ANALISI MATEMATICA B
LEZIONE 22 - 10.11.2021

$$\frac{\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)^2 - \frac{1+\frac{1}{x^2}}{1-\frac{1}{x}}}{\frac{x}{x-1} - \frac{x}{x+1}} \qquad x \to +\infty$$

$$= \frac{\left(\frac{x+1}{x-1}\right)^2 - \frac{x^2+1}{x^2-x}}{\frac{\cancel{x^2}+x-\cancel{x^2}+x}{x^2-1}} = \frac{\frac{(x+1)^2}{(x-1)^2} - \frac{x^2+1}{x(x-1)}}{\frac{2x}{x^2-1}} =$$

$$= \frac{\frac{x\cdot(x+1)^2-(x^2+1)\cdot(x-1)}{(x-1)^{\cancel{2}}x}}{\frac{2x}{\cancel{(x-1)}(x+1)}} = \frac{(x+1)\cdot[x(x^2+2x+1)-(x^2+1)(x-1)]}{2x\,(x-1)\cdot x}$$

$$= \frac{(x+1)[\cancel{x^3}+2x^2+\cancel{x}-\cancel{x^3}-\cancel{x}+x^2+1]}{2x^2(x-1)}$$

$$= \frac{(x+1)\,[3x^2+1]}{2x^2(x-1)} = \frac{\cancel{x}\cdot\left(1+\cancel{\frac{1}{x}}\right)\cancel{x^2}\left[3+\cancel{\frac{1}{x^2}}\right]}{2\cancel{x^2}\cdot\cancel{x}\left(1-\cancel{\frac{1}{x}}\right)} \qquad x \to +\infty$$

$$\longrightarrow \frac{3}{2}$$

$$\boxed{e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n}$$

$$\log x = \ln x$$

$$\ln x = \log_e x$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$a^x = e^{x \ln a}$$

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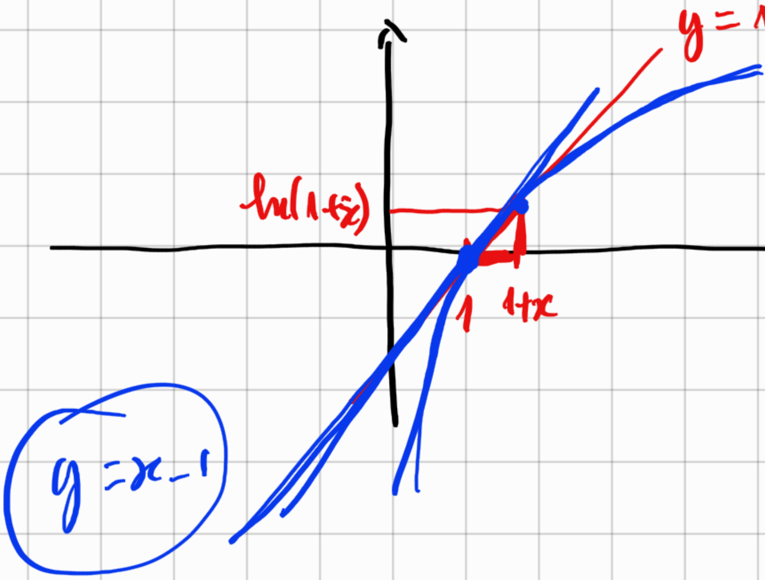
(1) $$\boxed{\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e}$$

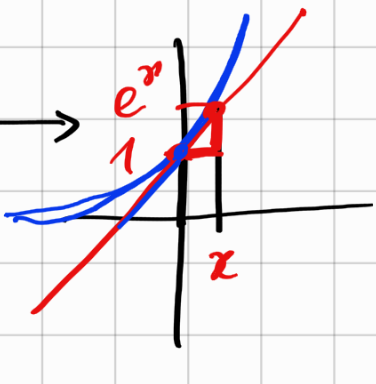
(3) $$\boxed{\lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n = e^x}$$

