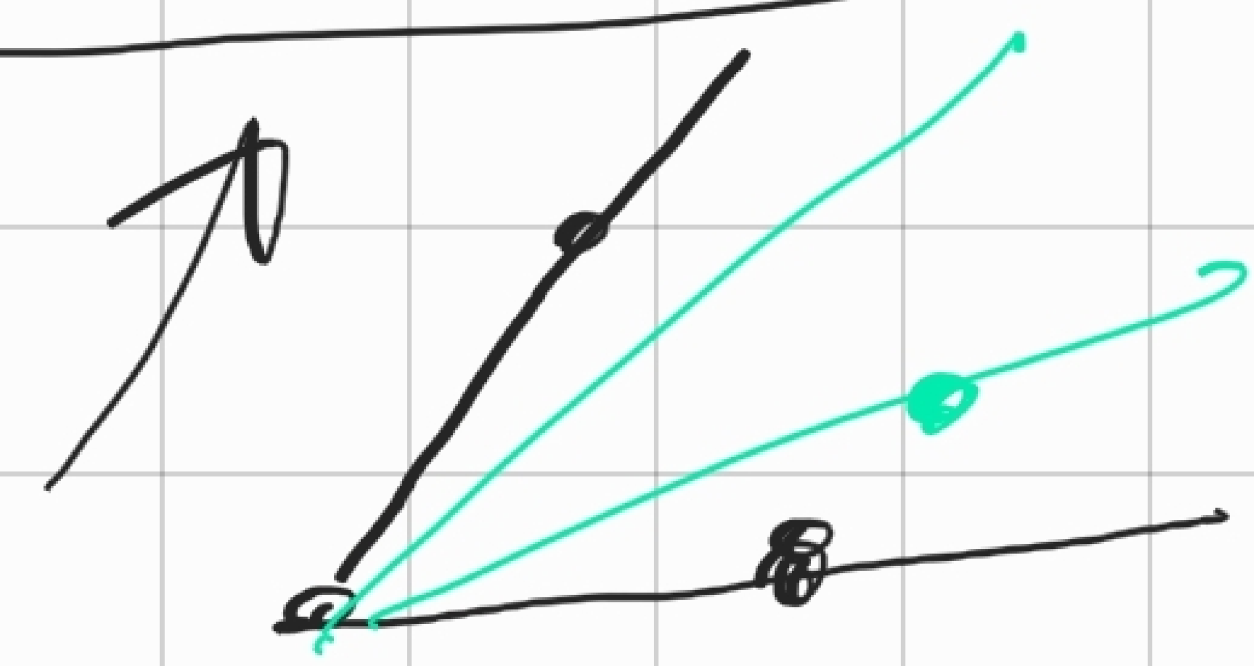
$\rightarrow \cos(45^\circ) = \frac{\sqrt{2}}{2} = \sin(45^\circ)$

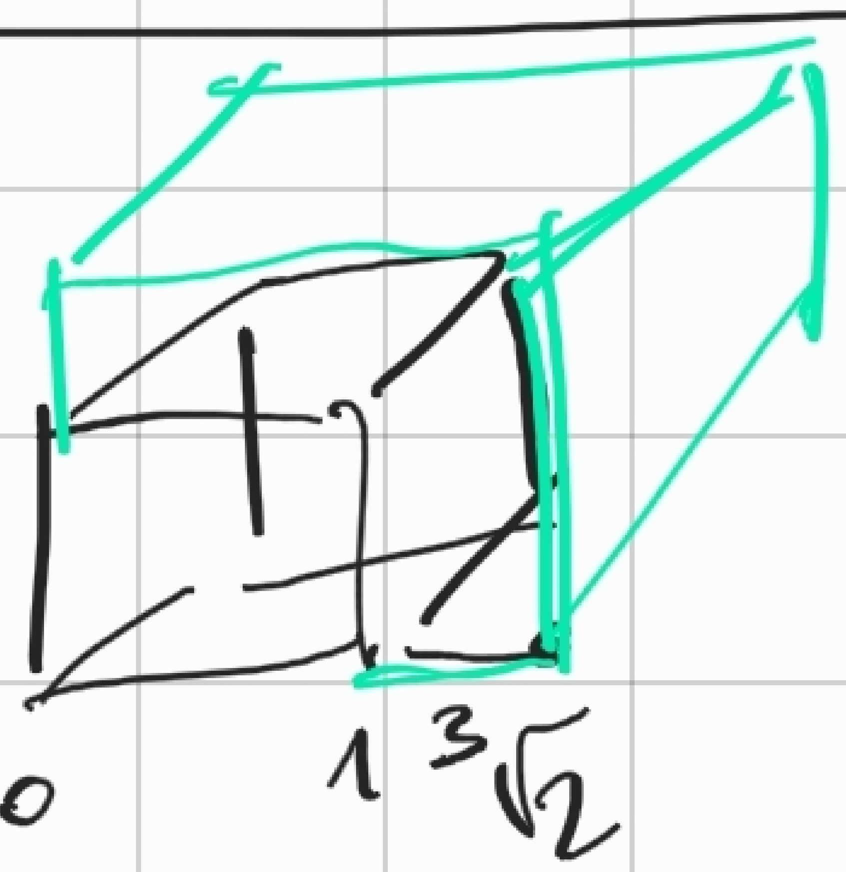
Cosa significa risolvere una equazione?

$$\boxed{x^5 - x - 1 = 0}$$

→ non ha una "formula risolutiva"

Trisezione dell'angolo:

