ANALISI MATEMATICA B
17.2.2021 - LEZIONE 54

Es. Giust. $\displaystyle \lim_{x\to 0} \frac{1-e^{-x^2}+x^3\sin\frac{1}{x}}{x^2} = \frac{\cancel{1}-(\cancel{1}-x^2+o(x^2))+x^3\sin\frac{1}{x}}{x^2} =$

$$\left[e^{-x^2} = 1-x^2+o(x^2) \qquad e^x = 1+x+o(x) \right]$$

$$= \frac{x^2+x^3\sin\frac{1}{x}+o(x^2)}{x^2} = 1+\underbrace{x\sin\frac{1}{x}}_{\to 0}+\underbrace{\frac{o(x^2)}{x^2}}_{\to 0}$$

$$\longrightarrow 1$$

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1. $\sin(0,3)$ $\qquad n=4:\quad P(x)=x-\dfrac{x^3}{6}$

2. Resto di Lagrange: $\forall x\ \exists y\in(0,x)$

$$f(x)=\sin(x) = P(x) + \frac{f^{IV}(y)}{4!}x^4$$

$x=0,3$

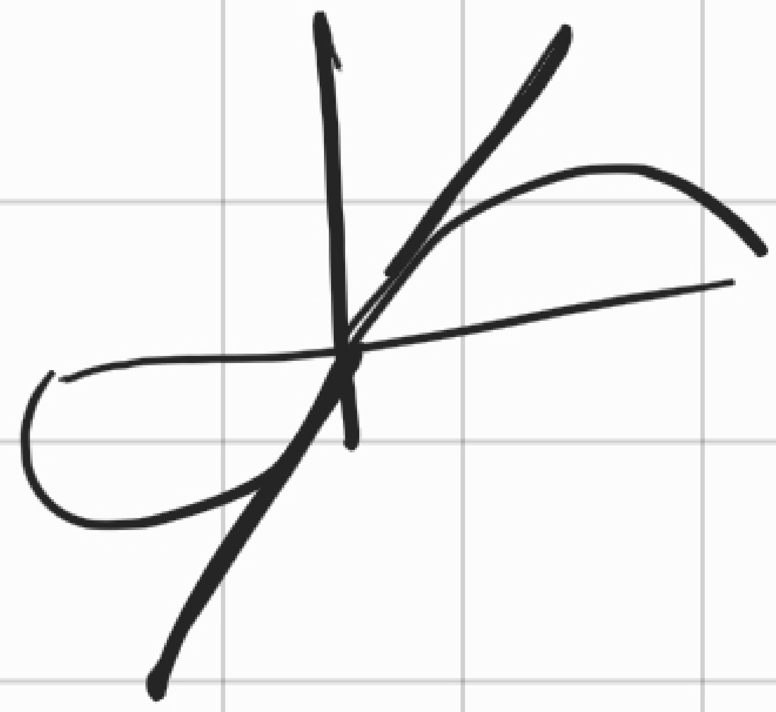
$$|\sin(0,3)-P(0,3)| \le \frac{|f^{IV}(y)|}{4!}\left(\frac{3}{10}\right)^4 =$$

$$= \frac{|\sin(y)|}{4!}\,\frac{3^4}{10^4} \le \frac{3}{10}\,\frac{1}{4!}\,\frac{3^4}{10^4} = \frac{3^5}{10^5}\,\frac{1}{4!}$$

$$0 \le |\sin(y)| \le \sin(0,3) \le \frac{3}{10} \quad \text{perché} \quad 0 < y < 0,3$$

$\sin$ è crescente

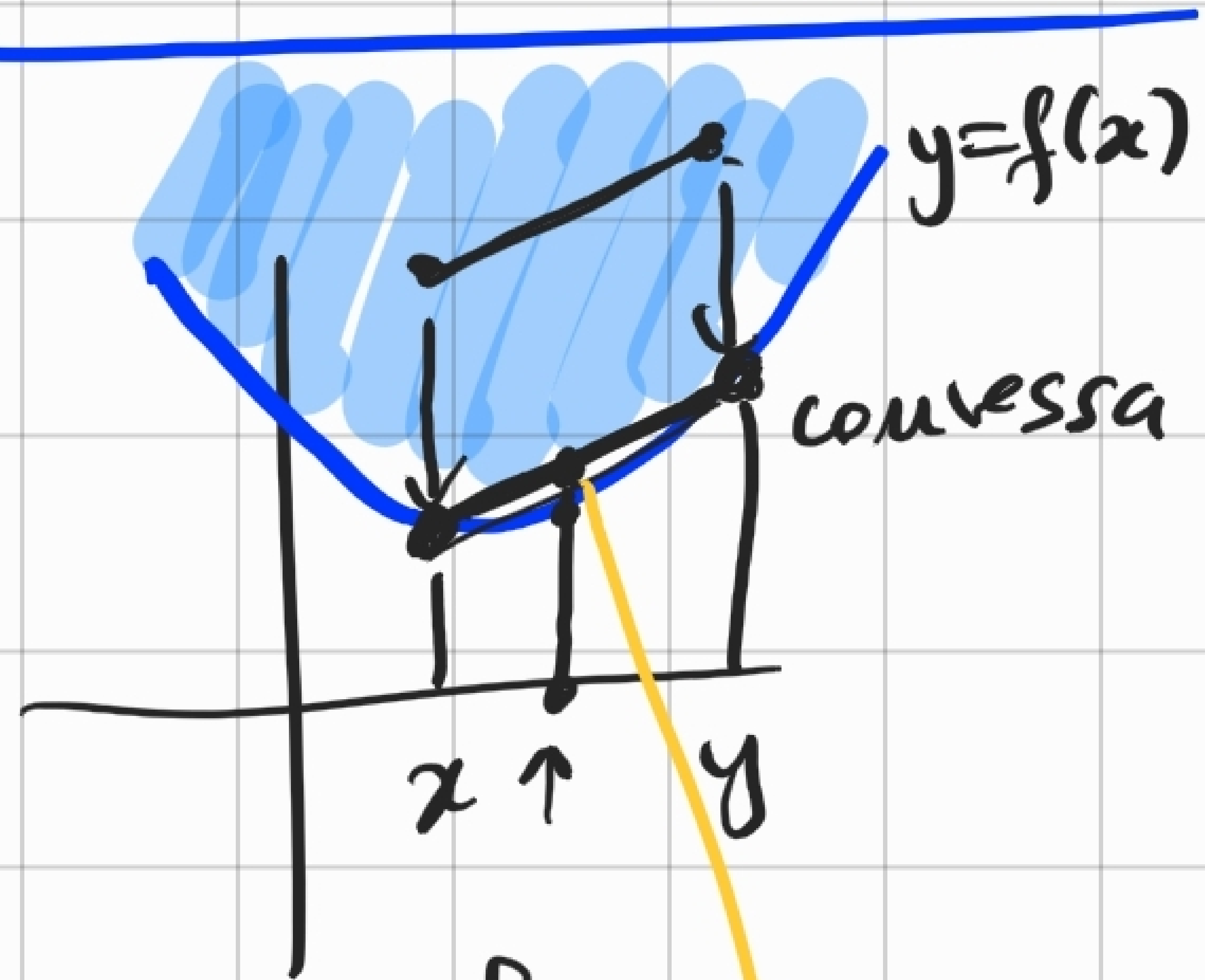
$|\sin x| \le |x|$



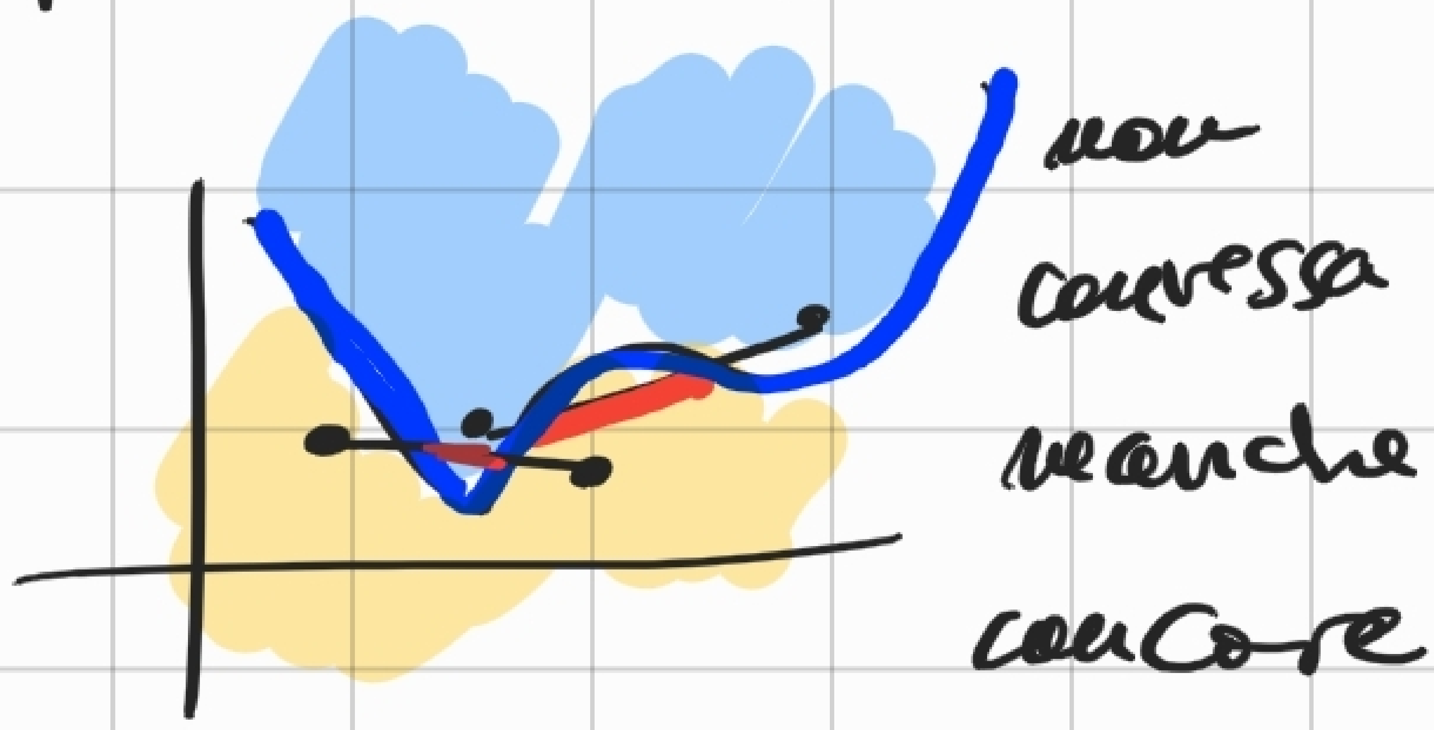
$$\frac{3^5}{4\cdot 3\cdot 2}\,\frac{1}{10^5} = \frac{3^4}{8}\,\frac{1}{10^5} \le 1.1 \cdot 10^{-4}$$

