Limiti (esercizi).

1°] $$\lim_{(x,y)\to(0,0)} \frac{\sin^2(xy)}{x^2+y^2}$$

$$|\sin\alpha| \leq |\alpha| \qquad \forall \alpha\in\mathbb{R}$$

$$\sin^2\alpha \leq \alpha^2$$

$$\alpha := xy \Rightarrow \sin^2 xy \leq x^2y^2 \qquad \forall (x,y)\in\mathbb{R}^2$$

$$0 \leq f(x,y) = \frac{\sin^2 xy}{x^2+y^2} \leq \frac{x^2y^2}{x^2+y^2}$$

$$\lim_{(x,y)\to 0} \frac{x^2y^2}{x^2+y^2} = \lim_{\rho\to 0^+} \frac{\rho^4 \sin^2\varphi\cos^2\varphi}{\rho^2} =$$

$$\boxed{\begin{array}{l} x=\rho\cos\varphi \\ y=\rho\sin\varphi \end{array}}$$

$$= \lim_{\rho \to 0^+} \rho^2 \sin^2\varphi \cos^2\varphi = 0$$

$$0 \leq |\rho^2 \sin^2\varphi \cos^2\varphi| \leq \rho^2 \to 0$$

Conclusione $\displaystyle \lim_{(x,y)\to 0} \frac{\sin^2 xy}{x^2+y^2} = 0.$

$\underline{2^\circ|}$

$$\lim_{(x,y)\to\infty} xy\, e^{-(x^2+y^2)} = \lim_{\rho\to+\infty} \rho^2 \sin\varphi\cos\varphi\, e^{-\rho^2}$$

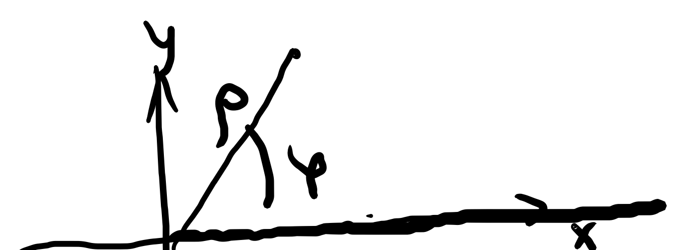
$$\boxed{\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}}$$

$$\left|\rho^2 \sin\varphi \cos\varphi\, e^{-\rho^2}\right| \le \rho^2 e^{-\rho^2} \xrightarrow[\rho \to +\infty]{} 0$$

$$\lim_{(x,y)\to\infty} xy\, e^{-(x^2+y^2)} = 0.$$

3°) $\displaystyle\lim_{(x,y)\to\infty} xy\, e^{x^2+y^2}$

$$f(x,y) = xy\, e^{x^2+y^2} = \rho^2 \sin\varphi\cos\varphi\, e^{\rho^2} \qquad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$



1) $\varphi = 0$ (ossia, $y=0$, $x \to +\infty$) $\quad f(x,y) = 0$

$$\lim_{\substack{(x,y)\to\infty \\ y=0}} f(x,y) = 0.$$

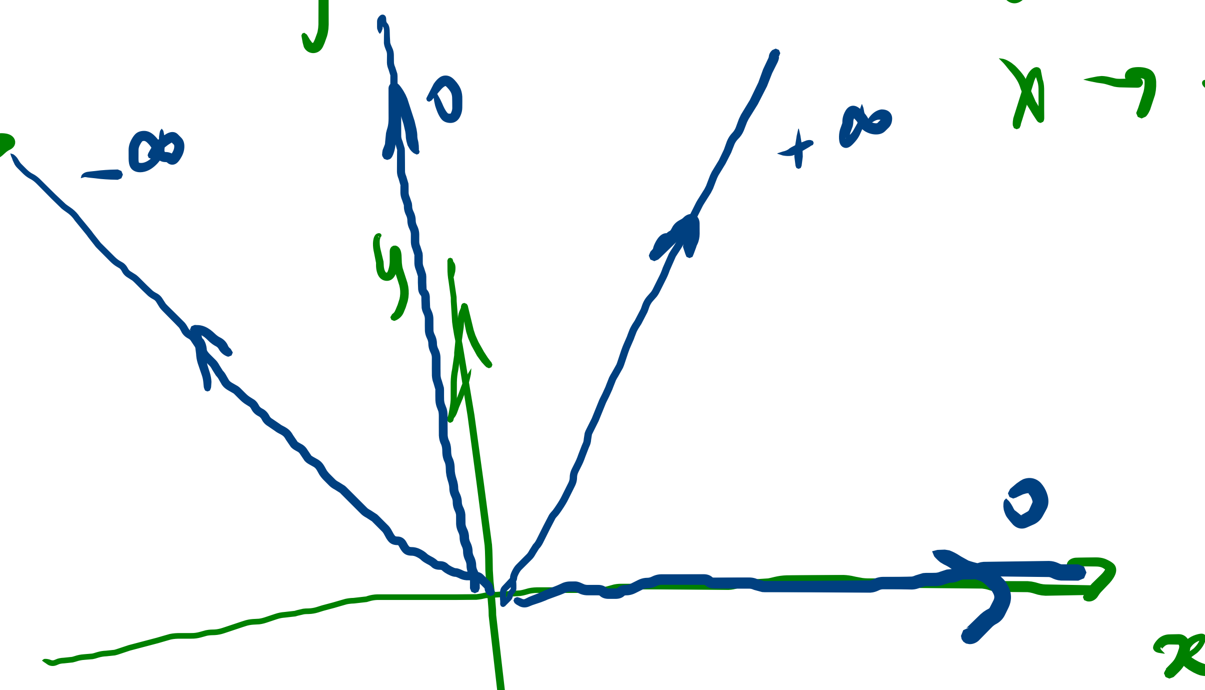
$\displaystyle\lim_{(x,y)\to\infty} f(x,y)$ non esiste

2). $\varphi = \pi/4$ (ossia, $y = x$, $x, y \to +\infty$)

$$\lim_{\substack{(x,y)\to+\infty \\ x=y}} xy\, e^{x^2+y^2} = \lim_{x\to+\infty} x^2 e^{2x^2} = +\infty.$$

Osservazione: se $\varphi = \frac{3\pi}{4}$ (ossia, $y = -x$)

$$\lim_{\substack{(x,y)\to\infty \\ y=-x}} xy\, e^{x^2+y^2} = \lim_{x\to+\infty} -x^2 e^{2x^2} = -\infty$$