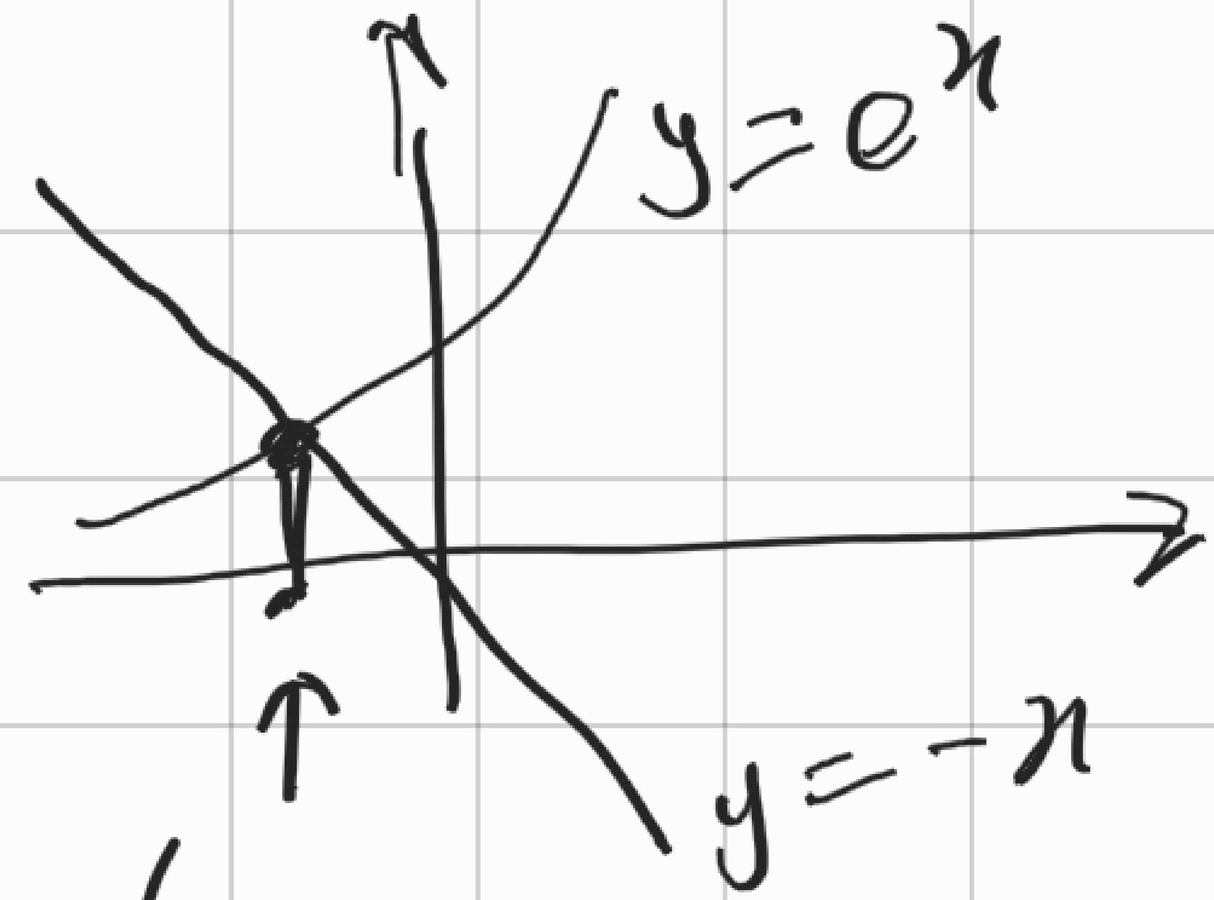




duplicazione del cubo

quadratura del cerchio:

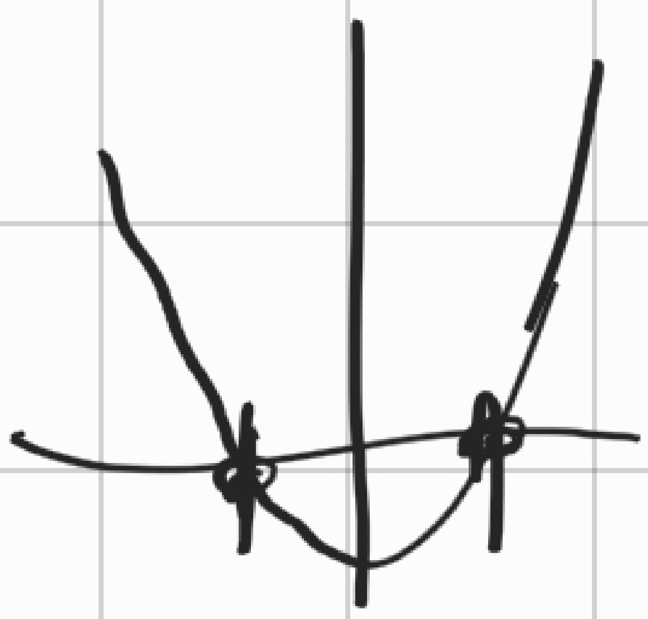
$e^x = x$



$x^2 = 2$

$x_1 = \sqrt{2}$

$x_2 = -\sqrt{2}$

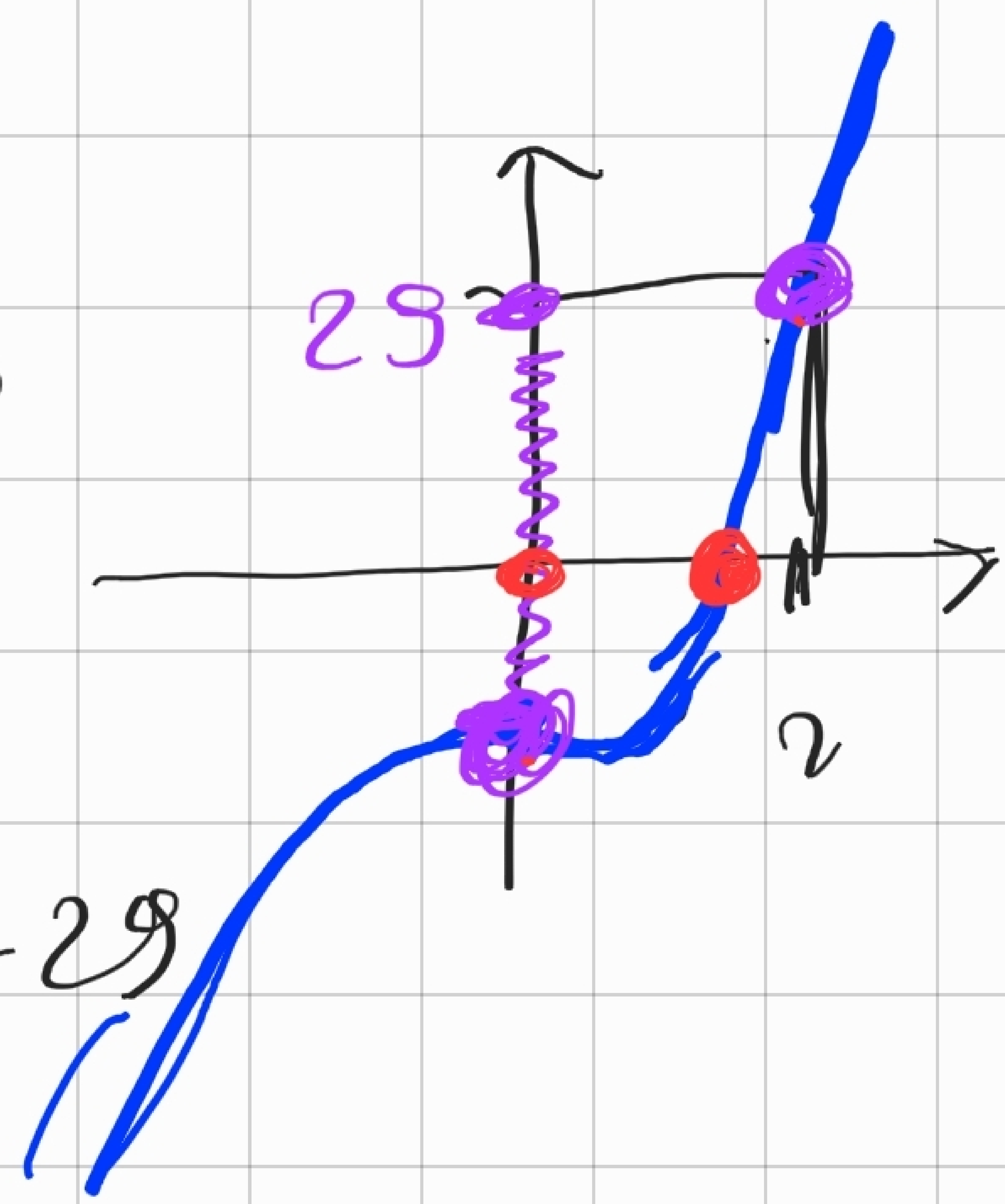


$\int e^{-x^2} dx$

ES $f(x) = x^5 - x - 1 = 0$

$f(0) = -1$

$f(2) = 32 - 2 - 1 = 29$



Teorema (degli zeri)