(2) $$\boxed{\lim_{x \to 0} \boxed{\frac{\ln(1+x)}{x}} = 1}$$

(4) $$\boxed{\lim_{x \to 0} \boxed{\frac{e^x - 1}{x}} = 1}$$

$y = m(x-1)$ $\quad m = \frac{\ln(1+x)}{x}$





(1) $\lim\limits_{\substack{x\to+\infty \\ x\in\mathbb{R}}} \left(1+\frac{1}{x}\right)^x = \lim\limits_{\substack{n\to+\infty \\ n\in\mathbb{N}}} \underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} = e$

$n = \lfloor x \rfloor$

$$\lfloor x\rfloor \le x < \lfloor x\rfloor + 1$$

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}$$

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \frac{\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}}{\left(1+\frac{1}{\lfloor x\rfloor+1}\right)} \to \frac{e}{1} = e$$

with $\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1} \to e$ and $\left(1+\frac{1}{\lfloor x\rfloor+1}\right) \to 1$

$$\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} = \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor} \cdot \left(1+\overset{0}{\frac{1}{\lfloor x\rfloor}}\right) \to e \cdot 1 = e$$

$\left(1+\frac{1}{x}\right)^x \to e$ (2 carabinieri)

$$\boxed{\lim_{x\to 0^+} \underbrace{(1+x)^{\frac{1}{x}}}_{1^+} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e}$$

① $\displaystyle\lim_{y\to 0^-}(1+y)^{\frac{1}{y}} = \lim_{x\to 0^+}(1-x)^{-\frac{1}{x}}$

$\not\exists$

$y=-x$

se $x\to 0^-$

$y=-x\to 0^+$

$$(1-x)^{-\frac{1}{x}} = \left(\frac{1}{1-x}\right)^{\frac{1}{x}} = \left(1+\frac{x}{1-x}\right)^{\frac{1}{x}}$$

$-x+x$

$$= \left[\boxed{\left(1+\frac{x}{1-x}\right)^{\frac{1-x}{x}}}\right]^{\frac{1}{1-x}} = \boxed{\left[(1+t)^{\frac{1}{t}}\right]}^{\frac{1}{1-x}}$$

$\to e$ $\quad$ $\to e$ $\quad$ $\frac{1}{1-x}\to 1$

$t=\frac{x}{1-x}$ se $x\to 0^+$

$t\to 0^+$

$a(x)^{b(x)}$ $\begin{cases} a(x)\to e \\ b(x)\to 1\end{cases}$

allora $a(x)^{b(x)}\to e^1$.

$\quad = e^{b(x)\ln a(x)}$

$\emptyset$

DIMOSTRAZIONE SBAGLIATA.

idea $\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^{x} = \lim_{y \to +\infty} \left(1-\frac{1}{y}\right)^{-y}$

$y = -x$

$$= \left(\frac{y-1}{y}\right)^{-y}$$

$$= \lim_{y \to +\infty} \left(\frac{y}{y-1}\right)^{y} = \lim_{y \to +\infty} \left(1+\frac{1}{y-1}\right)^{y}$$

$$= \lim_{y \to +\infty} \underbrace{\boxed{\left(1+\frac{1}{y-1}\right)^{y-1}}}_{\to e} \cdot \underbrace{\left(1+\frac{1}{y-1}\right)}_{\to 1} = e \quad (1)$$

$\square$

(1) $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

$$\ln\left[(1+x)^{\frac{1}{x}}\right] \longrightarrow \ln e = 1$$

$$\| $$

$$\frac{1}{x} \ln(1+x) = \boxed{\frac{\ln(1+x)}{x}} \quad (2)$$

(3) $\lim_{n \to +\infty} \left(1+\frac{x}{n}\right)^{n} \overset{?}{=} e^{x}$

$$\left(1+\frac{x}{n}\right)^n = e^{n\ln\left(1+\frac{x}{n}\right)} \longrightarrow e^{x}$$

$$n\ln\left(1+\frac{x}{n}\right) = \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} =$$

$$= \overbrace{\boxed{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}}}^{\overset{1}{\uparrow}\ ②} \cdot x \longrightarrow x$$

