LEZIONE 45

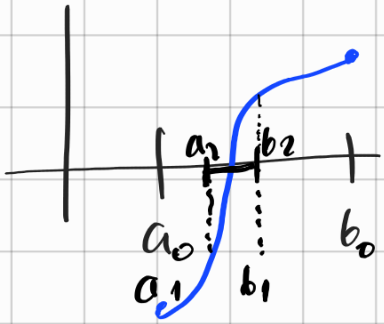
<u>Teorema degli zeri</u> $f:[a,b]\to\mathbb{R}$ continua

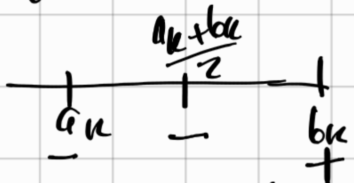
Se $f(a)\cdot f(b)\le 0$ allora $\exists\, x_0\in[a,b]$ tale che $f(x_0)=0$

<u>dim</u> algoritmo di bisezione.

$f(a)\le 0$, $f(b)\ge 0$ $\quad b>a$

$a_0=a$, $b_0=b$





$$[a_{k+1},b_{k+1}]=\begin{cases}\left[a_k,\frac{a_k+b_k}{2}\right] & \text{se } f\left(\frac{a_k+b_k}{2}\right)\ge 0\\ \left[\frac{a_k+b_k}{2},b_k\right] & \leftarrow \text{altrimenti}\end{cases}$$

$$\begin{cases} a_k \text{ è crescente},\ b_k \text{ decrescente},\ a_k\le b_k\\ b_k-a_k=\dfrac{b-a}{2^k} \qquad a_0\le a_k\le b_k\le b_0\\ f(a_k)\le 0,\ f(b_k)\ge 0\end{cases}$$

$a_k\to x_0\in[a,b]$

$$b_k=\underbrace{a_k}_{x_0}+\underbrace{(b_k-a_k)}_{\to 0}\longrightarrow x_0$$

f continua

$$\left.\begin{array}{l} \underset{\substack{\wedge\!\!\!| \\ 0}}{f(a_k)}\to f(x_0)\le 0\\ \underset{\substack{\vee\!\!\!| \\ 0}}{f(b_k)}\to f(x_0)\ge 0\end{array}\right\} f(x_0)=0$$

$\square$

Esempio $\quad x+x^3+x^7=2 \quad (*)$

$$f(x)=x^7+x^3+x-2$$

$$f'(x)=7x^6+3x^2+1\geq 1>0$$

f è strettamente crescente $\Rightarrow$ f iniettiva

$$f(0)=-2 \qquad f(1)=1$$

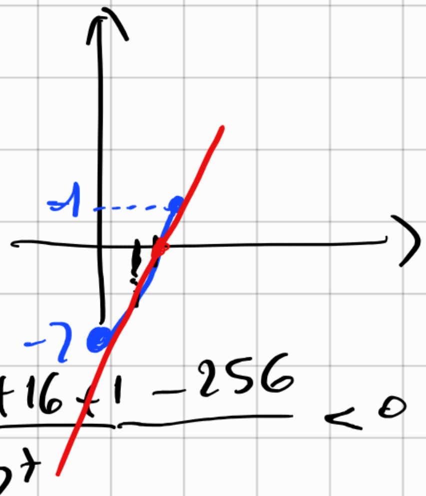
Teorema zeri $\Rightarrow \exists\, x_0\in(0,1)$ t.c. $f(x_0)=0$

x_0 risolve $(*)$

(siccome f è iniettiva $\Rightarrow$ la sol. è unica)