Se $f : [a, b] \to \mathbb{R}$ è continua,

$f(a) \leq 0$ e $f(b) \geq 0$ allora

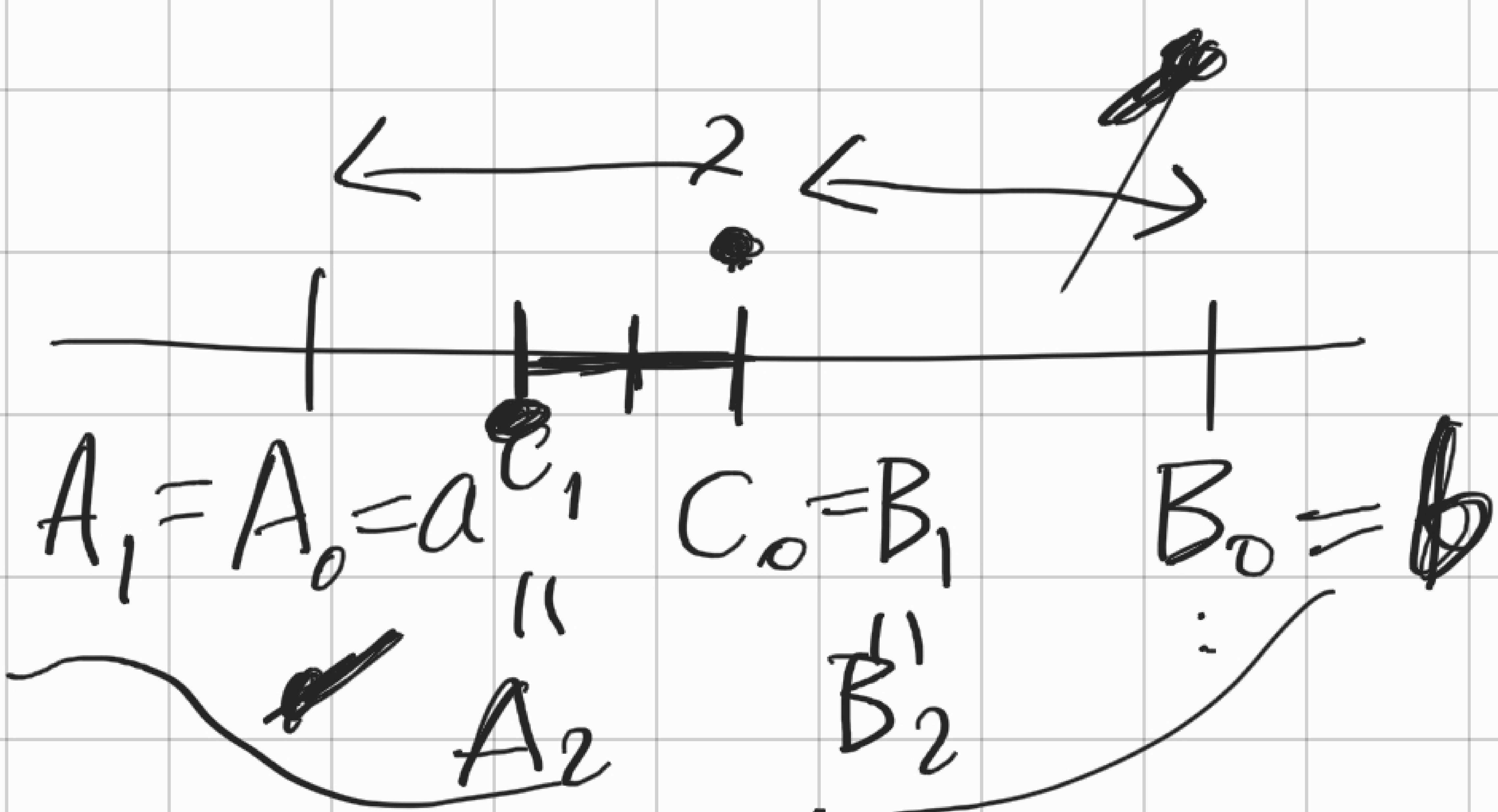
$\exists \; c \in [a, b]$ tale che

$$f(c) = 0$$

ovviamente vale lo stesso

se $f(a) \geq 0$ e $f(b) \leq 0$.

dim (metodo di bisezione)



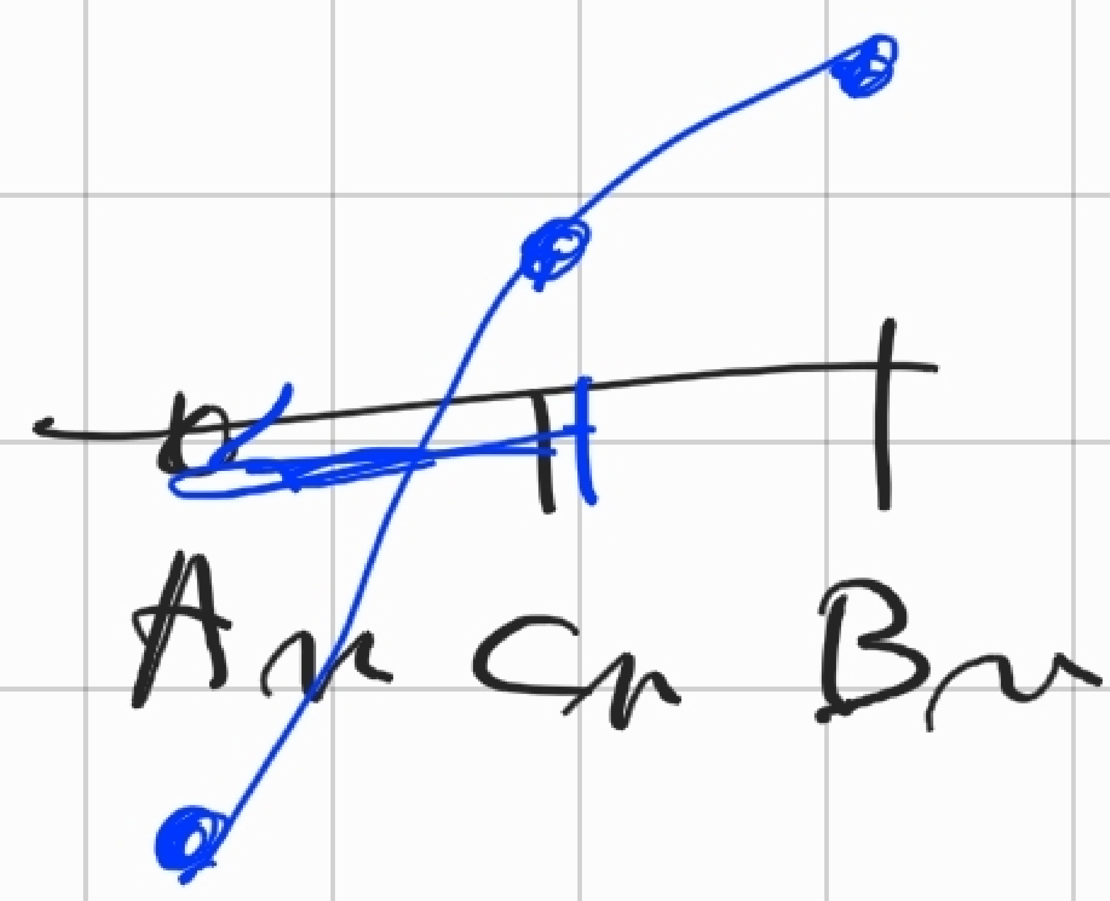
$A_0 = a \qquad B_0 = b \qquad C_0 = \frac{a+b}{2}$

$f(a) \leq 0 \qquad f(b) \geq 0$

Definisco ricorsivamente le successioni A_n, B_n, C_n

Se

$$\begin{cases} \begin{cases} A_{n+1} = A_n \\ B_{n+1} = C_n \\ C_{n+1} = \dfrac{A_{n+1}+B_{n+1}}{2} \end{cases} & \text{se } f(C_n) \geq 0 \\ \begin{cases} A_{n+1} = C_n \\ B_{n+1} = B_n \\ C_{n+1} = \dfrac{A_{n+1}+B_{n+1}}{2} \end{cases} & \text{se } f(C_n) < 0 \end{cases}$$



Per induzione si osserva che:

$f(A_n) \le 0$, $f(B_n) \geq 0$
$\forall n \in \mathbb{N}$.

$\forall n$: $B_n - A_n = \frac{b-a}{2^n} \geq 0$

A_n e crescente

B_n e decrescente

$a \le A_n \le B_n \le B_0 = b$

A_n crescente limitata

$c = \lim_{n \to +\infty} A_n$ (esiste!)