$\underline{4^\circ)}$

$$\lim_{(x,y)\to(2,1)} \frac{(y-1)^2 \sin \pi x}{(x-2)^2+(y-1)^2} = \lim_{(x,y)\to(2,1)} \frac{(y-1)^2}{(x-2)^2+(y-1)^2} \sin \pi x$$

$$0 \le \frac{(y-1)^2}{(x-2)^2+(y-1)^2} \le \frac{(y-1)^2+(x-2)^2}{(x-2)^2+(y-1)^2} = 1$$

$$|f(x,y)| = \left| \frac{(y-1)^2 \sin^2 x}{(x-2)^2+(y-1)^2} \right| \leq |\sin^2 x| \xrightarrow[x\to 1]{} 0$$

$$\lim_{(x,y)\to(2,1)} f(x,y) = 0 .$$

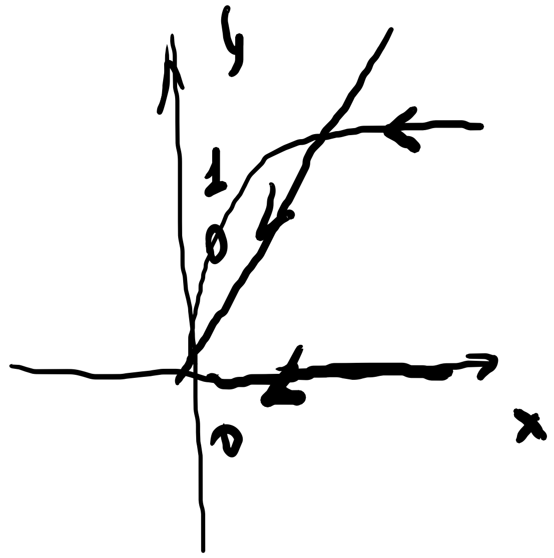
Osservazione: in realtà abbiamo dimostrato molto di più:

$$\lim_{\substack{x\to 1 \\ y\to a}} f(x,y) = 0 .$$

5°) $$\lim_{(x,y)\to 0} \frac{x^2+y^2}{x} = \lim_{(x,y)\to 0}\left(x+\frac{y^2}{x}\right) = $$ se entrambi esistono

$$= \cancel{\lim_{(x,y)\to 0} x}^{\,0} + \lim_{(x,y)\to 0} \frac{y^2}{x}$$

$$g(x,y) := \frac{y^2}{x}$$



1). $y = 0$, $x \neq 0$, $x \to 0$

$$g(x,0) = 0 \qquad \lim_{\substack{(x,y)\to 0\\ y=0}} \frac{y^2}{x} = 0.$$

2) $y = x \to 0$

$$\lim_{\substack{(x,y)\to 0\\ y=x}} \frac{y^2}{x} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

3). $y = \sqrt{x}$

$$\lim_{\substack{(x,y)\to 0\\ y=\sqrt{x}}} \frac{y^2}{x} = \lim_{x\to 0} \frac{(\sqrt{x})^2}{x} = 1.$$

Conclusione: $\lim_{(x,y)\to 0} \frac{x^2+y^2}{x}$ non esiste.

Osservazione: Si poteva fare direttamente

$$f(x,y) = \frac{x^2+y^2}{x}$$

$$\lim_{\substack{(x,y)\to 0 \\ y=0}} f(x,y) = \lim_{x\to 0} \frac{x^2}{x} = 0.$$

$$\lim_{\substack{(x,y)\to 0 \\ y=\sqrt{x}}} f(x,y) = \lim_{x\to 0} \frac{x^2+(\sqrt{x})^2}{x} = \lim_{x\to 0} \frac{x^2+x}{x} =$$

$$= 1.$$

Curve in $\mathbb{R}^n$

Def. $I \subset \mathbb{R}$ è un intervallo

$\Gamma : I \to \mathbb{R}^n$ si chiama curva parametrizzata in $\mathbb{R}^n$

(curva parametrica
curva in forma parametrica)

$t \in I$ tempo

$\Gamma(t) \in \mathbb{R}^n$ $\qquad \Gamma(\cdot)$ traiettoria di un punto

$$\Gamma(t) = (x_1(t), x_2(t), \dots, x_n(t)).$$

$\Gamma : I \to \mathbb{R}^n$ si dice continua, se lo sono $x_j : I \to \mathbb{R}$ $\forall j$

differenziabile, — " — " — " —

$$\lim_{t\to t_0} r(t) = \left(\lim_{t\to t_0} x_1(t), \dots, \lim_{t\to t_0} x_n(t)\right)$$