$$\boxed{\frac{81}{8} \le 11}$$

CONVESSITA'



$E_f = \{(x,y) : y \geq f(x)\}$



f concava se $-f$ convessa.

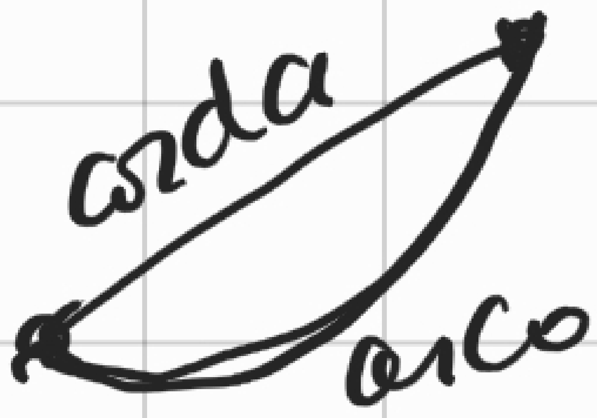
$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$

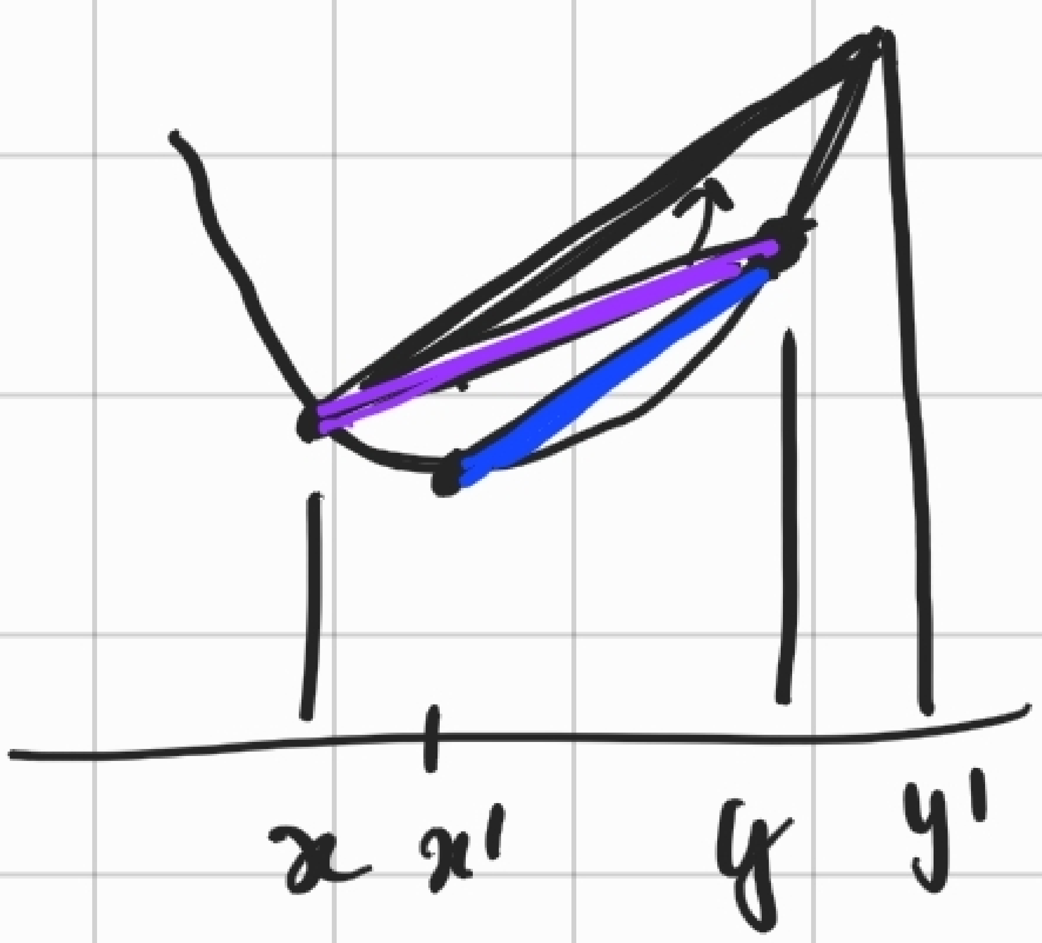
$(tx+(1-t)y,\ tf(x)+(1-t)f(y))$

Analiticamente: A intervallo.

$$\boxed{\forall x,y \in A\ \forall t \in [0,1] : f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).}$$

l'arco del grafico "sta sotto" la corda





$R(x,y) = \dfrac{f(y)-f(x)}{y-x}$ = pendenza della corda

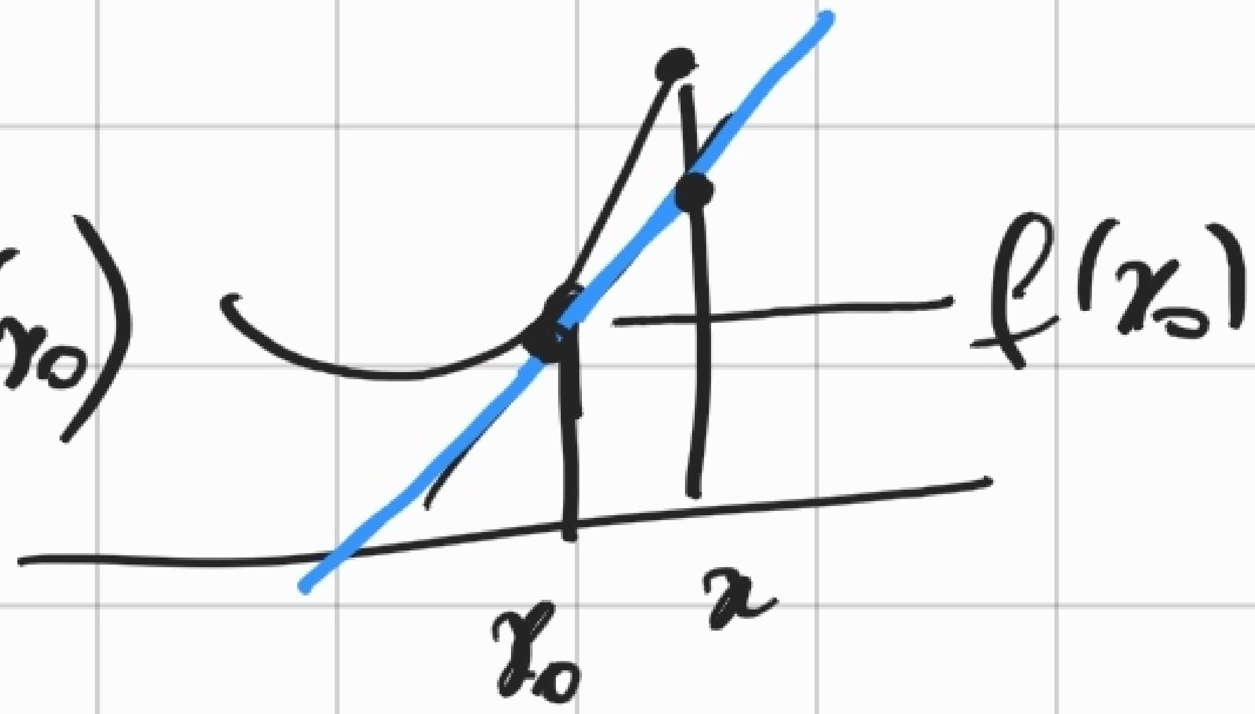
$R(x,y)$ è crescente sia in x che in y.