$\lim_{n \to +\infty} B_n = \lim_{n \to +\infty} \left(\underbrace{A_n}_{\to c} + \underbrace{\cancel{\frac{b-a}{2^n}}}_{\to 0} \right) = c$

$f(A_n) \leq 0$

$\Downarrow$

$f(c) \leq 0$

f continua

$A_n \to c$

$f(B_n) \geq 0$

$\Downarrow$

$f(c) \geq 0$

f continue

$B_n \to c$

$\Rightarrow f(c) = 0$

(se $f(a) \geq 0$ e $f(b) \leq 0$

prendo $-f$ al posto di f)

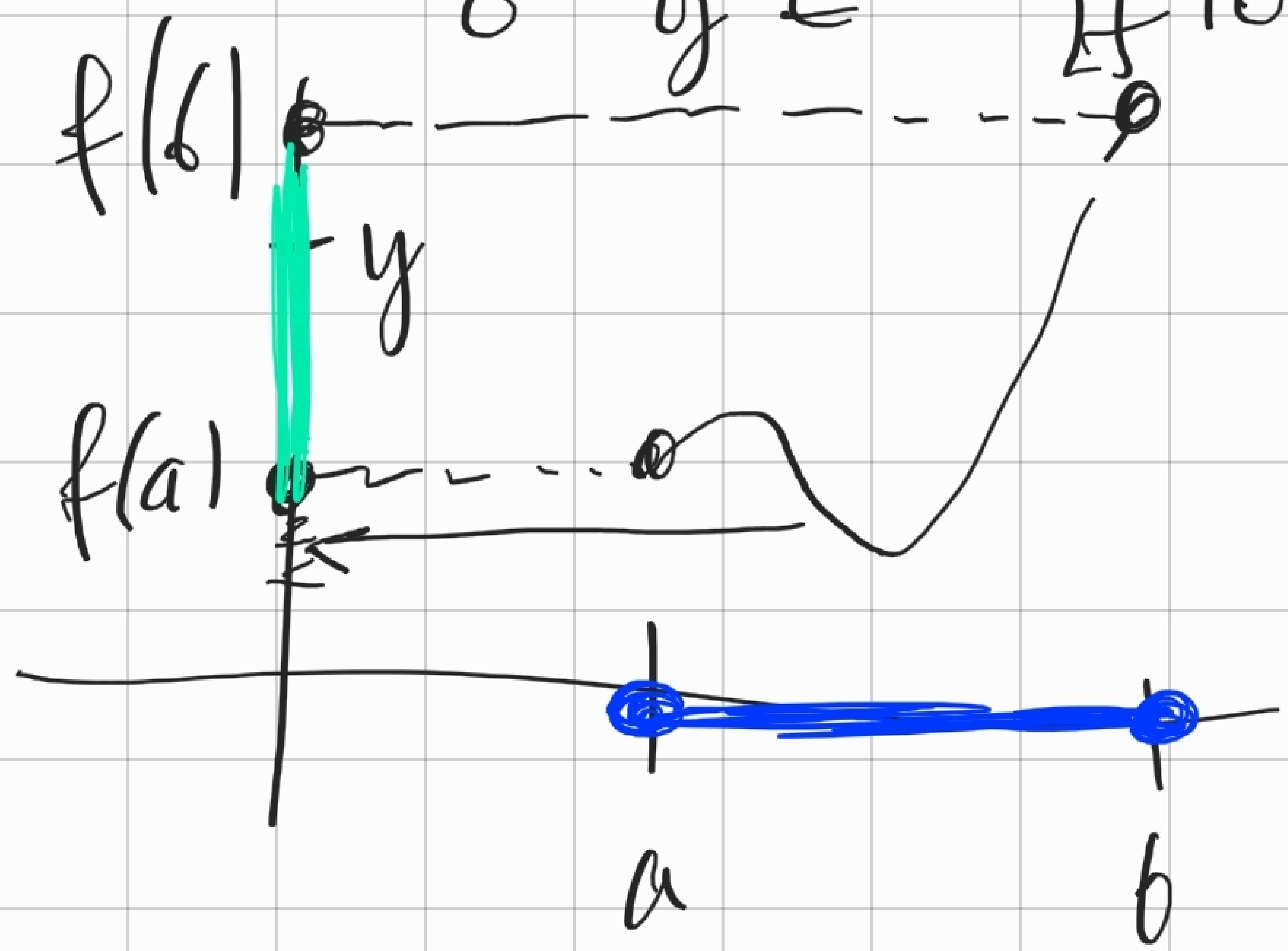
$\square$

Corollario (teorema dei valori intermedi)

Sia $f : [a,b] \to \mathbb{R}$ continua.

Allora se $y \in [f(a), f(b)]$

o $y \in [f(b), f(a)]$



allora $\exists\, x \in [a,b] : f(x) = y$

Ovvero:

$$f([a,b]) \supseteq [f(a), f(b)] \cup [f(b), f(a)]$$

Ovvero: f continua manda intervalli in intervalli:

[Se I è un intervallo, f continua $f(I)$ è un intervallo.]