Come trovo x_0? $\qquad a_0=0 \qquad a_1=1$

$$m_0=\frac{1}{2}$$

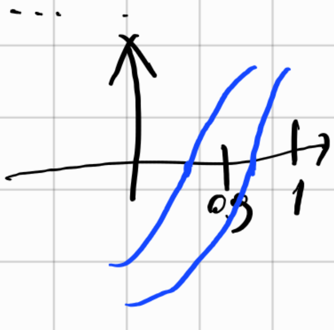


$$f\left(\frac{1}{2}\right)=\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^7}-2$$

$$=\frac{2^6+2^4+1-2^8}{2^7}=\frac{64+16+1-256}{2^7}<0$$

$a_1=\frac{1}{2}$, $b_1=1 \qquad m_1=\frac{3}{4} \qquad \frac{1}{2}\leq x_0\leq 1$

$$f\left(\frac{3}{4}\right)=\frac{3^7}{4^7}+\frac{3^3}{4^3}+\frac{3}{4}-2=\dots$$



dire se $x_0\geq 0.9$

$$f(0.9)$$

$$f\left(\frac{9}{10}\right) = \frac{3^{14}}{10^7} + \frac{3^6}{10^3} + \frac{3^7}{10} - 2$$

$$= \frac{3^{14} + 3^6 \cdot 10^4 + 3^7 \cdot 10^6 - 2 \cdot 10^7}{10^7}$$

$3^4 \sim 100 = 10^2$

$$\approx \frac{9 \cdot 10^6 + 9 \cdot 10^2 + 9 \cdot 10^6 - 2 \cdot 10^7}{10^7}$$

$$\approx 10^7 + 10^3 + 10^7 - 2 \cdot 10^7 \quad \approx 0 ?$$

Corollario (teorema dei valori intermedi) Se $f : I \to \mathbb{R}$ è continua, I intervallo

Allora $f(I)$ è un intervallo.

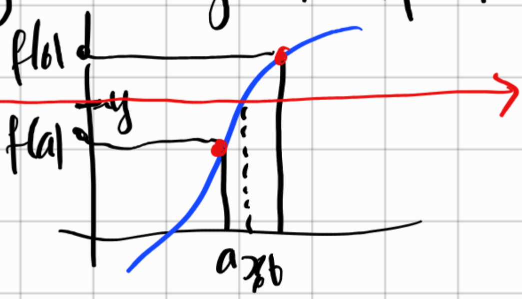
dim Se $a, b \in I$ $f(a), f(b) \in f(I)$

Se $f(a) < y < f(b)$ allora $y \in f(I)$

cioè $\exists\ x_0 \in I$ t.c. $f(x_0) = y$

$g(x) = f(x) - y$ $\quad g$ è continua

$g(a) = f(a) - y < 0 \qquad g(b) = f(b) - y > 0$



teo zeri $\exists\ x_0 \in [a,b] \cup [b,a]$

se $b > a$ $\quad$ se $b < a$

t.c. $g(x_0) = 0 \implies f(x_0) = y$

Es $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

è iniettiva perché strett. crescente

è surgettiva perché: $f\left(-\frac{\pi}{2}\right) = -1$

$$f\left(\frac{\pi}{2}\right) = 1$$

e per il teorema dei valori intermedi

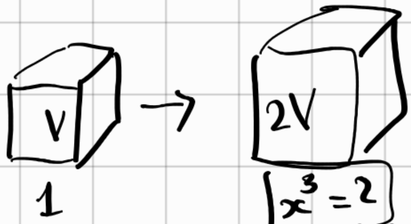
ogni $y \in [-1,1]$ è "assunto"

cioè f è suriettiva.

$\Rightarrow f$ bigettiva

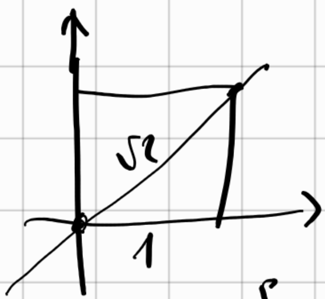
Es Risolvere $e^x = x^3$.

