ANALISI MATEMATICA B
LEZIONE 57 - 18.2.2022

10.12 $\quad$ P polinomio $\deg P \leq 3$

$$\begin{cases} P(1) = 0, & P'(1) = 2 \\ P''(1) = 0 & P'''(1) = 1. \end{cases}$$

$$P(x) = P(x_0) + P'(x_0)(x-x_0) + \frac{P''(x_0)}{2}(x-x_0)^2 + \frac{P'''(x_0)}{6}(x-x_0)^3$$
$$+ o\left((x-x_0)^3\right)$$

$x_0 = 1$

$$P(x) = \underbrace{0 + 2(x-1) + 0 + \frac{1}{6}(x-1)^3}\ \cancel{+o((x-1)^3)}$$
$$= 2(x-1) + \frac{1}{6}(x-1)^3.$$

$$P(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3.$$

$0 = P(1) = a_0$

$\vdots$

$P'(1) = a_1$

$\vdots$

$f(x) = 6e^x - e\cdot x^3$ è strett. convessa.

$$f'(x) = 6e^x - 3ex^2$$
$$f''(x) = 6e^x - 6ex = 6(e^x - e\cdot x) \overset{?}{>} 0$$

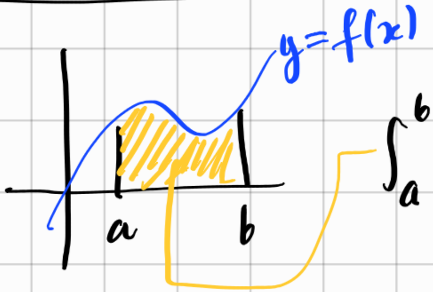
$$f'''(x) = 6e^x - 6e = 6(e^x - 1) > 0 \iff x > 0$$

		0	
f'''	$-$	0	$+$
f''	$\searrow$	min	$\nearrow$

$f''(0) = 6$

$f''(x) \geq 6 > 0.$

INTEGRALI



$\int_a^b f(x)dx$

Che cos'è l'area

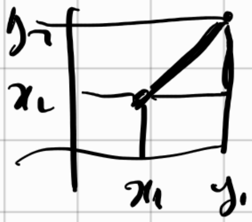
MISURA di PEANO-JORDAN $\longleftrightarrow$ INTEGRALE di RIEMANN

$\mathbb{R}^2$

Sia $E \subseteq \mathbb{R}^2$

$$d(\underline{x},\underline{y}) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

$\underline{x}, \underline{y} \in \mathbb{R}^n$

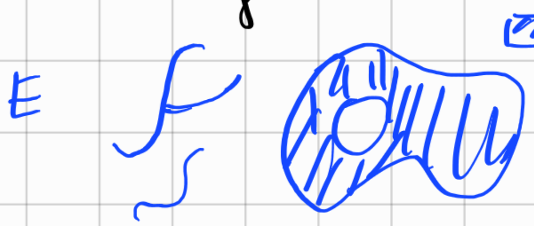


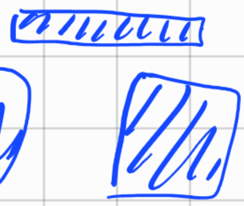
$$d(f,g) = \sqrt{\int_a^b |f(x)-g(x)|^2 dx}$$

Vogliamo definire

$m(E) \in \mathbb{R}$

Quali proprietà vogliamo che abbia m?





se $E\cap F=\emptyset$: $\quad m(E\cup F) = m(E)+m(F)$ (additività)

$\big(m(\lambda\cdot E) = \lambda^2\, m(E)$ (omogeneità)$\big)$

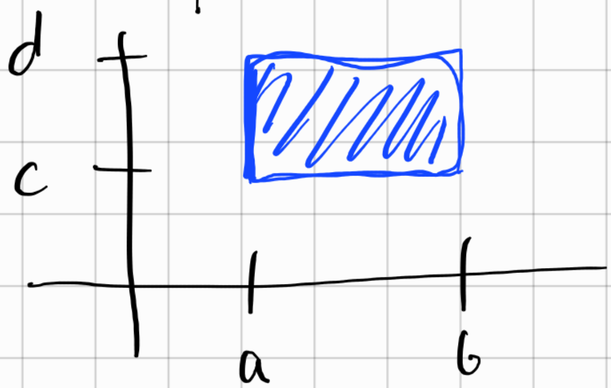
$\big(m(E\cup F) = m(E)+m(F)-m(E\cap F)$ (additività generale)$\big)$