dim

$$g(x) = f(x) - y$$

se $f(a) \leq y \leq f(b)$

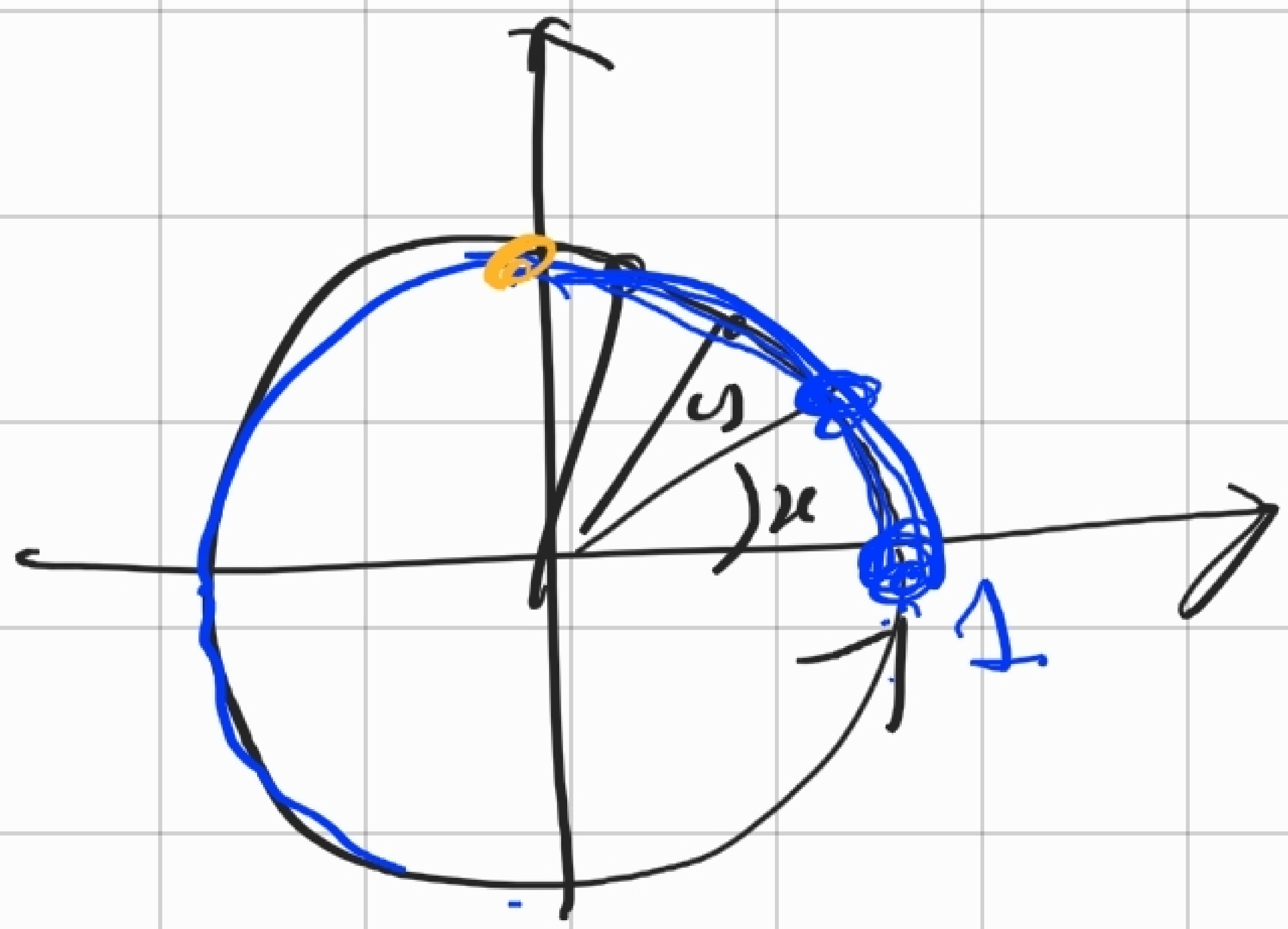
$$g(a) \leq 0 \qquad g(b) \geq 0$$

g continua $\exists\, c : \; g(c) = 0$

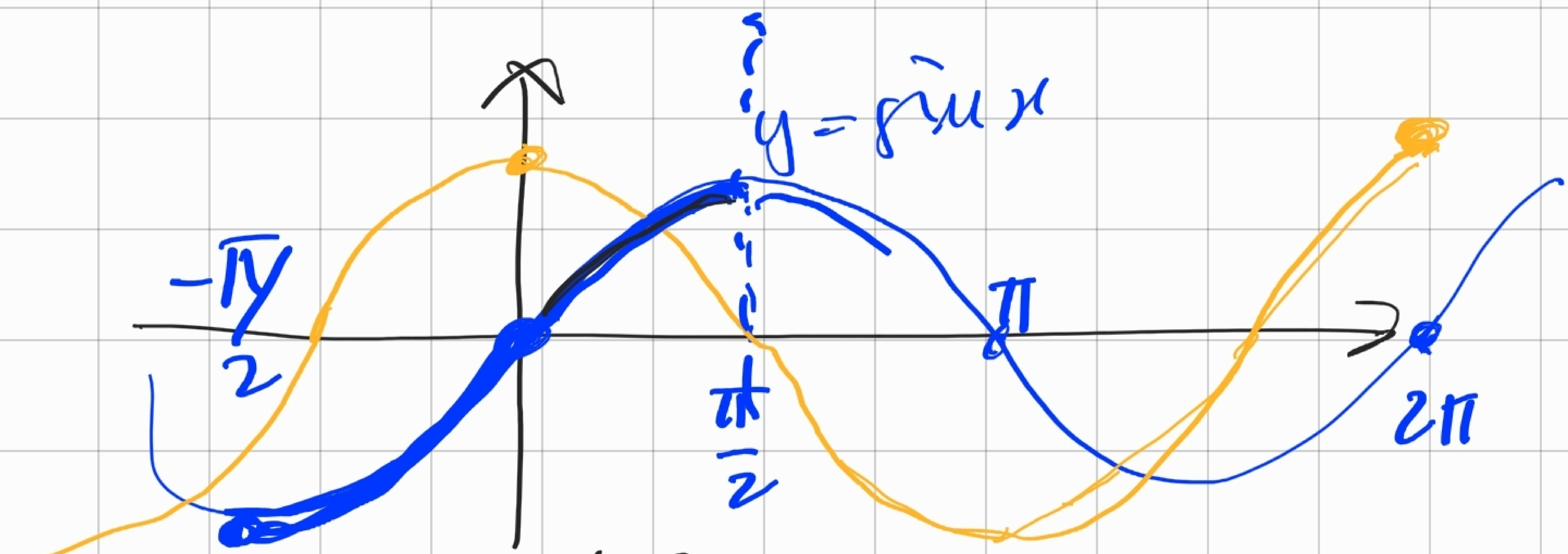
$$f(c) = y. \qquad \square$$

π (proprietà di sin, cos)

$$\underbrace{|e^{ix}| = 1}$$



$$\begin{cases} \cos x = \operatorname{Re} e^{ix} & |\cos x| \le 1 \\ \sin x = \operatorname{Im} e^{ix} & |\sin x| \le 1 \end{cases}$$



$$\cos x = \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

$$\sin x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Vogliamo mostrare che
sin e cos sono funzioni
periodiche. Posto π il
minimo
semi-periodo (il periodo è 2π)