Se f è derivabile sono equivalenti:

1) f convessa

2) $\boxed{f' \text{ crescente}}$ ← Se f'' esiste basta $f'' \geq 0$

3) $f(x) \geq f'(x_0)(x-x_0)+f(x_0)$



ES $f(x) = e^x$ è convessa.

dunque: $t = \frac{1}{2}$

$$\boxed{f\left(\frac{1}{2}x+\frac{1}{2}y\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)}$$

$$e^{\frac{x+y}{2}} \leq \frac{1}{2}e^x + \frac{1}{2}e^y$$

$$\sqrt{e^x \cdot e^y} \leq \frac{e^x+e^y}{2}$$

$a,b>0 \quad x = \ln a, \quad y = \ln b$

$$\boxed{\sqrt{a \cdot b} \leq \frac{a+b}{2}}$$

Es $\ln x$ è concavo

in quanto

$$(\ln x)'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0 \qquad \square$$

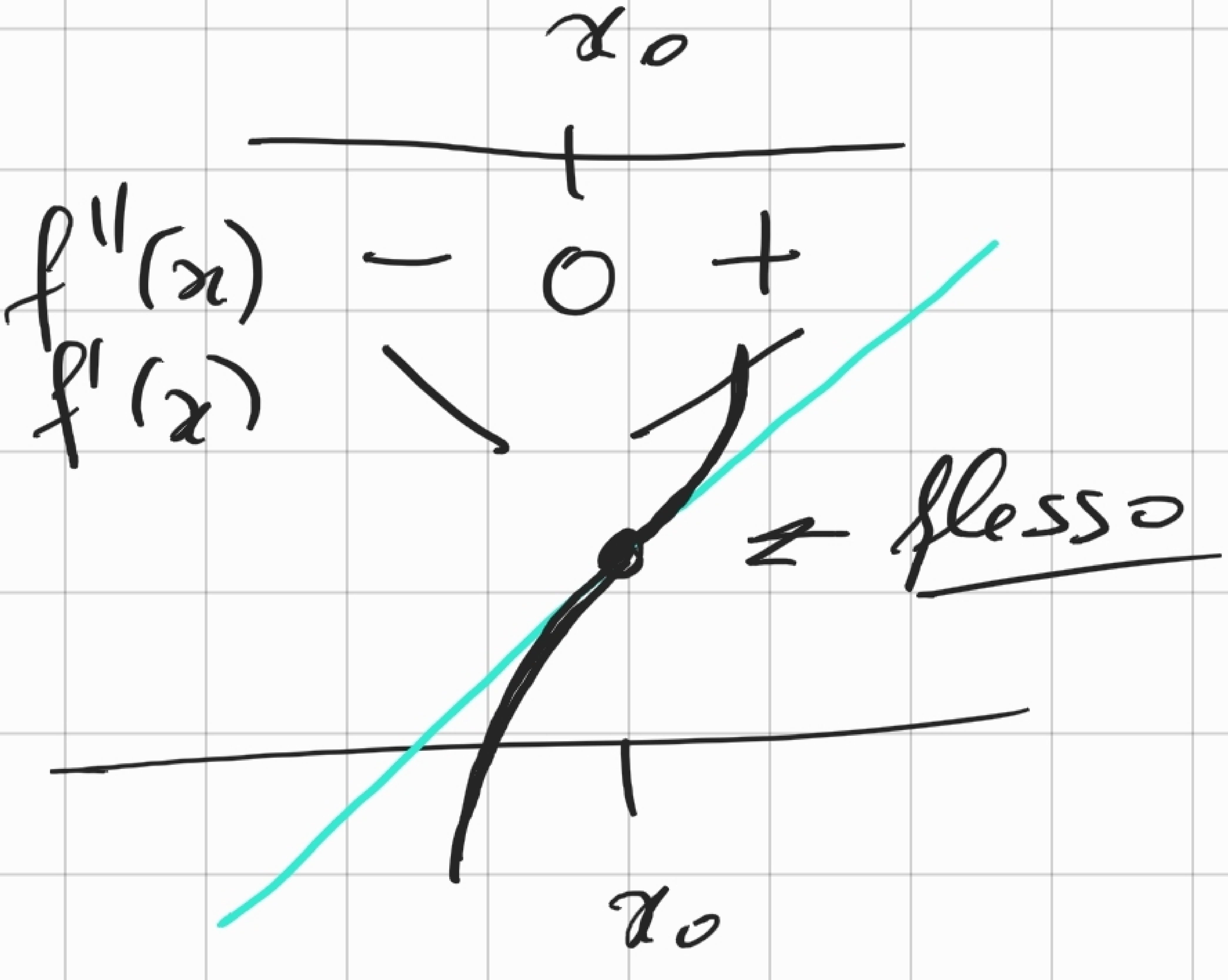
Es

$$f(x) = x^2$$

$$f'(x) = 2x, \quad f'' = 2 > 0$$

$\Rightarrow f(x)$ è convessa

Flessi



Osservazione $R(x,y)$ è crescente

$\longrightarrow$ $\lim_{x \to x_0^+} R(x_0, x) = m^+$ esiste

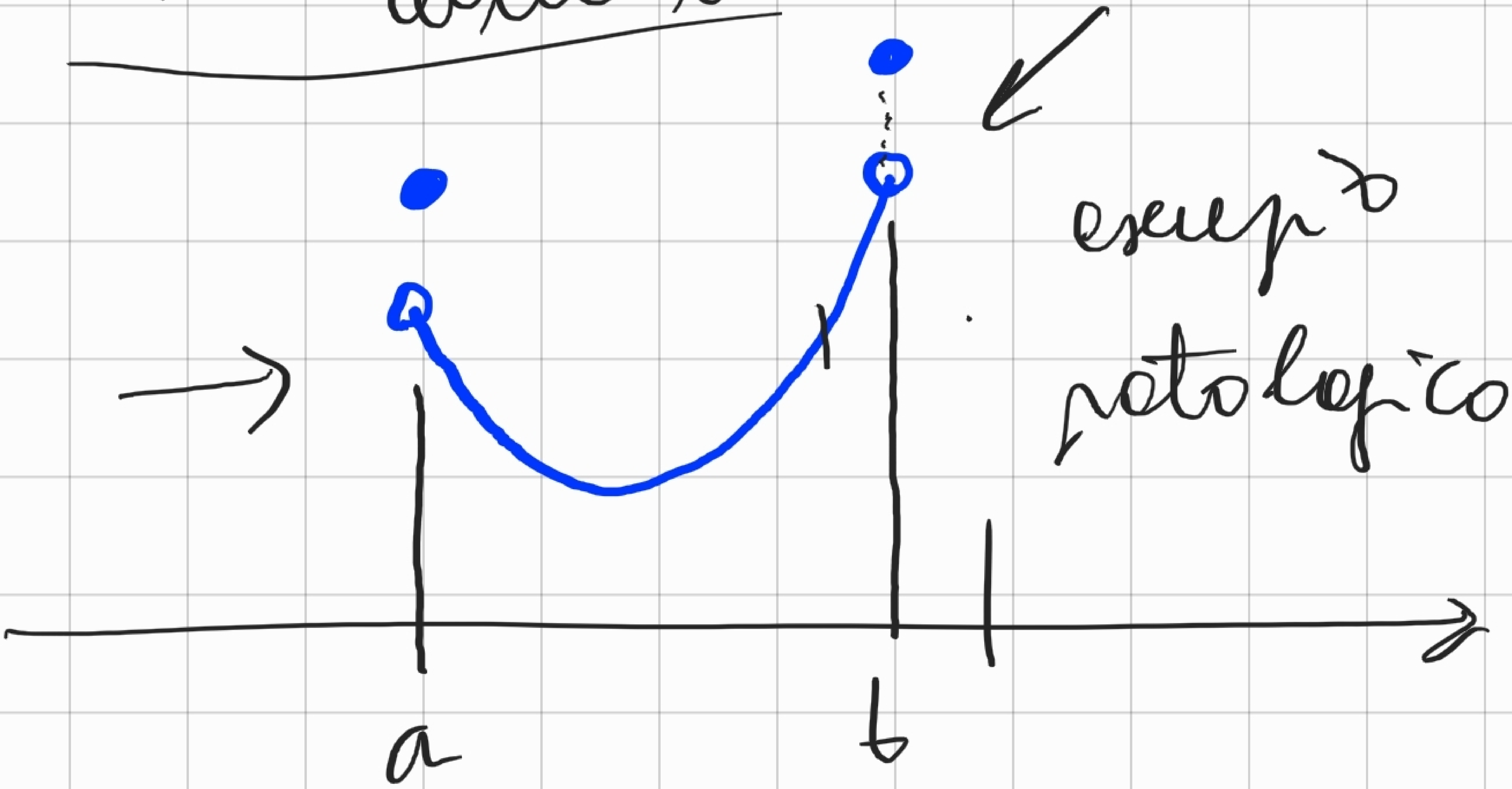
$\lim_{x \to x_0^-} R(x_0, x) = m^-$ esiste

$m^+ \geq m^-$

$$m^+ = \inf_{x > x_0} R(x_0, x) \geq m^- = \sup_{x < x_0} R(x_0, x) \Rightarrow m^+ \geq m^-$$