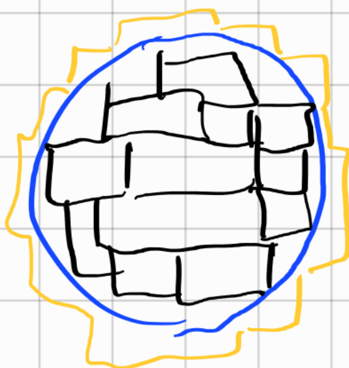
$E\subseteq\mathbb{R}$: $\quad m(E)=0 \quad \left(\begin{array}{l} m(E+v)=m(E) \\ m(R\cdot E)=m(E) \end{array}\right)$

(normalizzazione)

$$m\big([a,b]\times[c,d]\big) = |b-a|\cdot|d-c|$$

↑





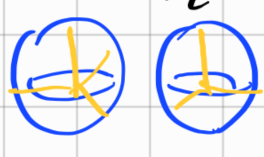
▷ monotonia: $\quad E\subseteq F \Rightarrow m(E)\le m(F)$

[Dehn e Banach-Tarski: opposti paradossi]

Banach-Tarski: Paradosso

$B_1\subseteq\mathbb{R}^3$ $\quad\longrightarrow 2$ $\qquad B_1 = A_1\cup A_2\cup A_3\cup A_4\cup A_5$

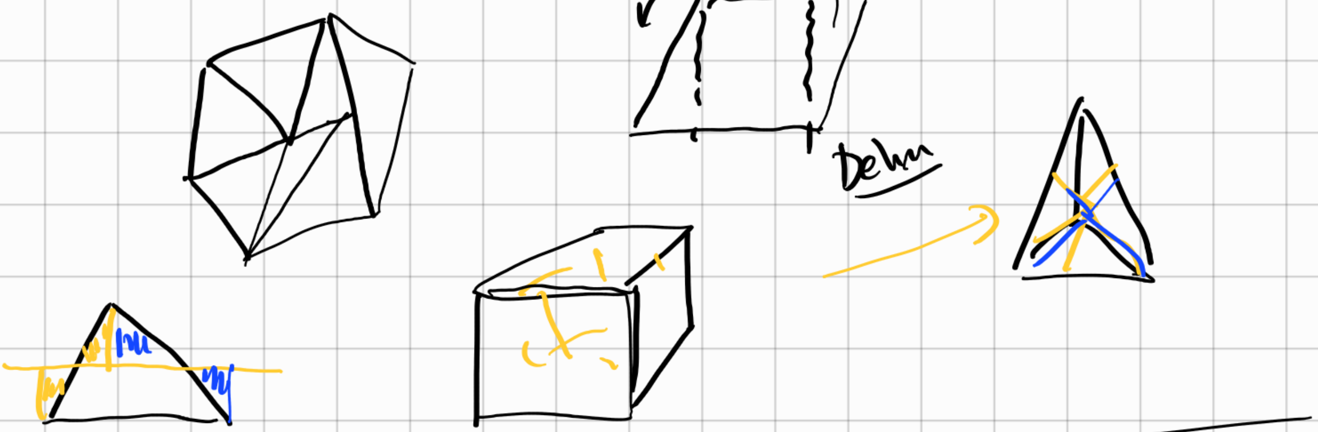
$A_i\cap A_j=\emptyset$ se $i\neq j$



$\exists\ \theta_1 \dots \theta_5$ isometrie di $\mathbb{R}^3$ $\quad B_i = \theta_i(A_i)$

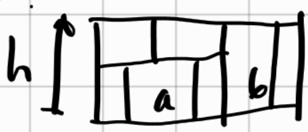
$B_i \cap B_j \neq \emptyset$ se $i \neq j$ $\qquad B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 = B_1 \cup (B_1 + (2,0,0))$

! Esistono insiemi non misurabili



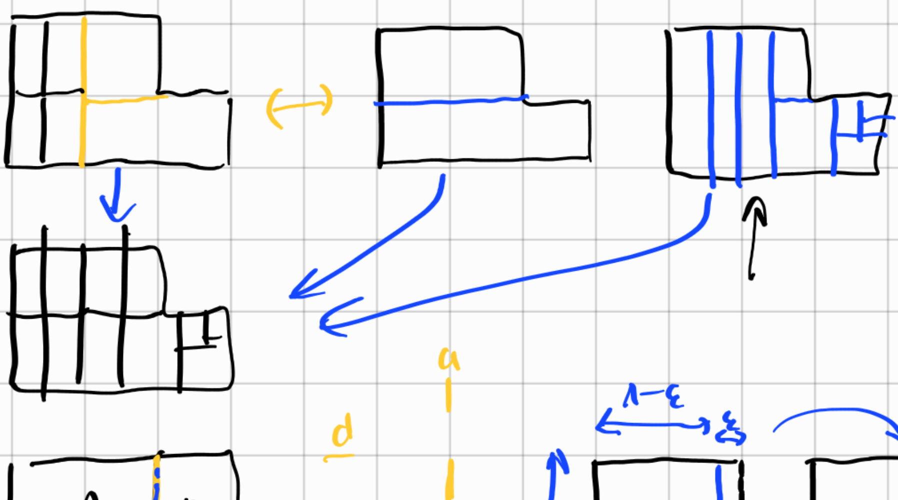
1. additività.
2. monotonia.

