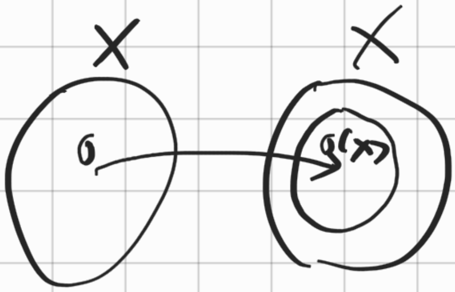
ANALISI MATEMATICA B
LEZIONE 8 — 6. 10. 2021

NUMERI NATURALI

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Assioma di infinito $\exists X : \exists \sigma : X \to X$

σ iniettiva ma non suriettiva.



$X \to \sigma(X) \subset X$

biunivoca $\neq$

$\mathbb{N}$ è un insieme con queste proprietà:
(assiomi di Peano)

$\exists\, 0 \in \mathbb{N}$, $\exists\, \sigma : \mathbb{N} \to \mathbb{N}$ tali che:

(i) $\forall n \in \mathbb{N} : \sigma(n) \neq 0$ (σ non è suriettiva)

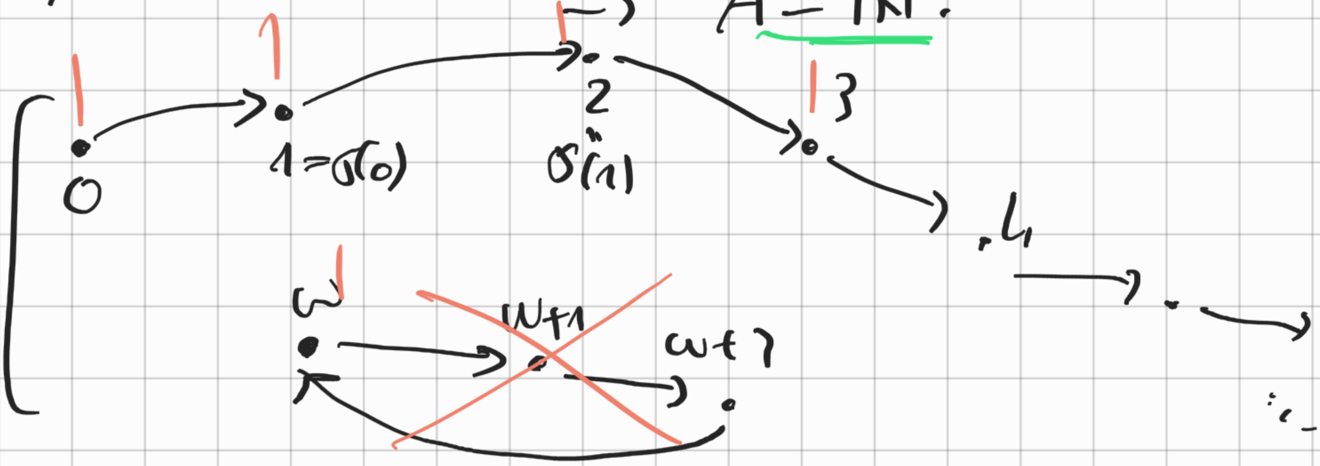
$(\forall n : n \in \mathbb{N} : \sigma(n) \neq 0)$

(ii) $\sigma(n) = \sigma(m) \Rightarrow n = m$ (σ iniettiva)
$(\forall n, m \in \mathbb{N})$

A induttivo

(iii) $\forall A \subseteq \mathbb{N}$: $(0 \in A,\ \forall n : n \in A \Rightarrow \sigma(n) \in A)$

$\Rightarrow A = \mathbb{N}$.



Idea: $\sigma(n) = n+1$

__Principio di induzione__ Sia $P(n)$ un predicato $(n \in \mathbb{N})$

Se

(i) $P(0)$ è valida

(ii) $\forall n \in \mathbb{N}: \quad P(n) \Rightarrow P(\sigma(n))$

Allora $\forall n : P(n)$.

__dim__ $A = \{ n \in \mathbb{N} : P(n) \}$

(i) $0 \in A$ (ii) $n \in A \Rightarrow \sigma(n) \in A$

$\Rightarrow A$ è induttivo

$\Rightarrow A = \mathbb{N}$

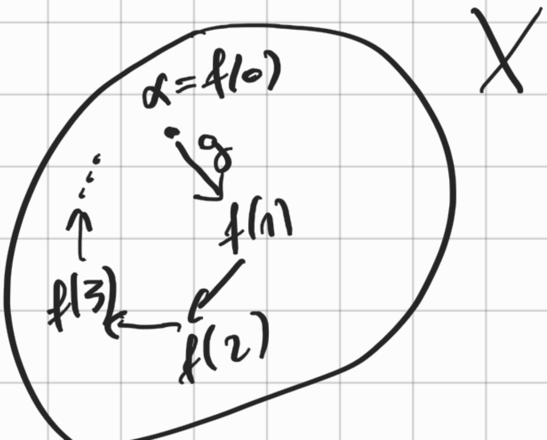
$\Rightarrow \forall n \in \mathbb{N} : P(n)$.