$m^+ = m^- \Rightarrow \exists f'(x_0) = m^+ = m^-$

m^+ e m^- sono finiti salvo eventuali punti agli estremi del dominio

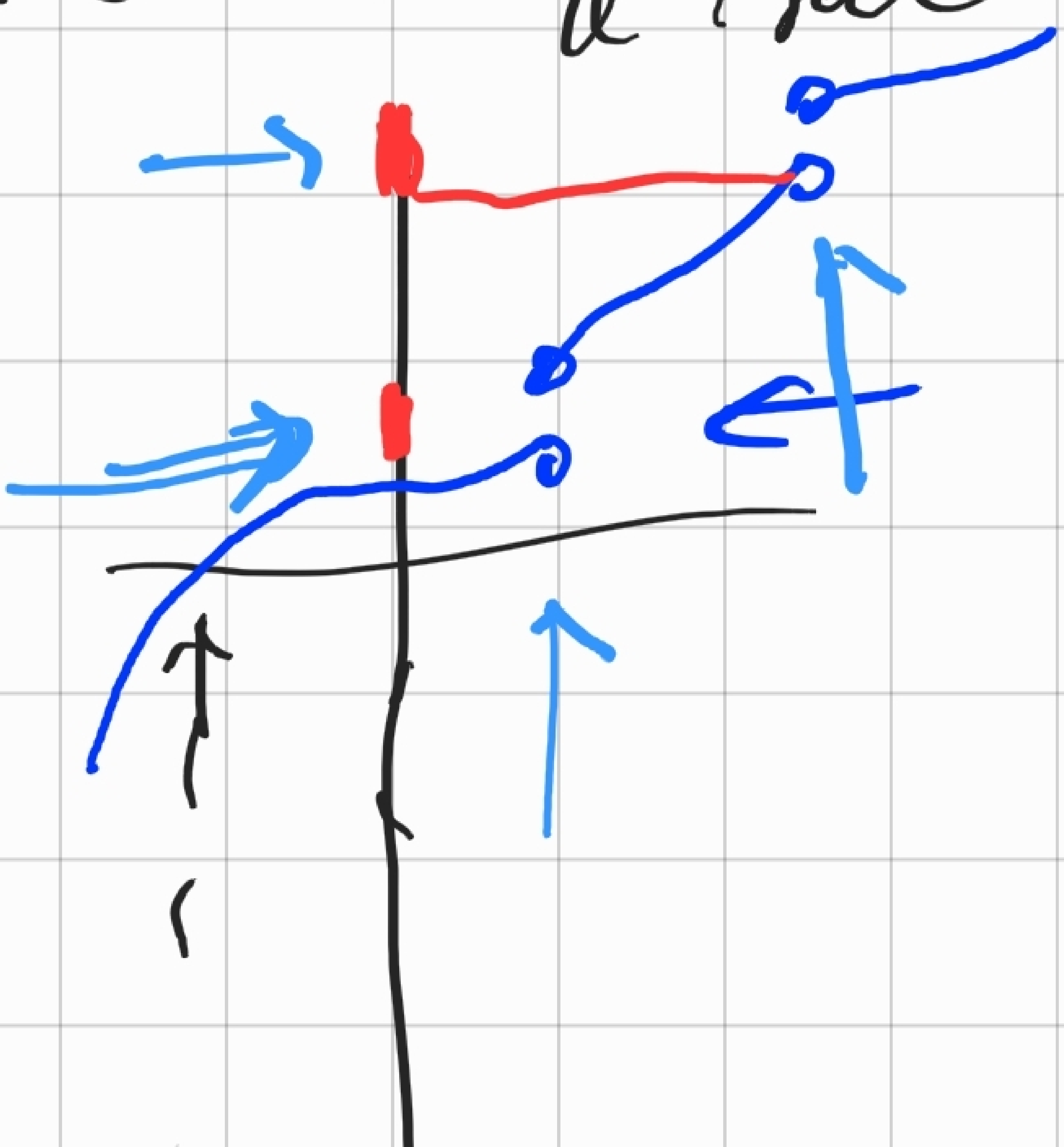


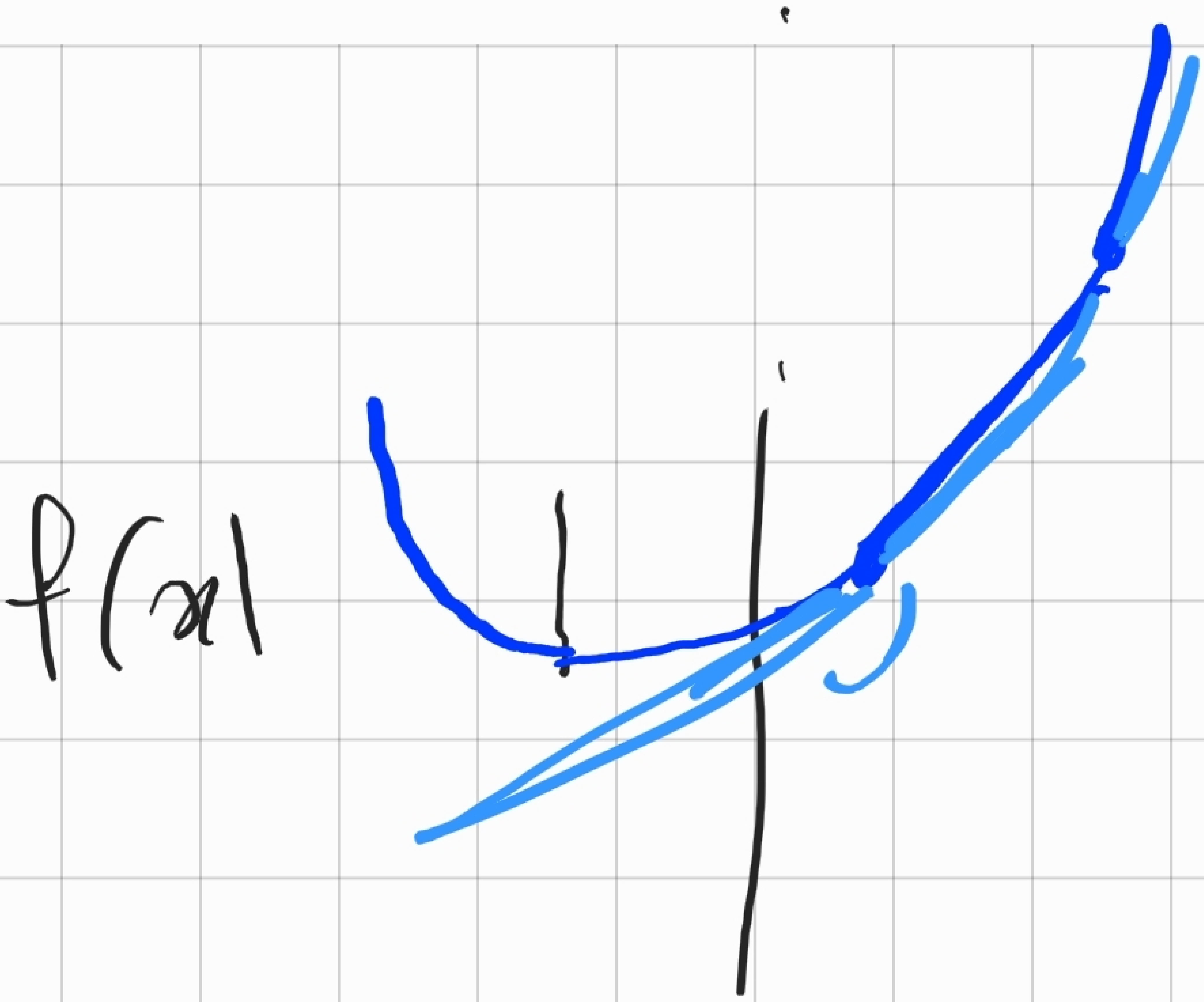
solo gli estremi

se f è convessa $\Rightarrow$ f continua.