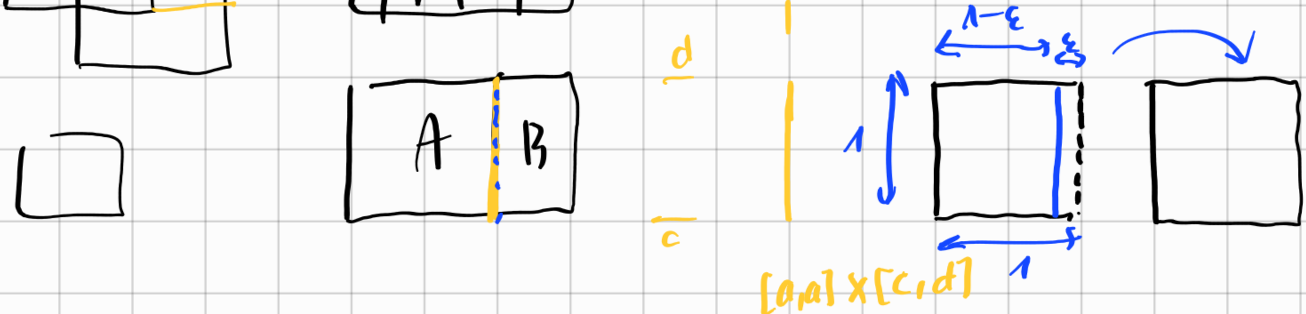
$\to$ 3. area del rettangolo cartesiano.

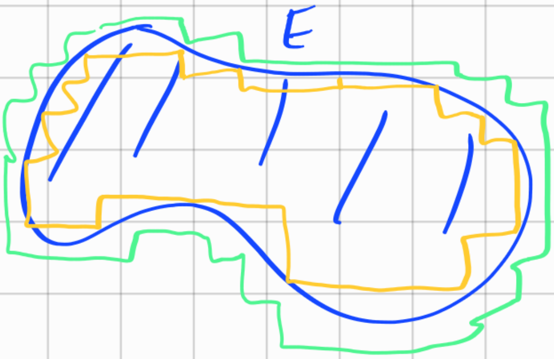


$(a+b)\cdot h = ah + b\cdot h$

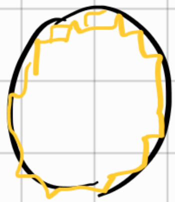
polirettangoli

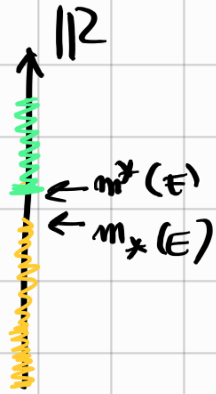






$E \subseteq \mathbb{R}^2$





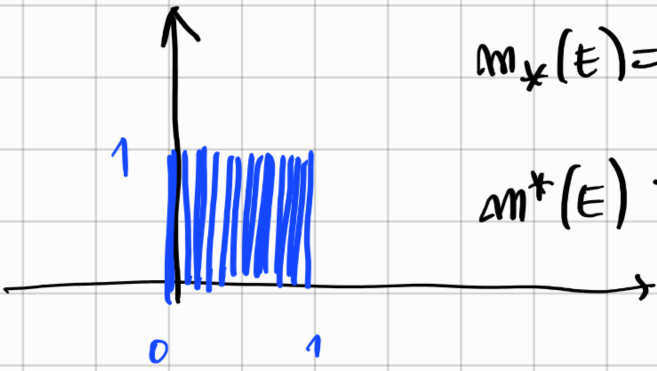
$$m_*(E) = \sup\{ m(E_*) : E_* \text{ è un plurirettangolo } E_* \subseteq E \}$$

$$m^*(E) = \inf\{ m(E^*) : E^* \text{ plurirettangolo } E^* \supseteq E \}$$

Se $m_*(E) = m^*(E)$ diremo che E è Peano-Jordan misurabile e poniamo $m(E) = m^*(E) = m_*(E)$.