$$\dot r(t)$$

$$\cancel{r'(t)} = \left(x_1'(t), \dots, x_n'(t)\right)$$

vettore di velocità istantanea (al tempo t) di moto

sostegno $\bigcup_{t\in I} r(t) = r(I)$

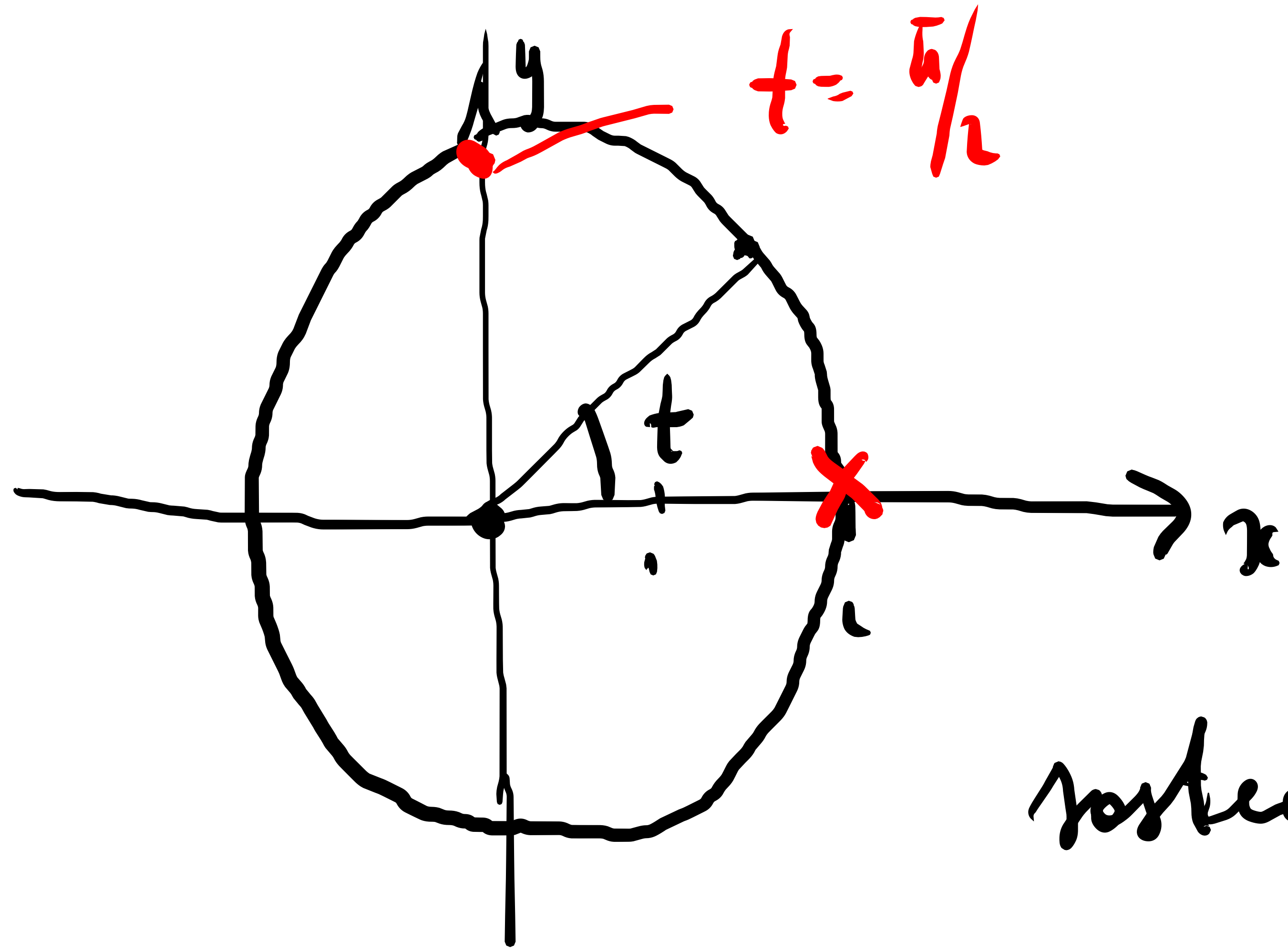
Esempi

i) $I = [0, 2\pi]$

$$\begin{cases} x(t) = 2\cos t \\ y(t) = 2\sin t \end{cases}$$

$$r(t) = (2\cos t, 2\sin t) \in \mathbb{R}^2 \qquad t\in[0, 2\pi]$$

In coordinate polari $\begin{cases} \rho(t) = 2 \\ \varphi(t) = t \end{cases}$

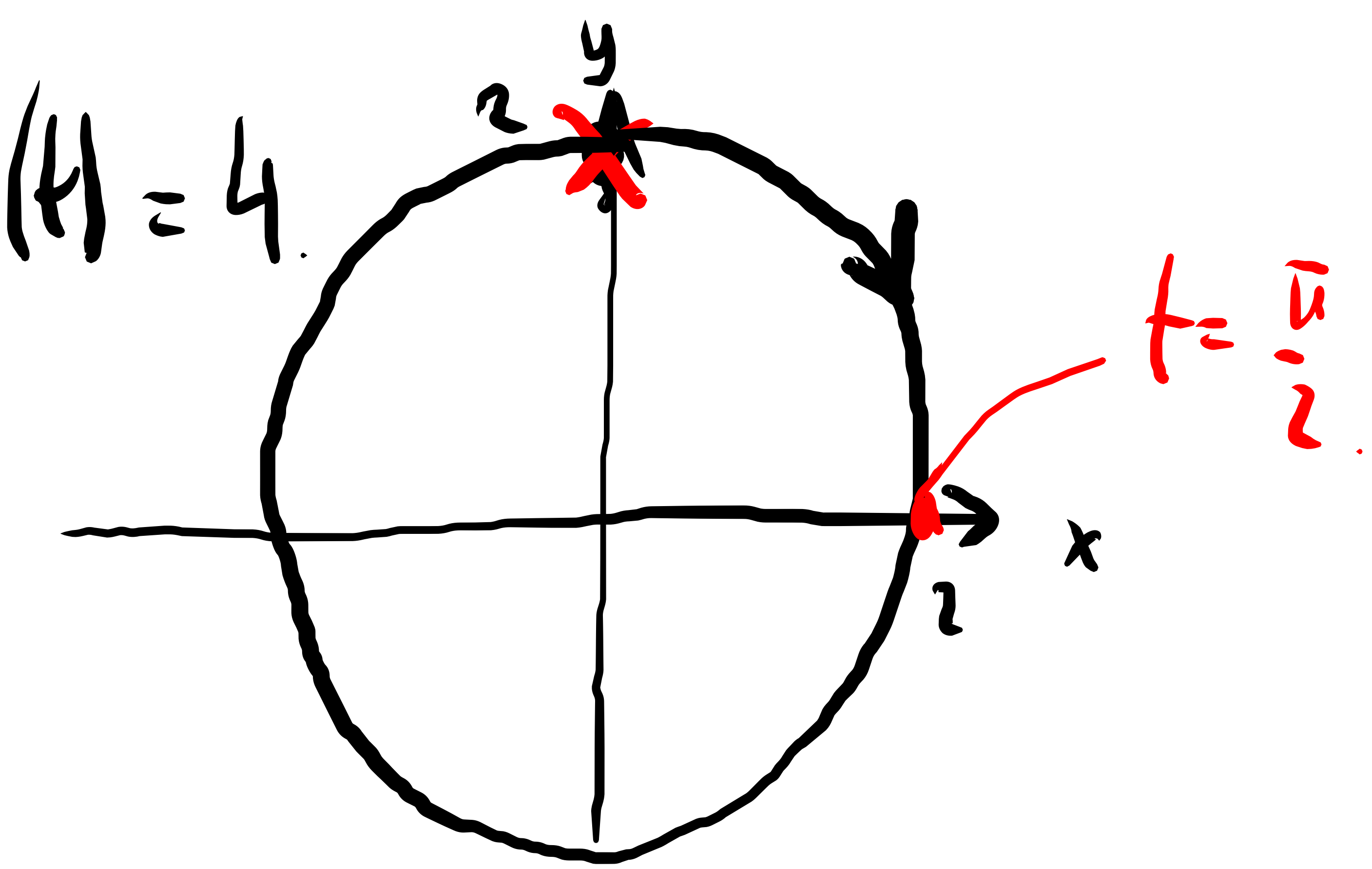


$$x^2(t) + y^2(t) = 4.$$

sostegno: $\{(x,y) : x^2 + y^2 = 4\}$.