Si trova che

$\sin x$ è strettamente
crescente su $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos x$ è strettamente
decrescente su $[0, \pi]$ $\square$

dim (idea)

$$e^{ix} = \sum_{k=0}^{+\infty} \frac{(ix)^k}{k!}$$

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \le \frac{|z|^{n+1}}{n \cdot n!}$$

se $|z| \le 1$

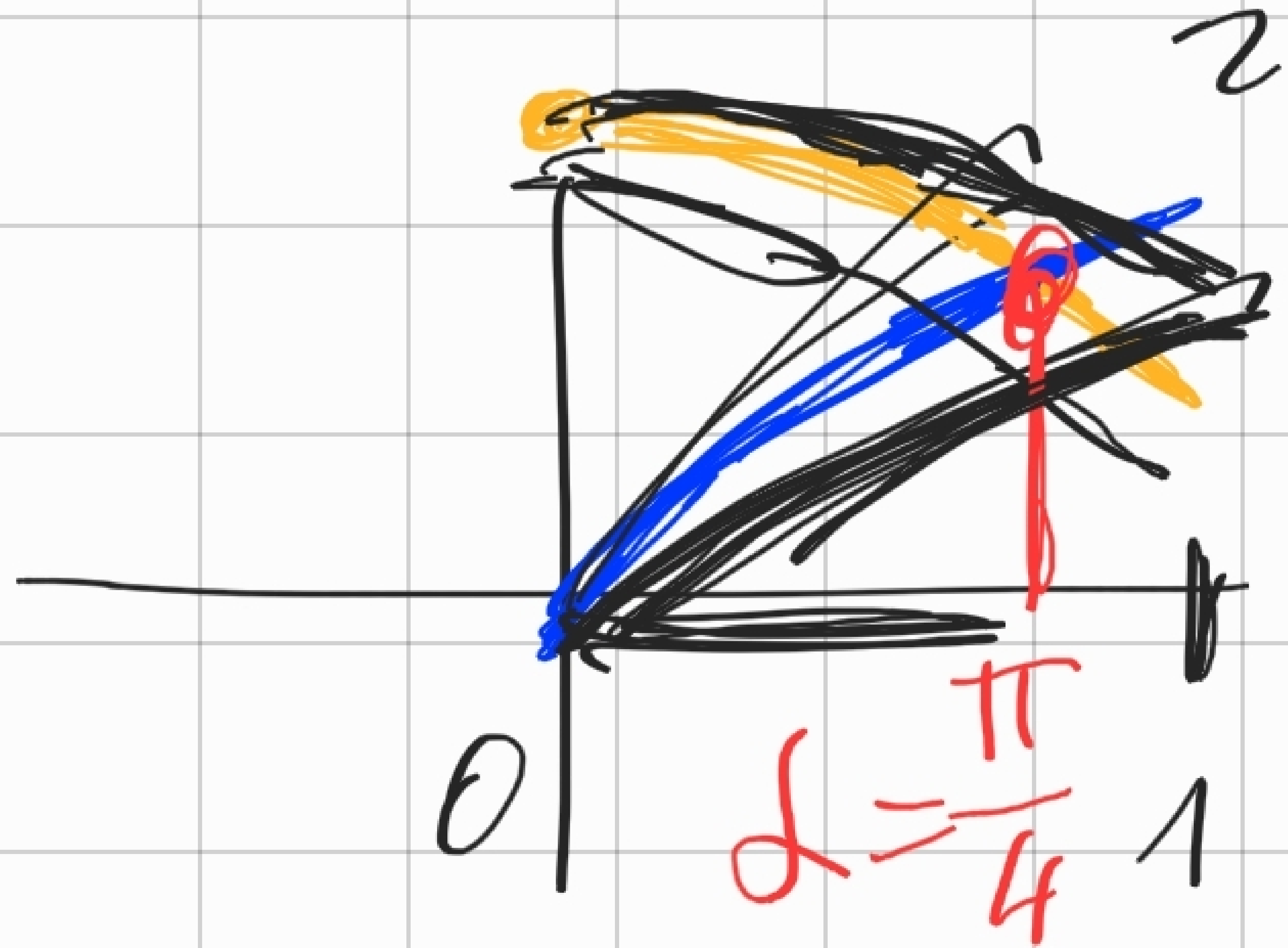
$z = ix$

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$n = 3$, x $|x| \le 1$

$$x - \frac{x^3}{6} - \frac{x^4}{18} \le \sin x \le x - \frac{x^3}{6} + \frac{x^4}{18}$$

$$1 - \frac{x^2}{2} - \frac{x^4}{18} \le \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{18}$$



$\sin x$ è strett crescente su $[0,1]$
$\cos x$ è strett. decrescente su $(0,1]$?

$\cos(x+t) = \cos x \cos t - \operatorname{sen} x \operatorname{sen} t$

$\operatorname{sen} x \geq 0$ se $x \in [0,1]$

$\operatorname{sen} t \geq 0$ se $t \in (0,1)$

$\cos t < 1$ se $t \in (0,1]$

$\cos(x+t) < \cos x$ se $t \in (0,1]$, $x \in [0,1]$

$\Downarrow$

$\cos$ è strett decrescente.