f' è crescente ha solo discontinuità a salto

$f'(x)$



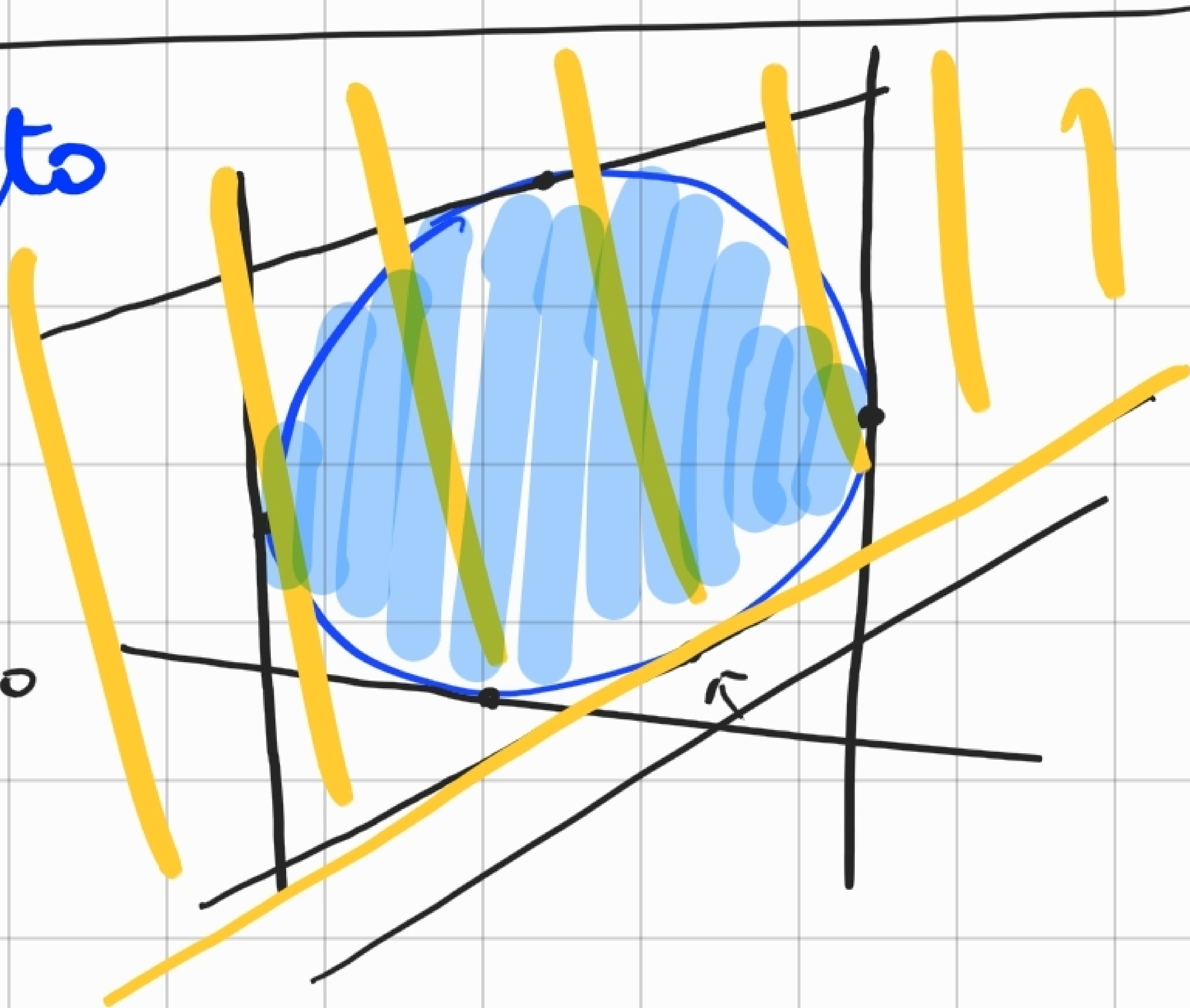


Così come se f è crescente il #{discontinuità} ≤ #ℕ

così se f convessa

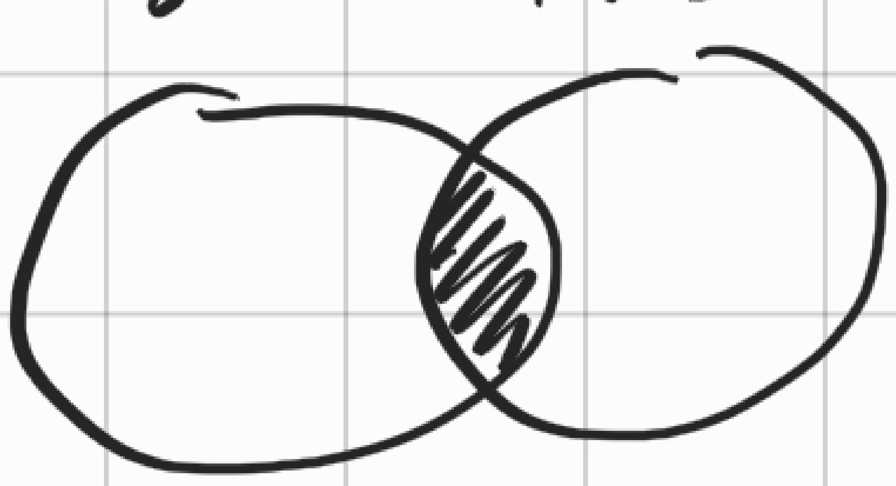
#{non è derivabile} ≤ #ℕ.

Rette di supporto



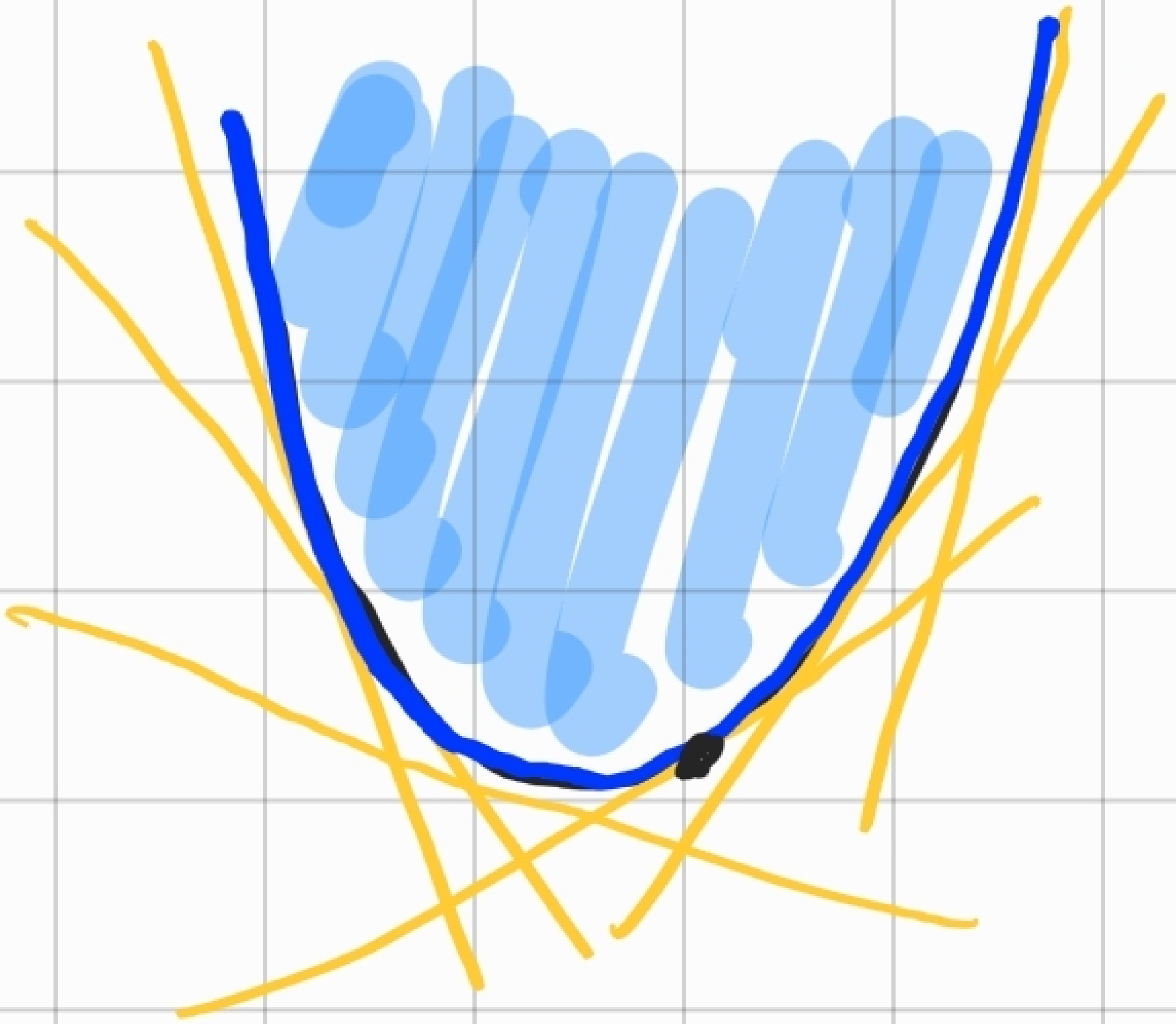
Esercizio 1

Se A, B sono insiemi convessi ⇒ $A \cap B$ è convesso



A_i convessi $\qquad \bigcap_i A_i$ è convesso

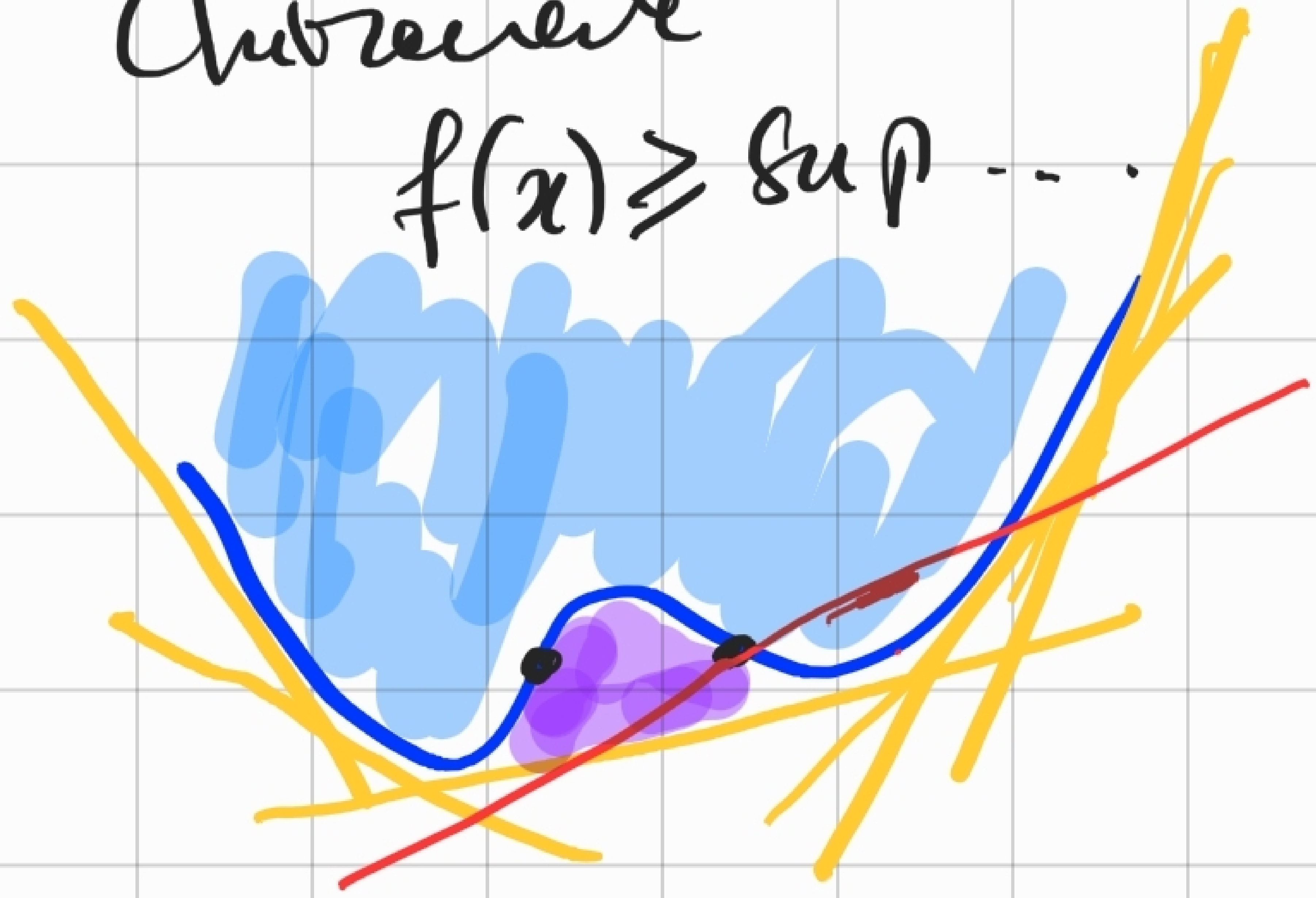
Se f è convessa $\quad f(x) = \sup_L \{L(x)\}$



$L(x)$ lineare $\leq f(x)$.

