ANALISI MATEMATICA B
LEZIONE 71 – 25.3.2022

ES 24.4.2021

$$u' = 2x \cdot (u + x^2) \qquad u = u(x)$$

$$u' \underline{- 2xu} = 2x^3$$

$$a(x) = -2x$$
$$A(x) = -x^2$$

fattore integrante: e^{-x^2}

$$u' e^{-x^2} - 2xu\, e^{-x^2} = 2x^3 e^{-x^2}$$

$$\left(u \cdot \underline{e^{-x^2}}\right)' = 2x^3 \cdot e^{-x^2}$$

$$u \cdot e^{-x^2} = \int \underbrace{2x^3 \cdot e^{-x^2}}\, dx = *$$

$$\left[\, 2x \cdot e^{-x^2} = -\left(e^{-x^2}\right)' \,\right]$$

$$* = \int x^2 \cdot 2x e^{-x^2}\, dx = -x^2 \cdot e^{-x^2} + \int 2x\, e^{-x^2}\, dx$$

$$= -x^2 e^{-x^2} - e^{-x^2} = -(1+x^2) e^{-x^2}$$

$$u\, e^{-x^2} = -(1+x^2) e^{-x^2} + c$$

$$\boxed{u = -(1+x^2) + c \cdot e^{x^2}}$$

Problema di Cauchy: $\begin{cases} u' = 2x(u+x^2) \\ u(0) = 0 \end{cases}$

condizione di Cauchy

$\begin{cases} u' = \lambda u \\ u(0) = y_0 \end{cases}$ ← esercizio esponenziale

$\begin{cases} u = -(1+x^2) + c \cdot e^{x^2} \\ u(0) = 0 \end{cases}$

$x = 0 \longrightarrow 0 = -1 + c \cdot 1$

$c = 1$

$$u(x) = -(1+x^2) + e^{x^2}.$$

Eq. lineari I ordine: $u'(x) + a(x)u(x) = b(x)$

EQ. A VARIABILI SEPARABILI

(I ordine)
(forma normale)
$u' = F(x, u(x))$

$u'(x) = f(x) \cdot g(u(x))$

$u' = f(x) \cdot g(u)$

$$\frac{u'}{g(u)} = f(x) \qquad (\text{se } g(u(x)) \neq 0)$$

$$\frac{u'(x)}{g(u(x))} = f(x)$$

Se $g(y_0) = 0$
Noto che $u(x) = y_0$
$u'(x) = 0$
è soluzione.

$$\int \frac{\boxed{u'(x)\,dx}}{g(u(x))} = \int f(x)\,dx$$

$$\| $$

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$

Formalmente $\begin{cases} y = u(x) \\ dy = u'(x)\,dx \end{cases}$ $\left[\int \frac{1}{g(y)}\,dy\right]_{y=u(x)} = \int f(x)\,dx$

$$H(y) = \int \frac{1}{g(y)}\,dy \qquad F(x) = \int f(x)\,dx$$

$$H(u(x)) = F(x) + c$$

$$u(x) = H^{-1}(F(x) + c)$$

Esempio

$$u'(x) = x \cdot u^2(x) + x$$

$$u'(x) = x \cdot (u^2(x) + 1)$$

$$u' = x \cdot (u^2 + 1)$$

$$\frac{u'(x)}{u^2(x)+1} = x$$

$$\to \int \frac{u'(x)\,dx}{u^2(x)+1} = \int x\,dx$$

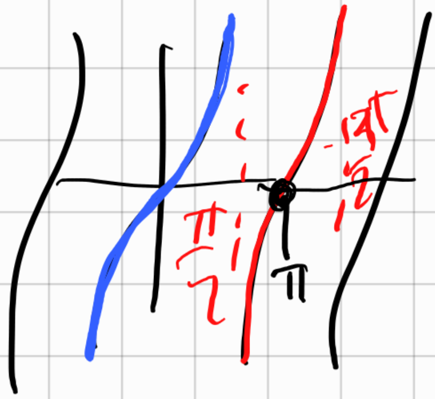
$$\int_{x_0}^{x} \frac{u'(t)}{u^2(t)+1}\,dt = \int_{x_0}^{x} t\,dt$$