2) $x(t) = 2\sin t$
$y(t) = 2\cos t$
$t \in [0, 2\pi]$

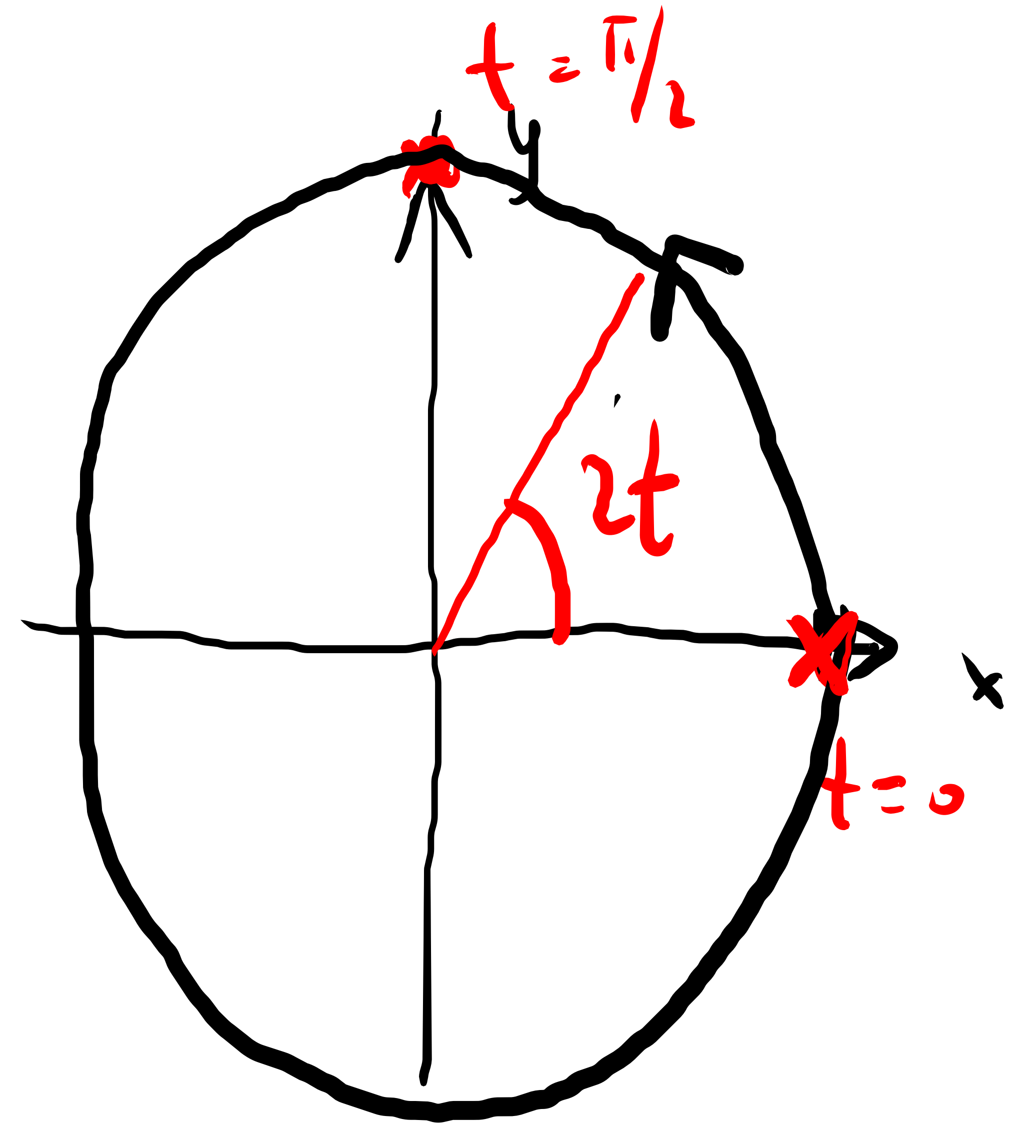
$$x^2(t) + y^2(t) = 4.$$



sostegno $\{(x,y) : x^2 + y^2 = 4\}$.

3) $\quad I = [0, \pi]$.

$x(t) = 2\cos 2t$
$y(t) = 2\sin 2t$.



sostegno $\{(x, y) : x^2 + y^2 = 4\}$.

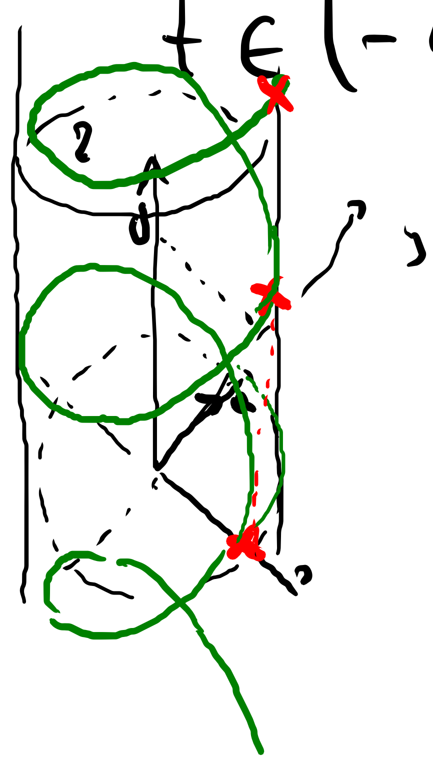
In coordinate polari (ρ, φ) : $\begin{cases} \rho(t) = 2 \\ \varphi(t) = 2t \end{cases}$

4°)

$$\begin{cases} x(t) = a\cos t \\ y(t) = a \sin t \\ z(t) = v t \end{cases} \qquad a, v > 0.$$

$$r(t) := (a\cos t, a \sin t, v t) \in \mathbb{R}^3$$

$$t \in (-\infty, \infty).$$



elica cilindrica.

In coordinate cilindriche

$$\underbrace{(x, y, z)}_{\text{cartesiane}} \longrightarrow (\rho, \varphi, z)$$

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi.$$

$$\begin{cases} \rho(t) = a \\ \varphi(t) = t \\ z(t) = v t \end{cases}$$

Sostegno (in forma implicita):

$$t = z/v.$$

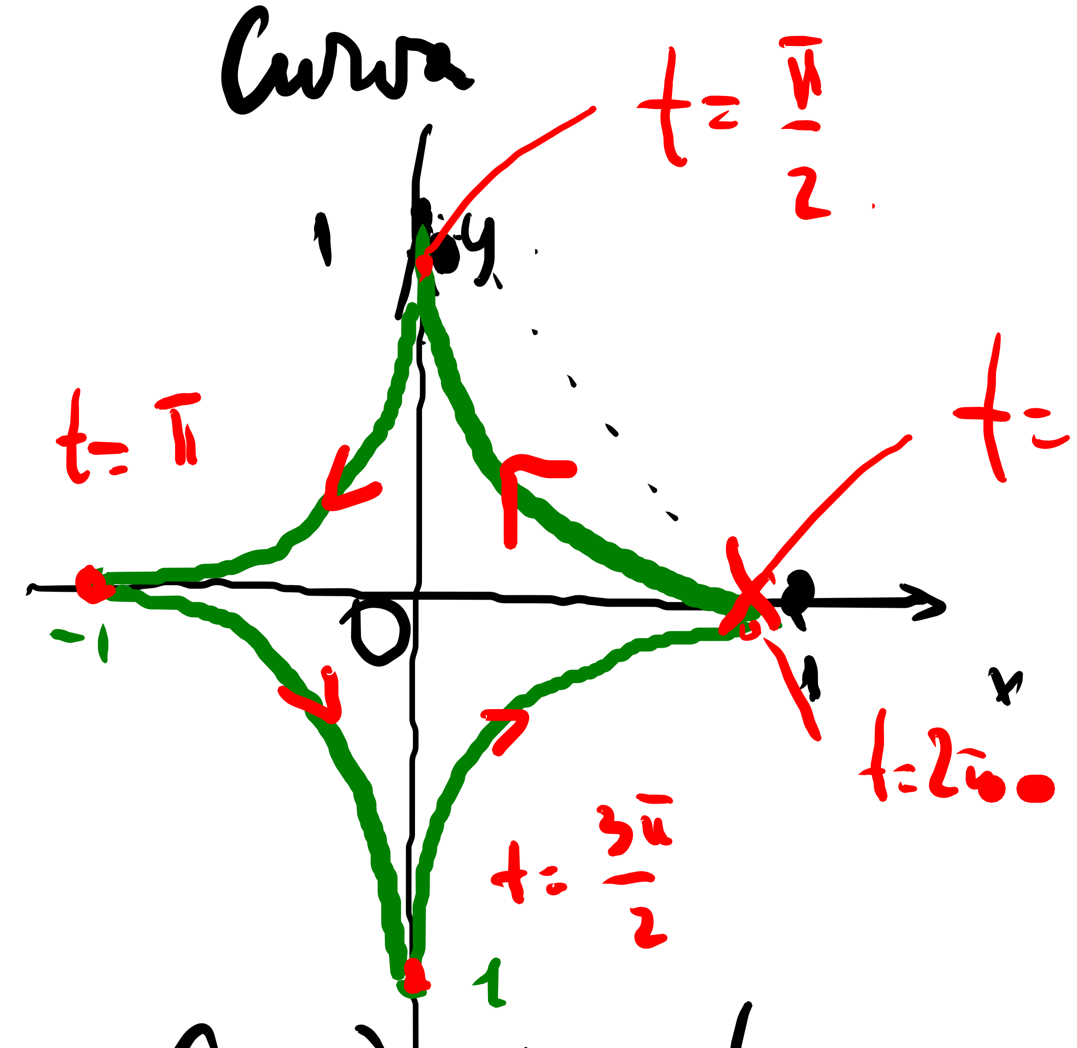
$$\begin{cases} x = a\cos z/v \\ y = a \sin z/v \end{cases}$$

O in coordinate polari

$$\begin{cases} \rho = a \\ \varphi = z/v. \end{cases}$$

5°) $x^{2/3} + y^{2/3} = 1$.