Chiaramente

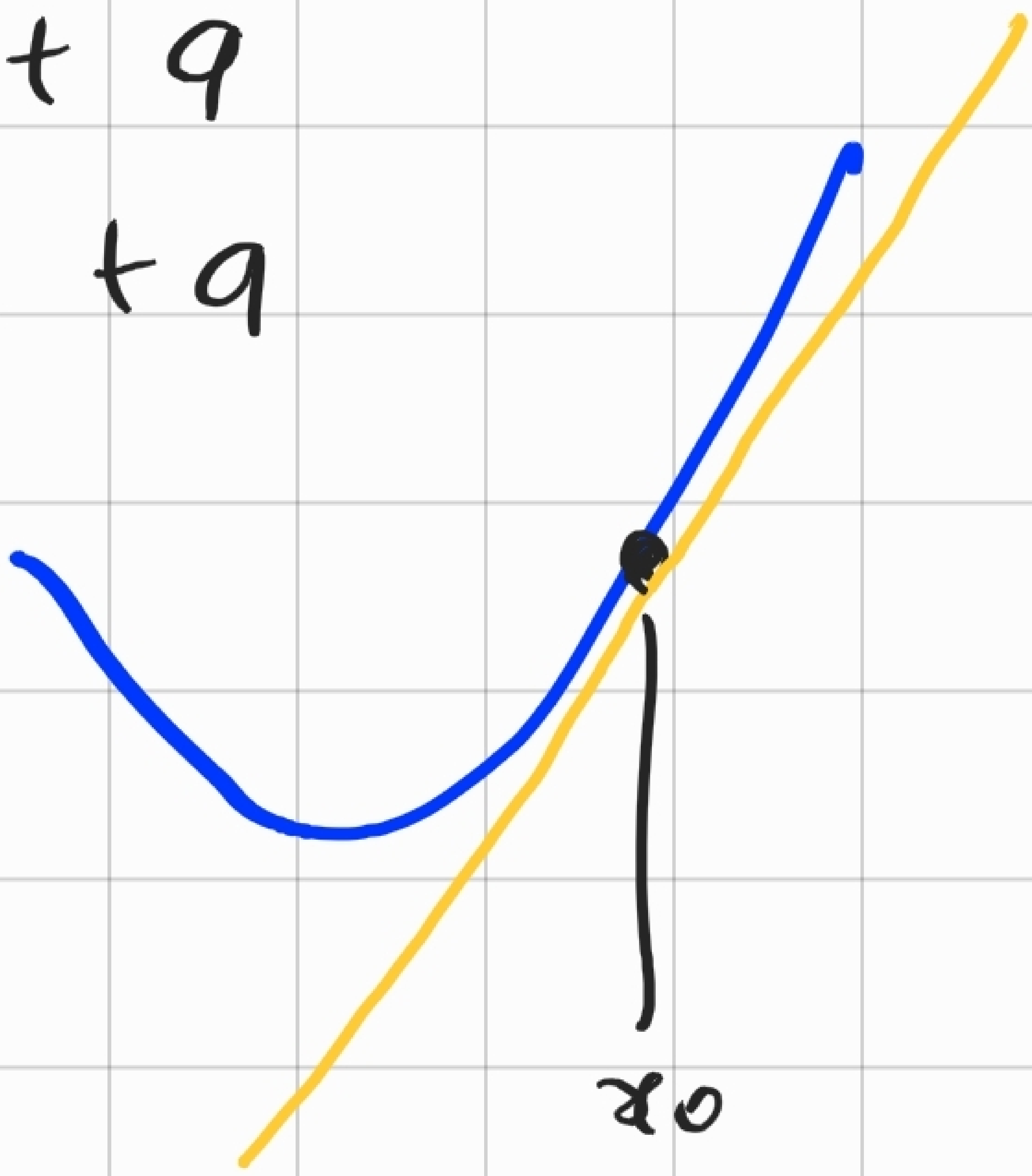
$f(x) \geq \sup \dots$



Teorema

Se f convessa $\forall x_0$ esiste una retta di supporto: ovvero $\exists\, m, q$ tali che

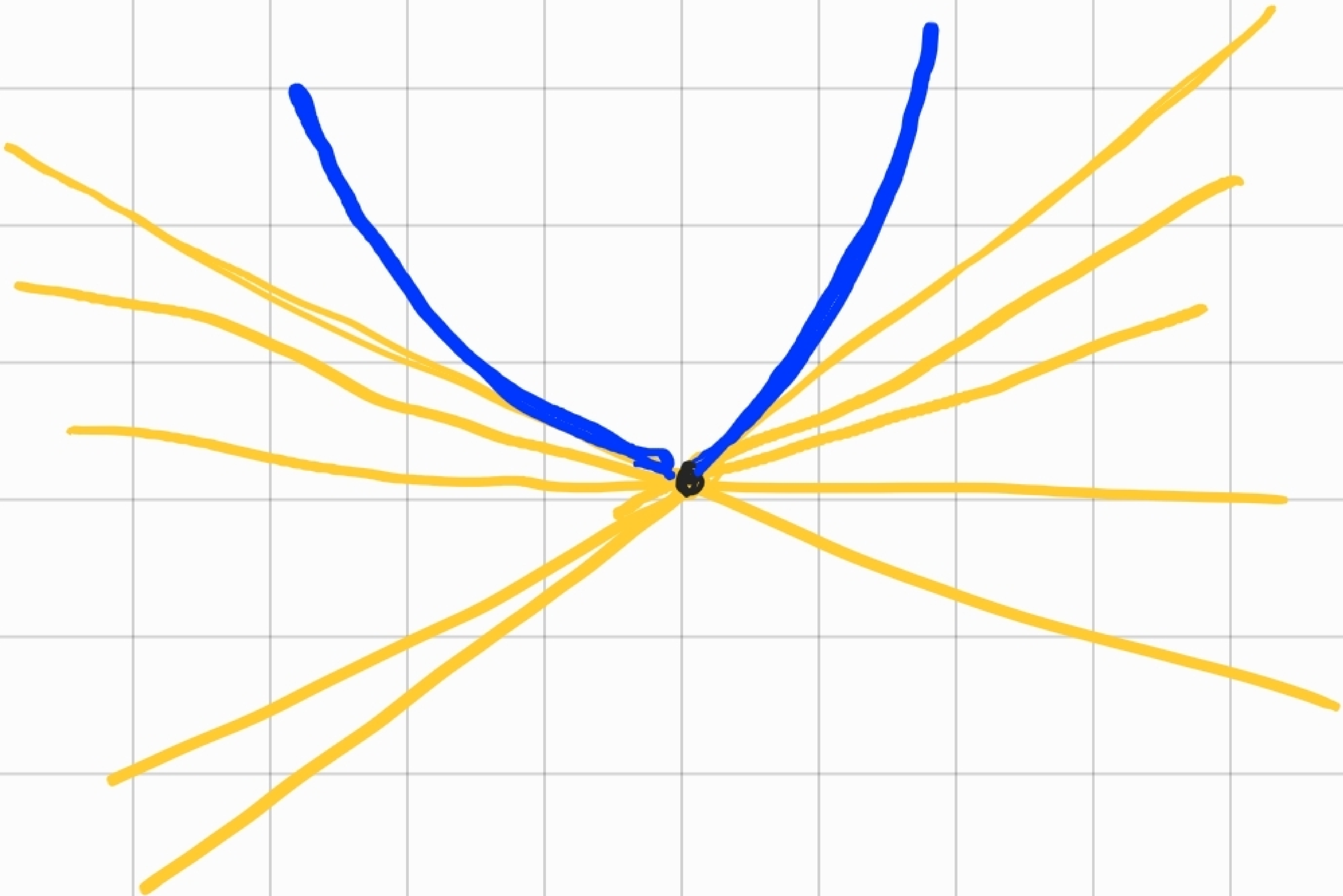
$$f(x) \geq mx + q$$
$$f(x_0) = m x_0 + q$$



dim

Se $f'(x_0)$ esiste allora $mx+q$ è la retta tangente

Se $f'(x_0)$ non esiste esistono derivata destra e sinistra:



ho infinite rette di supporto con ogni pendenza $m \in [m^-, m^+]$

Teorema (disuguaglianza di Jensen)

Se $f: I \to \mathbb{R}$ è convessa, $x_1, \dots, x_n \in I$

e se $\lambda_1, \dots, \lambda_n \geq 0$, $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$