x fissato

$$\lim_{n\to+\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}} = \lim_{y\to 0} \frac{\ln(1+y)}{y} \overset{②}{=} 1$$

$$y=\frac{x}{n} \qquad \begin{cases} n\to+\infty \\ y\to 0 \end{cases}$$

$$y=f(n) \qquad f(n)=\frac{x}{n}$$

$$g(y)=\frac{\ln(1+y)}{y} \to 1 \text{ per } y\to 0$$

$f(n)\to 0$ per $n\to+\infty$

④

$$\begin{cases} y = e^x - 1 \\ 1+y = e^x \\ x = \ln(1+y) \end{cases}$$

$$\frac{e^x - 1}{x} = \frac{y}{\ln(1+y)} = \frac{1}{\frac{\ln(1+y)}{y}} \longrightarrow \frac{1}{1} = 1$$

$$\frac{\ln(1+y)}{y} \xrightarrow{②} 1$$

Se $x \to 0$ $\quad y = e^x - 1 \to e^0 - 1 = 0$

\# $\displaystyle\lim_{n\to+\infty} \left(1 - \frac{n+1}{n!}\right)^{(n-1)!}$ $\quad$ "$1^{+\infty}$"

(per $n \to +\infty$ $\quad n! \to +\infty$ $\quad n! \geq n$)

$$\left[\frac{n+1}{n!} \leq \frac{n+1}{n(n-1)} = \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{n\cdot\cancel{n}\left(1-\frac{1}{n}\right)} \to 0 \quad n! \geq n(n-1)\right]$$ (se $n \geq 2$)

$$\left(1 - \frac{n+1}{n!}\right)^{(n-1)!} = e^{(n-1)!\cdot\ln\left(1-\frac{n+1}{n!}\right)}$$

$$\left[\quad \frac{\ln\left(1 - \frac{n+1}{n!}\right)}{-\frac{n+1}{n!}} \xrightarrow{②} e \quad\right]$$

$n \to +\infty$

$$(*)\quad (n-1)! \cdot \ln\left(1-\frac{n+1}{n!}\right) = (n-1)! \, \frac{\ln\left(1-\frac{n+1}{n!}\right)}{-\frac{n+1}{n!}} \cdot \left(-\frac{n+1}{n!}\right)$$

$\to 1$

$-1 \leftarrow$

$$= -\frac{\cancel{(n-1)!}}{n\cancel{!}}\,(n+1) = -\frac{n+1}{n} \to -1$$

$[\, n! = n \cdot (n-1)! \,]$

$$(*) \longrightarrow -1 \cdot 1 = -1$$

$$(\#)\quad \lim e^{(*)} = e^{-1} = \frac{1}{e} \qquad \square$$

Es $$\lim_{x \to 0^+} \left(1+x^2\right)^{\frac{1}{\sqrt{x}}} = 1$$

$$\left(1+x^2\right)^{\frac{1}{x^{1/2}}} = \left(1+x^2\right)^{x^{-1/2}}$$

$$\left(1+x^2\right)^{\frac{1}{\sqrt{x}}} = e^{\frac{1}{\sqrt{x}} \cdot \ln(1+x^2)} \longrightarrow e^0 = 1.$$

$$\frac{1}{\sqrt{x}} \cdot \frac{\ln(1+x^2)}{x^2} \cdot x^2 \longrightarrow 0$$

$0 \qquad 1$

$$\frac{x^2}{\sqrt{x}} = x\sqrt{x} \to 0$$

$$(1+x^2)^{\frac{1}{\sqrt{x}}} = \left[(1+x^2)^{\frac{1}{x^2}}\right]^{\frac{x^2}{\sqrt{x}}} \longrightarrow e^0$$

$e \qquad 0$

Lemma Se $f(x) \to a$ e $g(x) \to b$

$$f(x)^{g(x)} \to a^b$$

$a > 0$, $b < +\infty$

dim

$$f(x)^{g(x)} = e^{g(x) \cdot \ln g(x)} \quad \ldots\ldots \square$$