"Curva"



$$C := \{ (x,y) : x^{2/3} + y^{2/3} = 1 \}$$

$$(x,y) \in C \Rightarrow \begin{array}{l} (-x,y) \in C \\ (x,-y) \in C \\ (-x,-y) \in C \\ (y,x) \in C \end{array} \Rightarrow C \text{ è simmetrica risp. a } O_x,\ O_y \text{ e } O.$$

(e anche risp. a $\{x = y\}$)

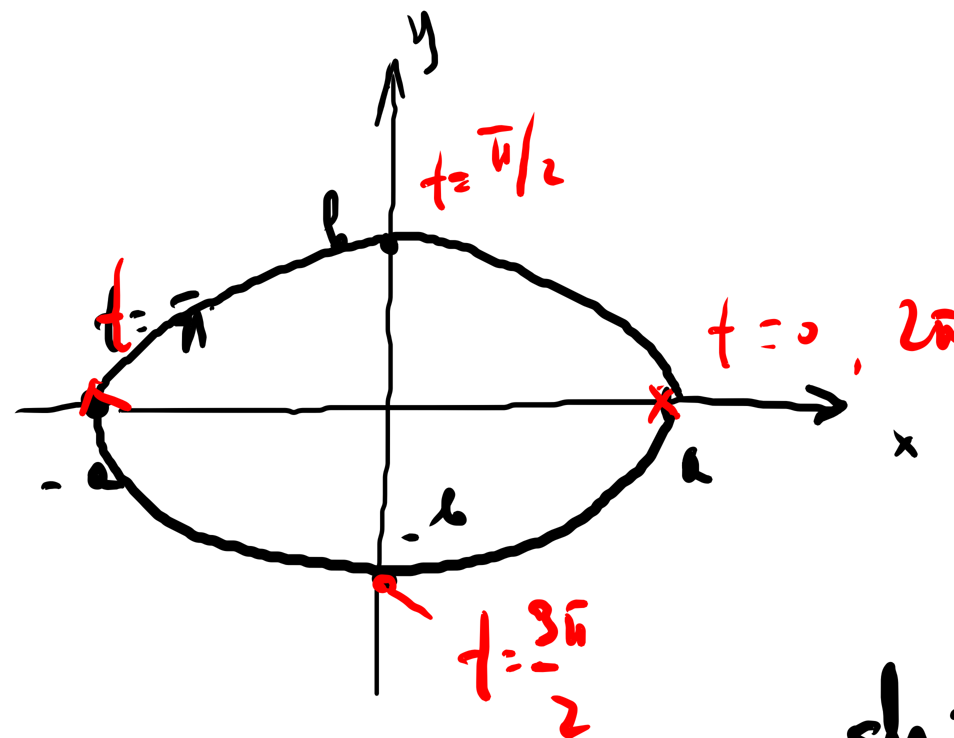
Nel primo quadrante: $y = \sqrt{(1 - x^{2/3})^3}$

$$\begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \end{cases} \quad t \in [0, 2\pi]$$

è una curva parametrizzata

6°) $t \in [0, 2\pi]$

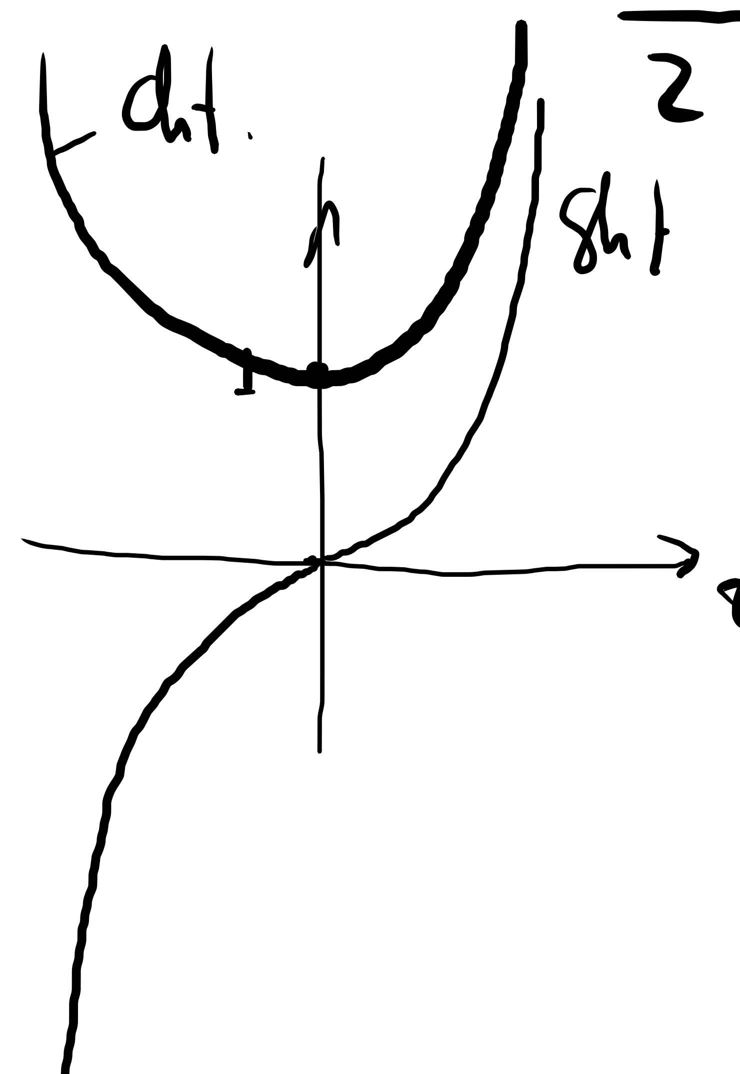
$$x(t) = a\cos t \qquad\qquad a, b > 0.$$
$$y(t) = b\sin t$$



ellisse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$$\text{sh}\, t = \frac{e^t - e^{-t}}{2}$$
$$\text{ch}\, t = \frac{e^t + e^{-t}}{2}.$$