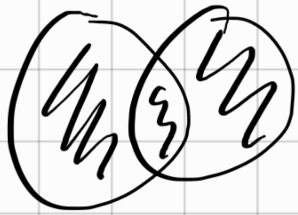
Esempio

$$E = \{ (x,y) \in [0,1] \times [0,1] : x \in \mathbb{Q} \}$$



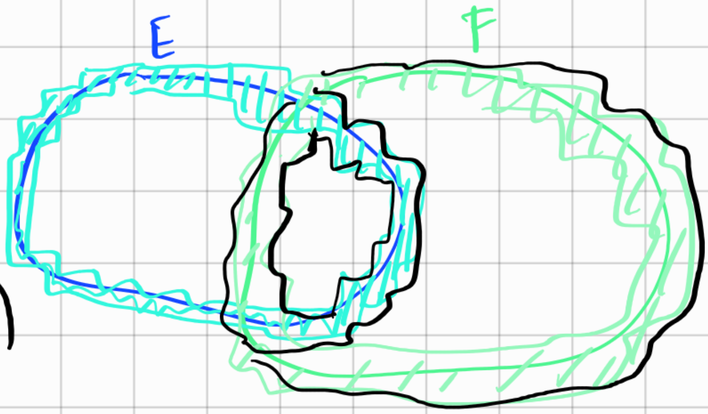
$m_*(E) = 0$

$m^*(E) = 1$



E non è Peano-Jordan misurabile.

m è additiva?



$$m(E \cup F) = m(E \setminus F) + m(E \cap F) + m(F \setminus E)$$

Se E è misurabile $\forall \varepsilon > 0$ esistono $E_* \subseteq E \subseteq E^*$ (pluri-rettangoli)

tali che $m(E^*) - m(E_*) < \varepsilon$.

$$E_* \subseteq E \subseteq E^*$$

$$F_* \subseteq F \subseteq F^*$$

$$E_* \cap F_* \subseteq E \cap F \subseteq E^* \cap F^*$$

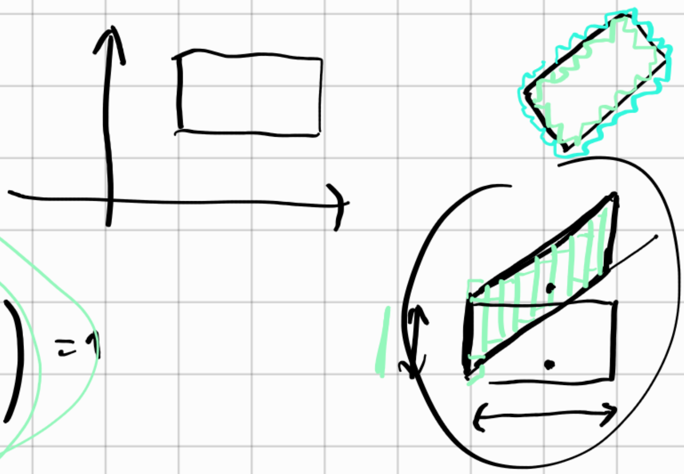
$$E_* \cup F_* \subseteq E \cup F \subseteq E^* \cup F^*$$

$$E_* \setminus F^* \subseteq E \setminus F \subseteq E^* \setminus F_*$$

$$F_* \setminus E^* \subseteq F \setminus E \subseteq F^* \setminus E_*$$

$$m(E_* \cup F_*) = m(E_* \cap F_*) + m(E_* \setminus F^*) + m(F_* \setminus E^*)$$

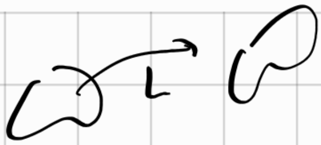
$$\pm 2\varepsilon$$



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y + mx \end{bmatrix}$$

$$\det \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = 1$$

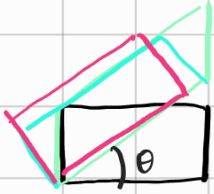
$$m(L(E)) = |\det L| \cdot m(E).$$



$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}}_{|\lambda \cdot \mu|} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

$$|\det L| = |\lambda \cdot \mu|$$



$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underbrace{=}_{m = \frac{\sin\theta}{\cos\theta}} \begin{pmatrix} \cos\theta + m \cdot \sin\theta & 0 \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} \cos\theta + m\sin\theta & 0 \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta + m\sin\theta & 0 \\ 0 & \cos\theta \end{pmatrix}$$

$$k = -\frac{\sin\theta}{\cos\theta + m\sin\theta}$$

$$= \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\lambda \cdot \mu = \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)\cos\theta = \cos^2\theta + \sin^2\theta = 1.$$