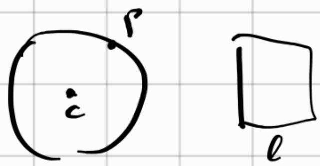
$$\boxed{x^7 + x^3 + x - 2 = 0}$$



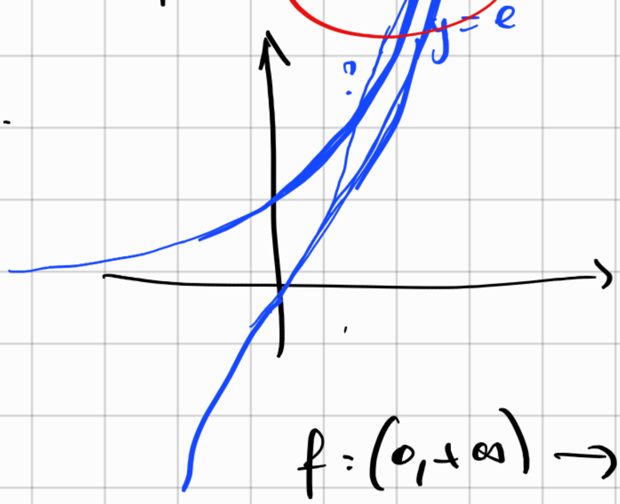
$\boxed{x^3 = 2}$

$x = \sqrt[3]{2}$





(*) $e^x = x^3$ $\qquad f(x) = e^x - x^3$



Se x è soluzione $x > 0$.

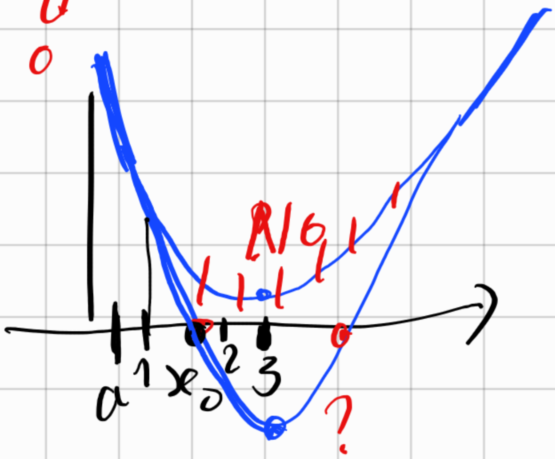
$x = \ln x^3 = 3\ln x$

$f(x) = x - 3\ln x$ $\qquad f : (0, +\infty) \to \mathbb{R}$

$\lim_{x \to 0^+} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x - 3\ln x)$

$= \lim_{x \to +\infty} x\left(1 - 3\underbrace{\frac{\ln x}{x}}_{0}\right) = +\infty$



$f'(x) = 1 - \frac{3}{x}$

x		3	
f'	$-$	0	$+$
f	$\searrow$	min	$\nearrow$

$f(3) = 3 - 3\ln 3 < 0$

$\ln 3 > \ln e = 1$

$$\boxed{\exists!\ x_0 \in (0,3) \text{ t.c. } f(x_0) = 0}$$

Infatti: $\lim_{x \to 0^+} f(x) = +\infty$

$\exists\, a < 3$ t.c. $f(a) > 0$

$f(3) < 0$ $\quad f'(x) < 0$ se $0 < x < 3$

$\Downarrow$

f strett. decrescente

su $(0,3]$

f è iniettiva su $(0,3]$

$f : [a,3] \to \mathbb{R}$ soddisfa le ipotesi

del teorema degli zeri: f continua

$f(a) > 0$, $f(3) < 0$.

$\exists\, x_0 \in [a,3) \subseteq (0,3]$ t.c. $f(x_0) = 0$

è unico perché f è iniettiva.

$\exists!\, x_1 \in [3, +\infty)$ t.c. $f(x_1) = 0$.

L'eq $(*)$ ha 2 soluzioni x_0, x_1

t.c. $x_0 < 3$, $x_1 > 3$.

Calcolare $\lfloor x_0 \rfloor$ e $\lfloor x_1 \rfloor$

$f(1) = 1 > 0 \quad \Rightarrow \quad x_0 > 1$

$f(2) = 2 - 3\ln 2 < 0$

$3\ln 2 > 2$

$\ln 8 > 2 \qquad\qquad 8 > e^2$ sí

$\Rightarrow \quad x_0 < 2$

$\lfloor x_0 \rfloor = 1.$