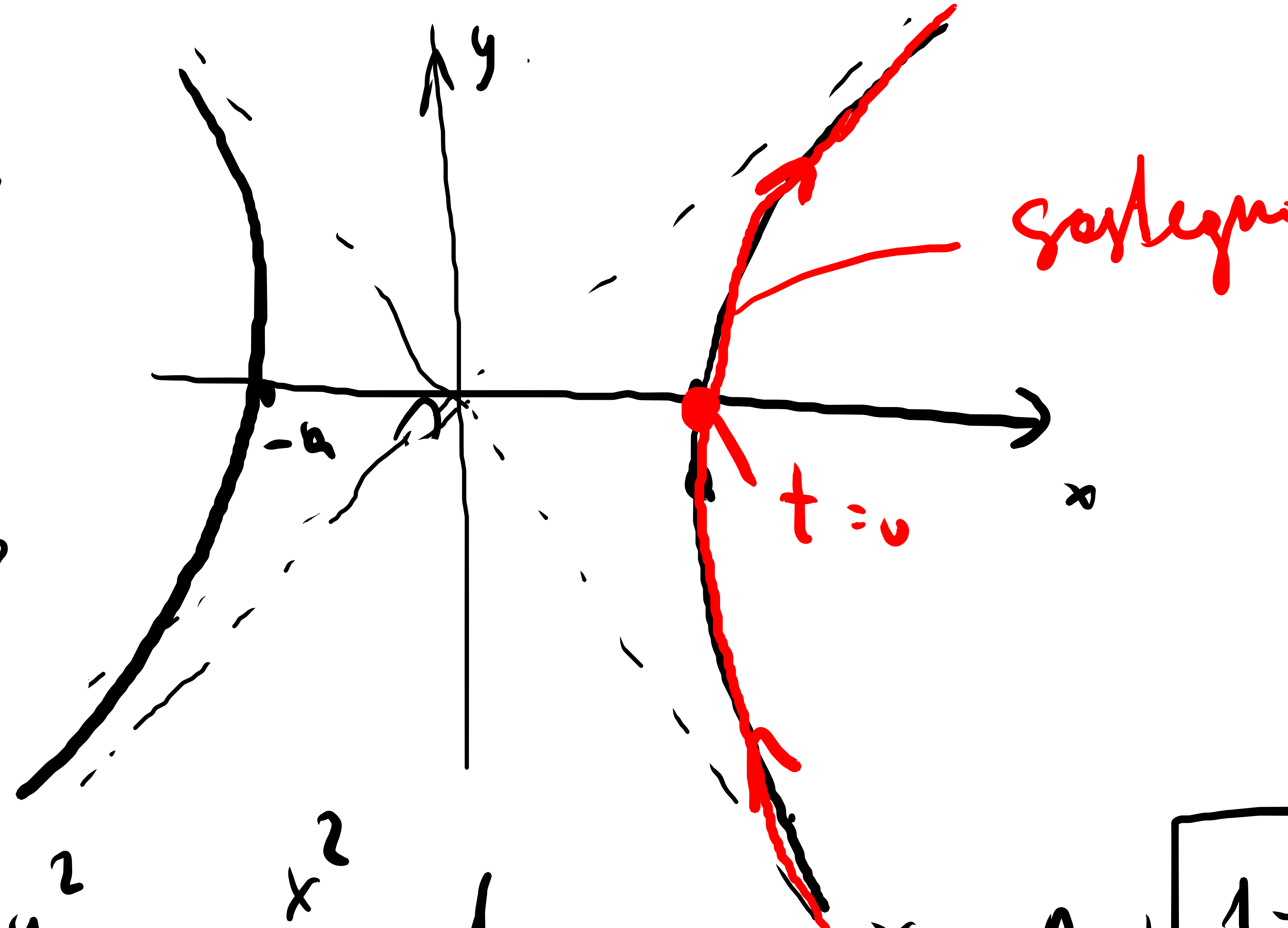
7°) $t \in \mathbb{R}$ $\quad \begin{cases} x(t) = a\,\text{ch}\, t \\ y(t) = b\,\text{sh}\, t \end{cases}$



$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\text{ch}^2\alpha - \text{sh}^2\alpha = 1. \quad \text{(es.)}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b} = 1.}$$ — iperbole.

$$C := \left\{ (x,y) : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$



C è simmetrico risp a O_x, O_y e O.

Nel Imo quadrante

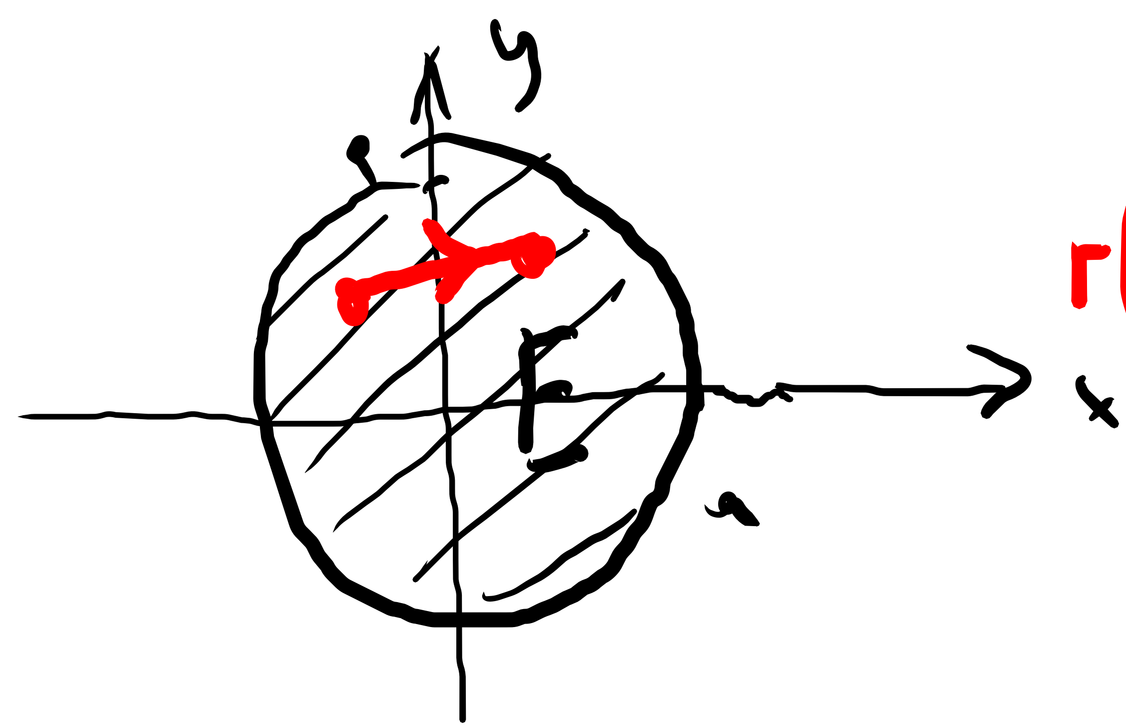
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$x = a\sqrt{1 + \frac{y^2}{b^2}}$$

$$y = b\sqrt{\frac{x^2}{a^2} - 1}$$

Def. $E \subset \mathbb{R}^n$ si chiama connesso per archi, se $\forall x, y \in E$ $\exists\, r : [a,b] \to E$ continua, $r(a) = x$, $r(b) = y$.