$u = u(x)$

$$\int \frac{1}{u^2+1}\,du\,\Big|_{u=u(x)} = \frac{x^2}{2} + c$$

$$\operatorname{arctg}(u(x)) = \frac{x^2}{2} + c$$

$$\boxed{u(x) = \operatorname{tg}\left(\frac{x^2}{2} + c\right)}$$



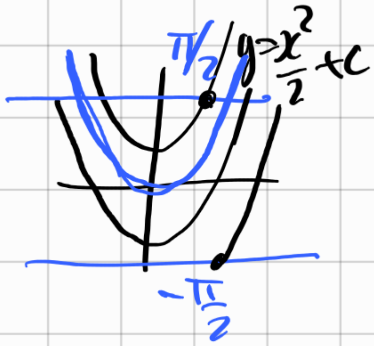
con $-\frac{\pi}{2} < \frac{x^2}{2} + c < \frac{\pi}{2}$

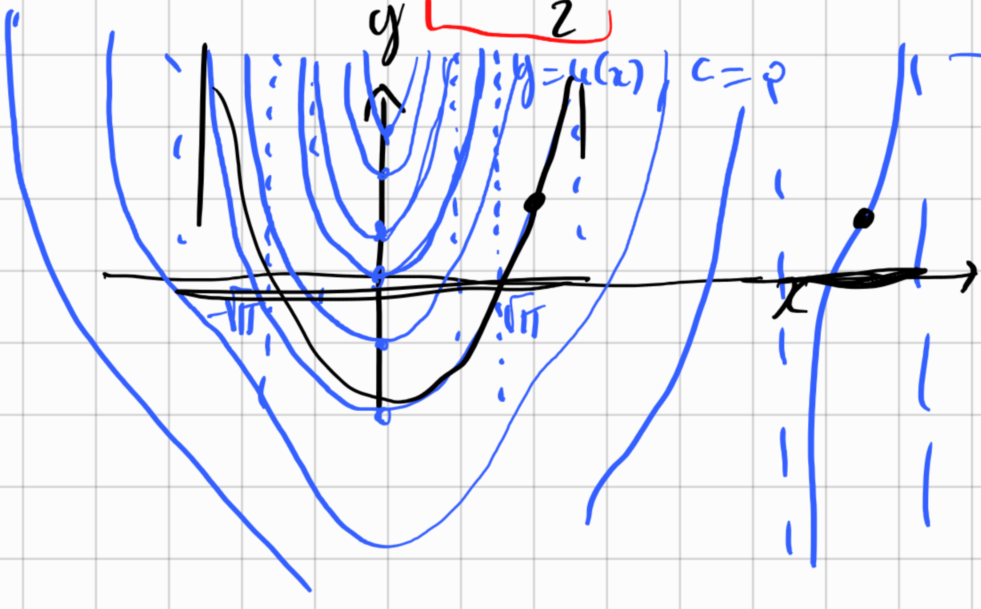
$$-\pi - 2c < x^2 < \pi - 2c$$

$c < \frac{\pi}{2}$

$$\sqrt{-\pi - 2c} < |x| < \sqrt{\pi - 2c}$$

se $c < -\frac{\pi}{2}$





Esempio 2

$$\begin{cases} u' = u^2 \\ u(0) = 1 \end{cases}$$

$$u'(x) = (u(x))^2$$

$$u' = 1 \cdot u^2$$

Se $u(x) \neq 0$ divido per $u^2(x)$

$$\frac{u'(x)}{u^2(x)} = 1.$$

$$\Rightarrow \int_0^x \frac{u'(t)}{u^2(t)}\, dt = \int_0^x 1 \cdot dt = [t]_0^x = x$$

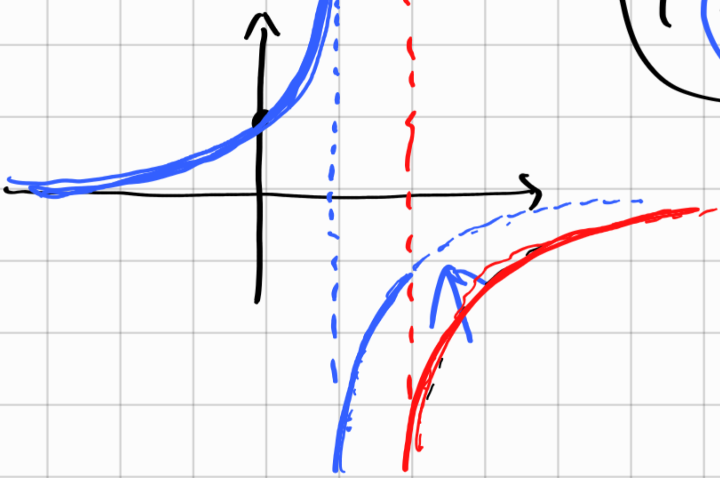
$$\begin{cases} u = u(t) \\ du = u'(t)\, dt \end{cases}$$