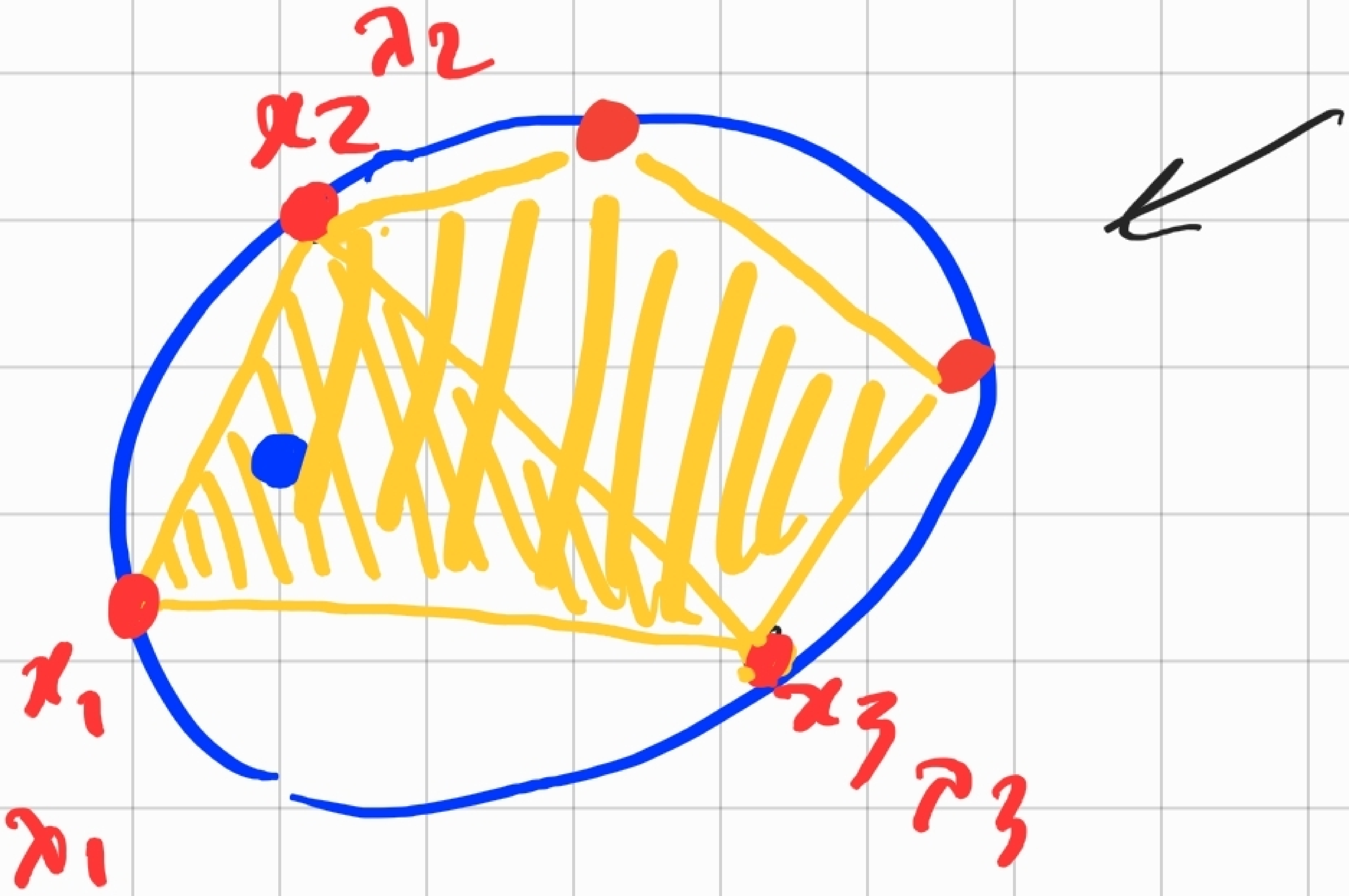
$$f\left(\boxed{\sum_{k=1}^{n} \lambda_k x_k}\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k)$$

Oss se $n=2$ questa è la definizione:

$\lambda_1 + \lambda_2 = 1$ $\lambda_2 = 1 - \lambda_1$

$t = \lambda_1 \in [0,1]$

$$f(t x_1 + (1-t) x_2) \le t f(x_1) + (1-t) f(x_2)$$



dim

$$\bar{x} = \sum_{k=1}^{n} \lambda_k x_k$$

$L(x) = mx + q$ tale che:

$$L(\bar{x}) = f(\bar{x})$$
$$L(x) \le f(x) \quad \forall x.$$

$$f(\bar{x}) = L(\bar{x}) = L\left(\sum_{k=1}^{n} \lambda_k x_k\right) = \sum_{k=1}^{n} \lambda_k L(x_k)$$

$$\leq \sum_{k=1}^{n} \lambda_k f(x_k) \qquad \square$$

Applicazione

$$\exp\left(\sum_{k=1}^{n} \frac{1}{n} x_k\right) \leq \sum_{k=1}^{n} \frac{1}{n} e^{x_k}$$

$$\exp\left(\sum_{k=1}^{n} \frac{1}{n} x_k\right) = \prod_{k=1}^{n} \sqrt[n]{e^{x_k}} = \sqrt[n]{\prod_{k=1}^{n} e^{x_k}}$$

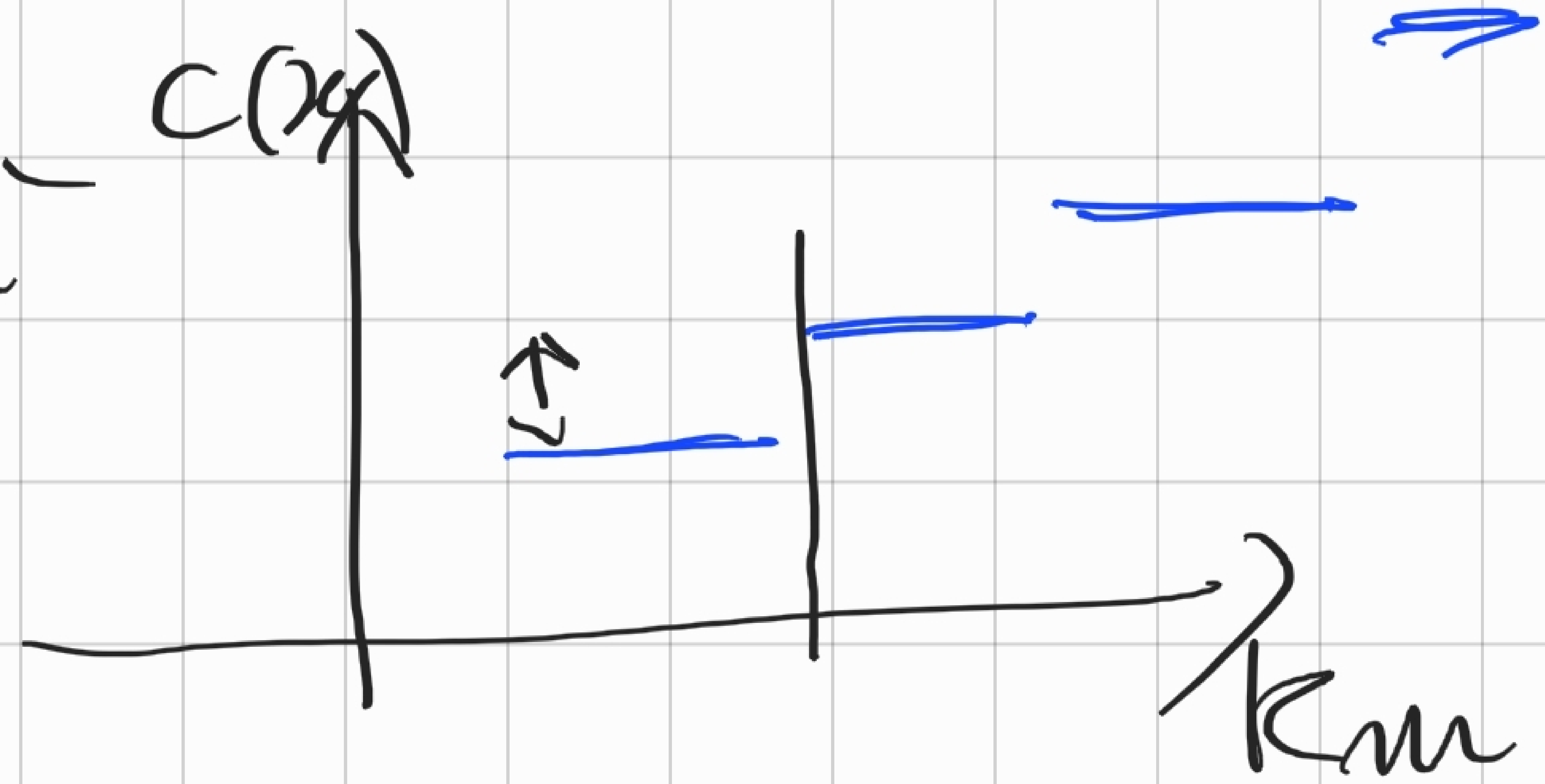
$$\sum_{k=1}^{n} \frac{1}{n} e^{x_k} = \frac{1}{n} \sum_{k=1}^{n} e^{x_k}$$