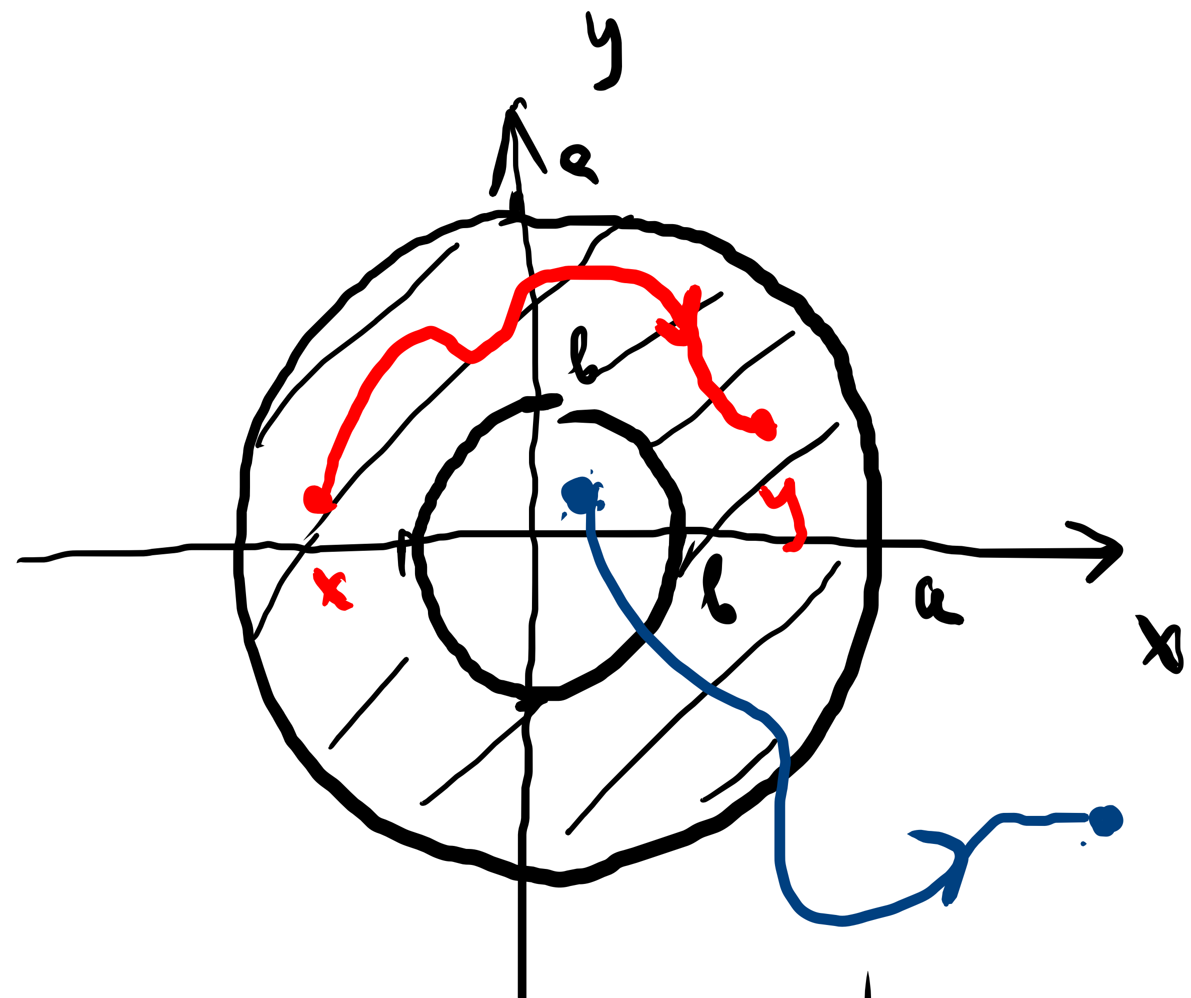
Esempi. 1°) $E := \{x^2 + y^2 \le a^2\}$ è connesso per archi



$r(t) := (1-t)x + ty$
$t \in [0,1]$

2°) $E := \{(x, y) : b^2 \le x^2 + y^2 \le a^2\}$. connesso per archi

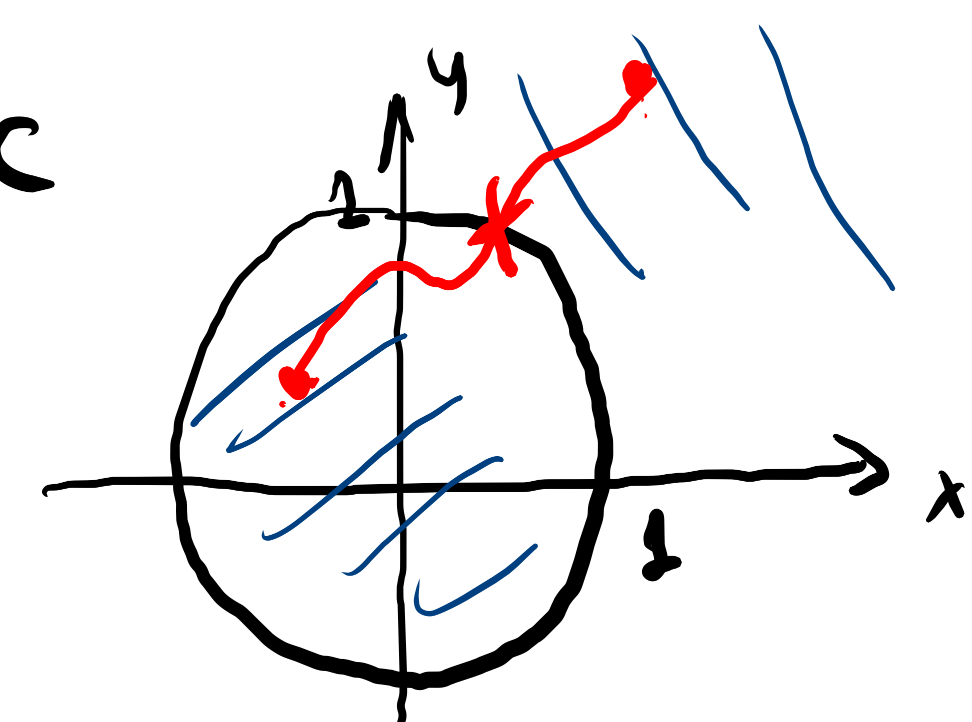
$(0 < b < a)$



3°). $E := \{(x, y) : x^2 + y^2 > a^2$ oppure $x^2 + y^2 < b^2\}$

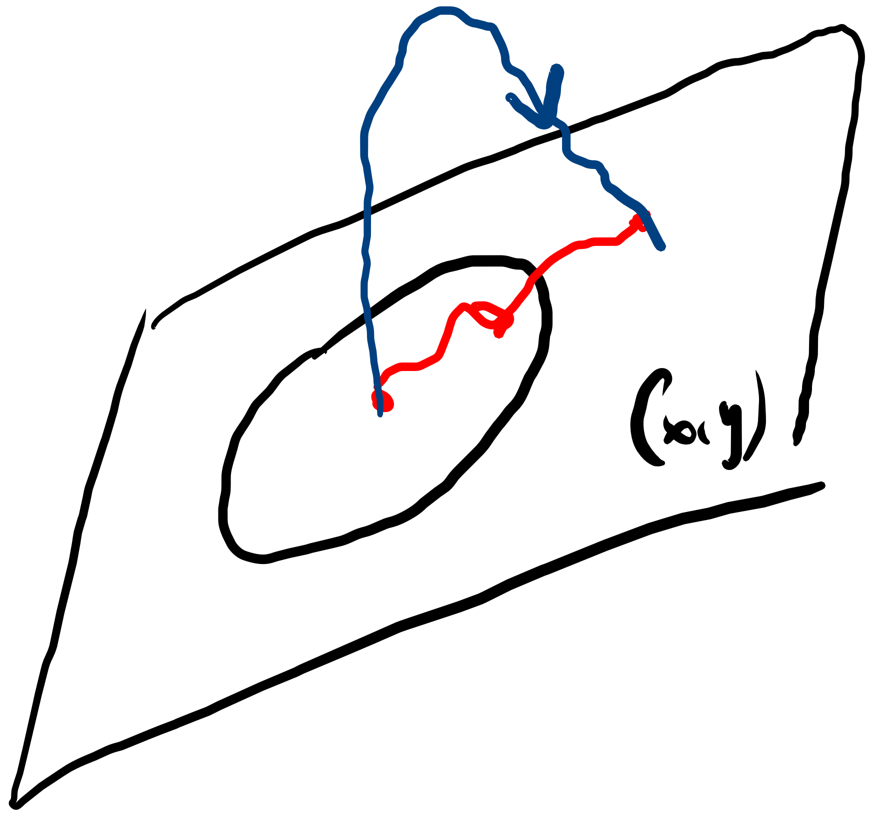
non è connesso per archi.