__Definizioni per induzione:__

Posso definire $f : \mathbb{N} \to X$ $\left\{ \begin{array}{l} \text{data } g : X \to X \\ \text{dato } \alpha \in X \end{array} \right.$

in modo unico affinché valga:

$$\begin{cases} f(0) = \alpha \\ f(\underbrace{\sigma(n)}_{n+1}) = g(f(n)) \end{cases}$$



Fissato α $\quad f(n) = $ "$\underbrace{g(\cdots(g(g(\alpha))))}_{n \text{ volte}}$" $\doteq g^n(\alpha)$

$= \underbrace{g \circ g \cdots \circ g}_{n}(\alpha)$ iterata n-esima.

$$\begin{cases} g^0(\alpha) = \alpha \\ g^{n+1}(\alpha) = g(g^n(\alpha)). \end{cases}$$

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$$\left[\begin{array}{l} n+0 = n \\ n+1 = \sigma(n) \\ \\ n+k = \sigma^k(n) \end{array}\right.$$

definisco la somma.

$$\left[\begin{array}{l} n \cdot 0 = 0 \\ n \cdot 1 = n \\ n \cdot 2 = n + n \\ \vdots \end{array}\right.$$

$$\begin{cases} n \cdot 0 = 0 \\ n \cdot (k+1) = n \cdot k + n \end{cases}$$

definisco la moltiplicazione

$$\begin{cases} n^0 = 1 \\ n^{k+1} = n^k \cdot n \end{cases}$$

elevamento a potenza.

[BTW: $\quad 0^0 = 1$]

Fattoriale: $\quad n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$$\rightarrow \begin{cases} 0! = 1 \\ \underbrace{(n+1)! = n! \cdot (n+1)}_{\text{riconoscenza}} \end{cases}$$

$1! = (0+1)!$
$= 0! \cdot 1 = 1 \cdot 1$
$= 1.$

$0! = 1$

$1! = 0! \cdot 1 = 1 \cdot 1 = 1$

$2! = 1! \cdot 2 = 1 \cdot 2 = 2$

$3! = 2! \cdot 3 = 1 \cdot 2 \cdot 3 = 6$

$4! = 3! \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$\vdots$

$n \geq m \iff \exists k \in \mathbb{N} : n = m + k.$

Esercizio $\quad 2^n \geq n^2 \quad$ FALSO perché

$2^{n+1} := 2 \cdot 2^n$

n	2^n	n^2
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25
6	64	36
$\vdots$		

$2^3 < 3^2$, $\quad 2^3 = 8$, $\quad 3^2 = 9$

Esercizio $\quad P(n): \; 1 + 2^n \geq n^2$

Per induzione $\quad$ (i) $P(0)$? $\quad 1 + 2^0 \overset{?}{\geq} 0^2$

$2 \geq 0$ SI!

(ii) $P(n) \overset{?}{\Longrightarrow} P(n+1)$

$$\underbrace{1+2^n \geq n^2}_{} \overset{?}{\Longrightarrow} 1+2^{n+1} \geq (n+1)^2$$

Supponiamo (per ipotesi induttiva) che $1+2^n \overset{!}{\geq} n^2$.

Devo dimostrare che vale: $1+2^{n+1} \overset{?}{\geq} (n+1)^2$

$$1+2^{n+1} = 1+2\cdot 2^n$$

$$\underline{(n+1)^2} = (n+1)(n+1) = (n+1)\cdot n + (n+1)\cdot 1$$

$$= n^2+n+n+1 = \underline{n^2+2n+1}.$$

$$1+2\cdot 2^n \overset{?}{\geq} n^2+2n+1$$

$$2^n \overset{!}{\geq} n^2-1$$

$$1+2\cdot 2^n \overset{!}{\geq} \underbrace{1+2\cdot(n^2-1) \overset{?}{\geq} n^2+2n+1}_{}$$

$$1+2n^2-2 \overset{?}{\geq} n^2+2n+1$$

$$2n^2 \overset{?}{\geq} n^2+2n+2$$

$$\underbrace{n^2 \overset{?}{\geq} 2n+2}_{}$$

Se $n \geq 3$ $\quad n^2 \geq 3 \cdot n \geq 2n + n \geq 2n+3$
$> 2n+2.$

Se $n \geq 3$: $\quad P(n) \Rightarrow P(n+1)$
Se vale $P(3)$
$\Rightarrow P(n)$ $\forall n \geq 3.$

$P(0), P(1), P(2)$ sono anch'esse vere

$\Rightarrow \quad \forall n : P(n).$

Morolmente ho dimostrato:

$$1 + 2^{n+3} \geq (n+3)^2 \quad \forall n \in \mathbb{N}$$

$$\boxed{2^n > n^2 \qquad \forall n \geq 5}$$

$$f : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$
$$n \longmapsto (k \longmapsto k+n)$$

$$\begin{cases} +(n)(0) = n \\ +(n)(k+1) = \sigma(+(n(k))) \end{cases}$$

$$a+b = (+(a))(b)$$