$$\int_{u(0)}^{u(x)} \frac{1}{u^2}\, du = \left[-\frac{1}{u}\right]_{u(0)}^{u(x)} = -\frac{1}{u(x)} - \left(-\frac{1}{u(0)}\right)$$

$u(0) = 1$

$$-\frac{1}{u(x)} + 1 = x \qquad \frac{1}{u(x)} = 1 - x$$

$$u(x) = \frac{1}{1-x}$$



$$\begin{cases} u' = u^2 \\ u(0) = 1 \end{cases}$$

$$\frac{1}{2-x}$$

Notazione Leibniz

$$\frac{du}{dx} = u^2$$

$g(u) = u^2$

$g(0) = 0$

$$du = u^2 \, dx$$

$u \neq 0$

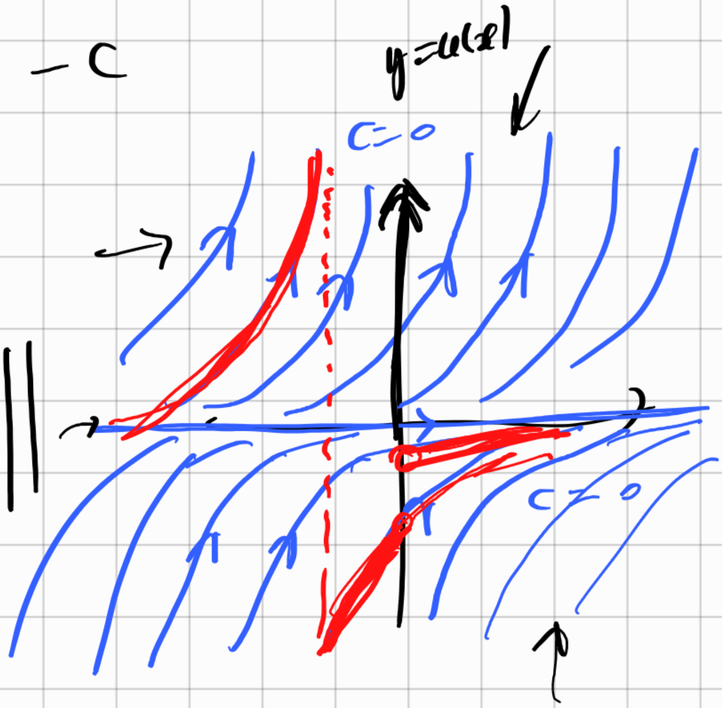
$u(x) \equiv 0$ è sol.

$$\int \frac{du}{u^2} = \int dx$$

$$-\frac{1}{u} = x - c$$

$$\frac{1}{u} = -x + c$$

$$u(x) = \frac{1}{c - x}$$



Equazioni autonome:

I ordine $u'(x) = g(u(x))$

II ordine $u''(x) = F(u(x), u'(x))$.

ES

$$u'(x) = u(x) \cdot x$$

$u(x) = 0$ è soluzione

se $u(x) \neq 0$ divido per $u(x)$:

$$\frac{u'(x)}{u(x)} = x$$

$$\int \frac{u'(x)\,dx}{u(x)} = \int x\,dx$$

$$\int \frac{du}{u} = \frac{x^2}{2} + c$$

$$\ln|u| = \frac{x^2}{2} + c$$

$$|u| = e^{\frac{x^2}{2} + c} = e^c \cdot e^{\frac{x^2}{2}}$$

$$u = \pm e^c e^{\frac{x^2}{2}}$$

$$u = k \cdot e^{\frac{x^2}{2}} \qquad k \neq 0$$

ma $k = 0$ $\quad u = 0$

$$u(x) = k \cdot e^{\frac{x^2}{2}} \qquad k \in \mathbb{R}.$$