$\sin x = \sqrt{1-\cos^2 x}$

$\Rightarrow \sin x$ è strett crescente

Voglio trovare $\alpha = \frac{\pi}{4}$

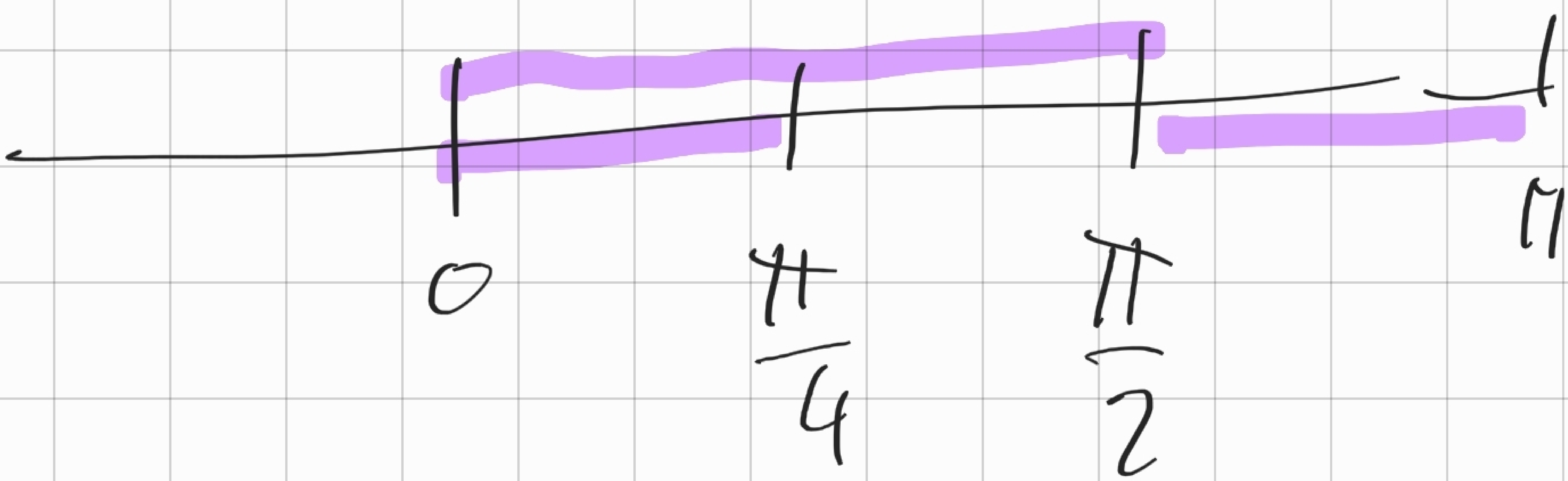
così $\sin \alpha = \frac{\sqrt{2}}{2}$

$\sin 1 > \frac{\sqrt{2}}{2}$ $\quad \sin 0 = 0$

$\sin x$ continuo ✓

$\exists \alpha : \sin \alpha = \frac{\sqrt{2}}{2}$.

$\Rightarrow \cos \alpha = \frac{\sqrt{2}}{2}$

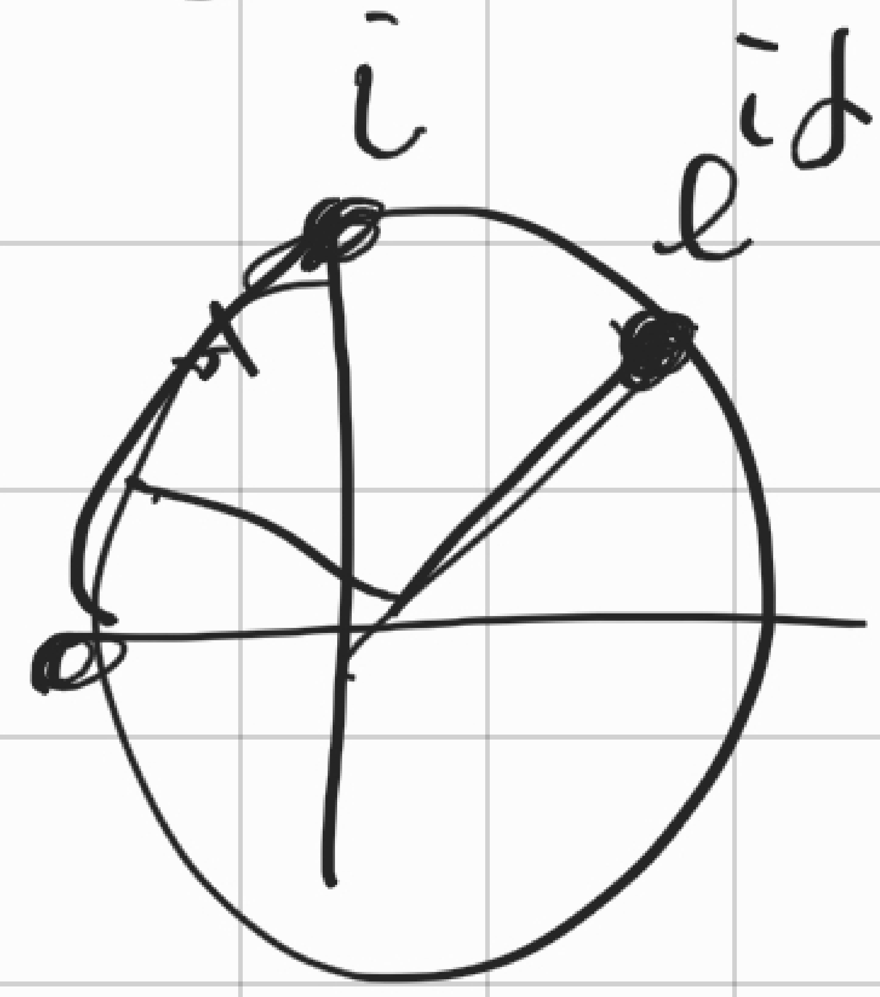


$$\begin{cases} \sin\left(\frac{\pi}{2} - x\right) = \cos x \\ \cos\left(\frac{\pi}{2} - x\right) = \sin x \end{cases}$$

$$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\begin{cases} \cos(2\alpha) = 0 \\ \sin(2\alpha) = 1 \end{cases}$$

$$\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = i$$



$$\left[\frac{1}{2}(\cancel{1} - \cancel{1} + 2i) = i\right]$$

$$\begin{cases} \sin(\pi - x) = \sin x \\ \cos(\pi - x) = -\cos x \end{cases}$$

$$e^{i4\alpha} = \left(e^{i2\alpha}\right)^2 = i^2 = -1$$

- - -

$$e^{2i\pi} = (-1)^2 = 1$$

$$e^{i(x+2\pi)} = \underbrace{\cancel{e^{i2\pi}}}_{1} \cdot e^{ix} = e^{ix} .$$

$\square$