4°) $\{(x, y) : x^2 + y^2 = 1\}^C$



non è connesso per archi.

5°). $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1,\ z=0\}^c$ è connesso per archi.



Th. (degli zeri di una funzione continua).

Sia $E \subset \mathbb{R}^n$ connesso per archi, $f : E \to \mathbb{R}$ continua.

$x, y \in E$: $f(x) < 0$, $f(y) > 0$.

Allora $\exists\, z \in E$ tale che $f(z) = 0$

Inoltre per ogni curva $\Gamma : [a,b] \to E$ continua con $\Gamma(a) = x$, $\Gamma(b) = y$ $\exists\, t \in [a,b]$ $f(\Gamma(t)) = 0$.

Dim: Sia $r : [a,b] \to E$ curva continua,

$$r(a) = x, \ r(b) = y.$$

$$g : [a,b] \to \mathbb{R} \quad , \ g(t) := f(r(t))$$

•) g è continua

•) $g(a) = f(r(a)) = f(x) < 0$

$g(b) = f(r(b)) = f(y) > 0.$

$$\exists \ \tilde{t} \in [a,b] : \quad \underset{f(r(\tilde{t}))}{\underset{\parallel}{g(\tilde{t})}} = 0$$

$$z := r(\tilde{t})$$

$$f(z) = 0 \quad , \ \underline{\underline{q.e.d.}}$$

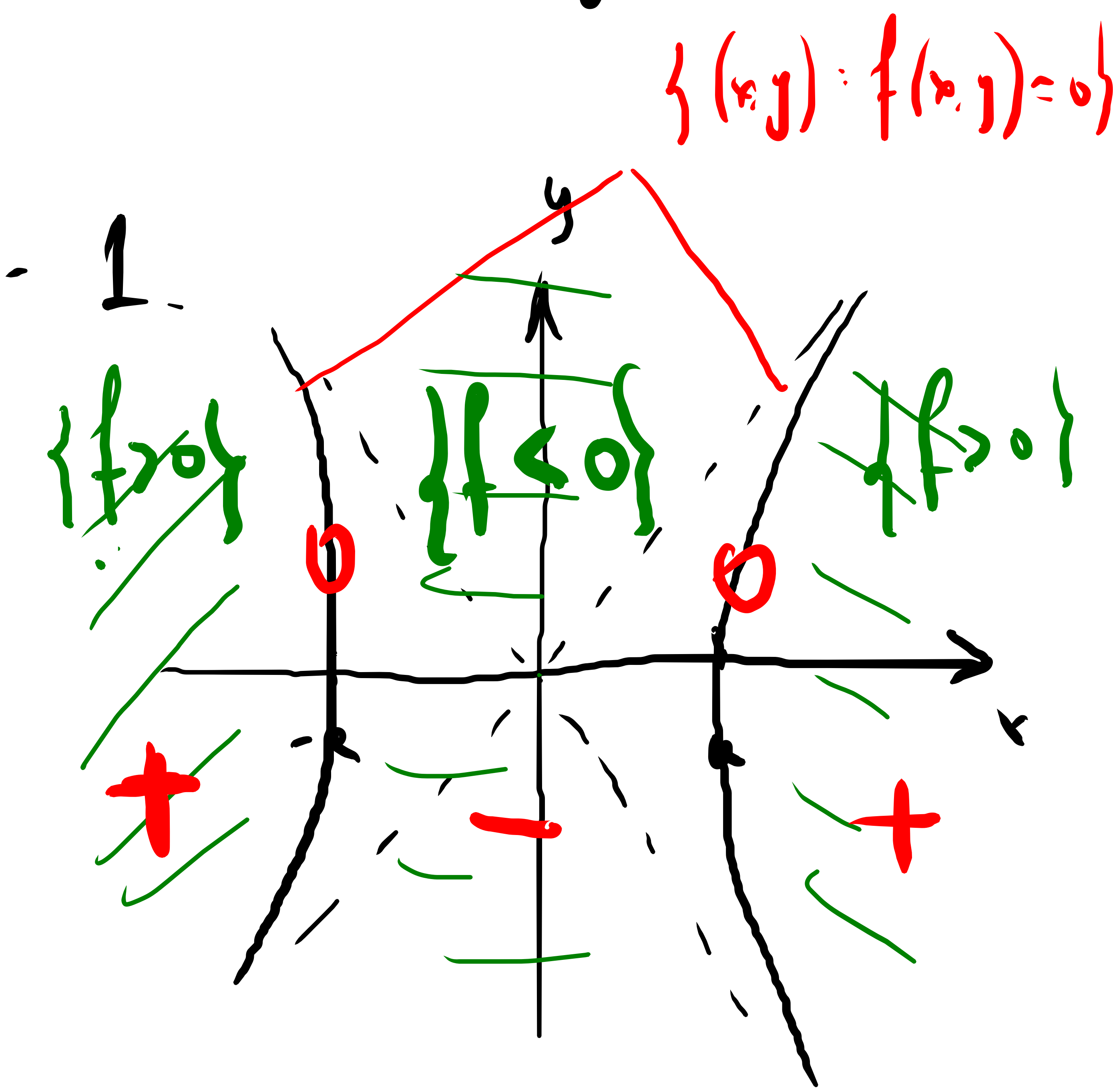
Corollario. Sia $f : \mathbb{R}^n \to \mathbb{R}$ continua.

Se $E \subset \{x \in \mathbb{R}^n : f(x) = 0\}^c$ è connesso per archi,

allora f su E non cambia segno.

Esempio: $f(x,y) = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1$

$$\{(x,y) : f(x,y) = 0\}$$



Th (Weierstraß). Sia $E \subset \mathbb{R}^n$ chiuso e limitato.

$f: E \to \mathbb{R}^n$ continua.

Allora $\exists\ \bar{x}, \underline{x} \in E$:

$$\min_{x \in E} f(x) = f(\underline{x}) \qquad f(\bar{x}) = \max_{x \in E} f(x)$$

<u>Dim.</u> (dell'esistenza del min):

$$\inf_{x \in E} f(x) + \frac{1}{k} \geq f(x_k) \quad : \quad x_k \in E$$

Per th. di Bolzano-Weierstraß: $\exists \{k_\ell\}_{\ell=1}^{\infty}$

$$\lim_{\ell \to +\infty} x_{k_\ell} = \underline{x} \in E$$

$$f(x_{k_\ell}) \leq \inf_{x \in E} f(x) + \frac{1}{k_\ell}$$

$$\lim_{\ell \to \infty} f(x_{k_\ell}) = \boxed{f(\underline{x}) \leq \inf_{x \in E} f(x)}$$

Quindi, $f(\underline{x}) = \inf_{x \in E} f(x) = \min_{x \in E} f(x)$, q.e.d.