$$x_k = \ln a_k$$

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \dots + a_n}{n}$$

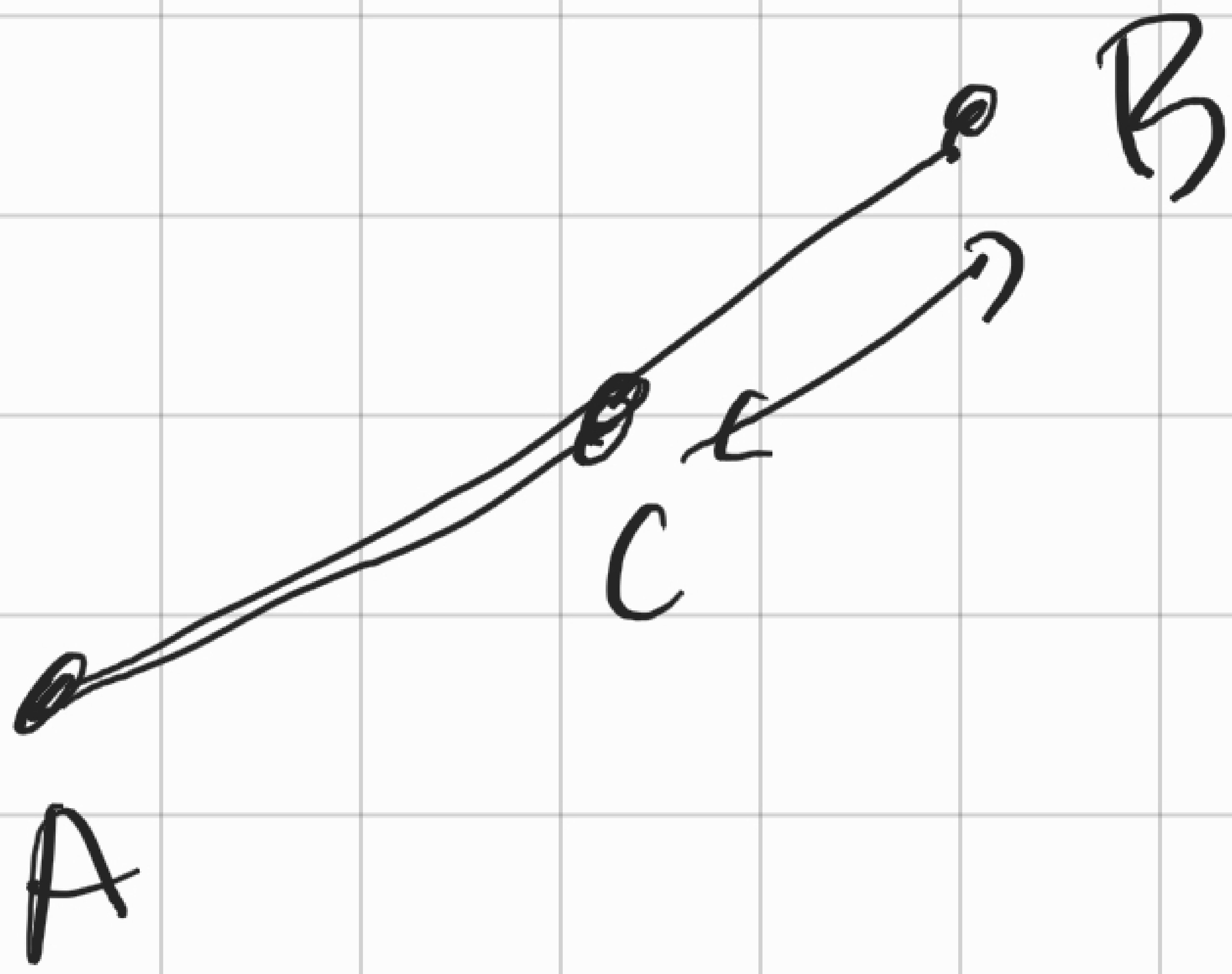
GM AM $\square$

Curiosità



$c(x)$ = costo di un biglietto per x chilometri.

non può essere convessa $\square$



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e^{z} $\qquad$ z^{w} ?

$$\boxed{1 = 3^{\frac{1}{x}}}$$

$$1^x = 3$$

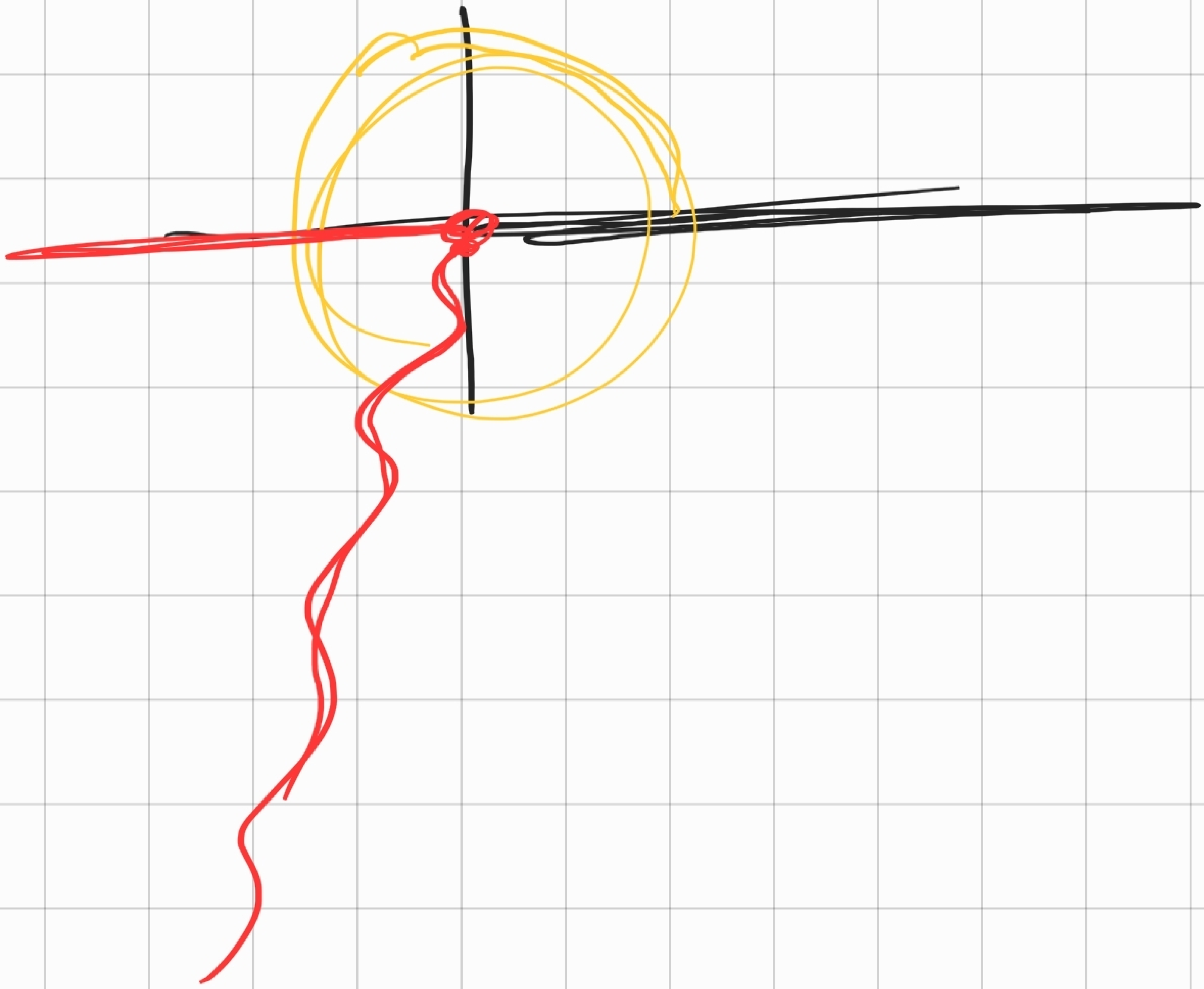
$$3^z = e^{z \underbrace{\ln 3}}$$

e^z $\qquad x = \ln 3$

$$e^{x+2\pi i} = e^x \cdot \overset{1}{\overset{\parallel}{e^{2\pi i}}}$$

$$1^z = e^{z \ln 1}$$

$$2\pi i$$



$1^n = 1$

$1^x = 1$

$\left(1+\frac{1}{n}\right)^n \leftarrow$

1^n

$$f(x) = \sin x = \underbrace{x - \frac{x^3}{6}}_{P(x)} + \frac{f^{v}(y)}{5!} x^5$$

$$\left(f^{v}(x) = \cos x\right)$$

$$\left| \sin(0,3) - P(0,3) \right| = \left| \frac{\cos(y)}{5!} (0,3)^5 \right| \quad \text{con } y \in (0, 0{,}3)$$

$$|\cos(y)| \le 1$$

$$\le \left(\frac{3}{10}\right)^5 \frac{1}{5!} = \frac{3\cdot 3\cdot 3\cdot 3\cdot \cancel{3}}{10^5\cdot 5\cdot 4\cdot \cancel{3}\cdot 2}$$

$$= \frac{81}{40} 10^{-5} = \varepsilon \qquad \frac{81}{40} = 2{,}$$

$$10^{-5} < 2\cdot 10^{-5} \le \varepsilon \le 2.1\cdot 10^{-5} < 10^{-4}$$

$$\square$$

$$\left| \sin(0,3) - P_4(0,3) \right| \le (0,3)$$

$$\left| \sin(0,3) - P_5(0,3) \right|$$

$$P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$$

$$\boxed{P_5(0,3) = \frac{3}{10} - \frac{3^3}{6\cdot 10^3} + \frac{3^5}{5!\cdot 10^5}}$$

$$=$$

$$\sin x = P_5(x) + \frac{f^{VI}(y)}{6!}x^6$$

$$P_4(0.3) = 0.2955$$

$$P_5(0,3) = 0,29552025$$

$$\sin(0,3) = 0,2955 \pm 10^{-5}$$

$$= 0,29552025 \pm 10^{-6}$$