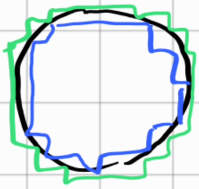
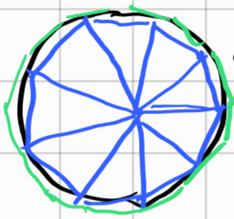
ANALISI MATEMATICA B
LEZIONE 43 – 22.1.2024

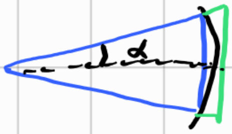
ricevimento domani: B1

Area (misura di Peano-Jordan)

Esercizio area del cerchio







$$m(\triangleleft) = \cos\alpha \cdot \sin\alpha$$

$$\alpha = \frac{\pi}{n}$$

$$m(\triangleleft) = \operatorname{tg}\alpha$$

$$m(\bigcirc) = n\cos\frac{\pi}{n}\sin\frac{\pi}{n}$$

$$m(\bigcirc) = n\operatorname{tg}\frac{\pi}{n}$$

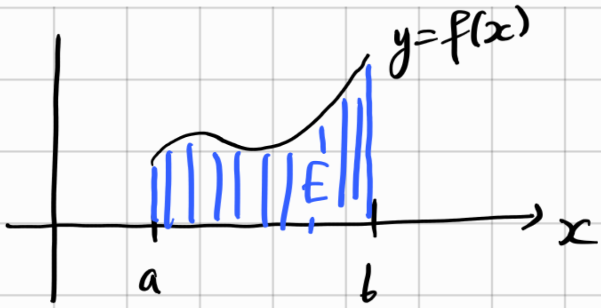
$$n\cos\frac{\pi}{n}\sin\frac{\pi}{n} \le m(\bullet) \le n\operatorname{tg}\frac{\pi}{n} = \underbrace{\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1}\frac{\pi}{\underbrace{\cos\frac{\pi}{n}}_{\to 1}} \to \pi$$

$$\| $$

$$\pi\underbrace{\cos\frac{\pi}{n}}_{\to 1}\underbrace{\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1} \to \pi$$

$$m(\bullet) = \pi.$$

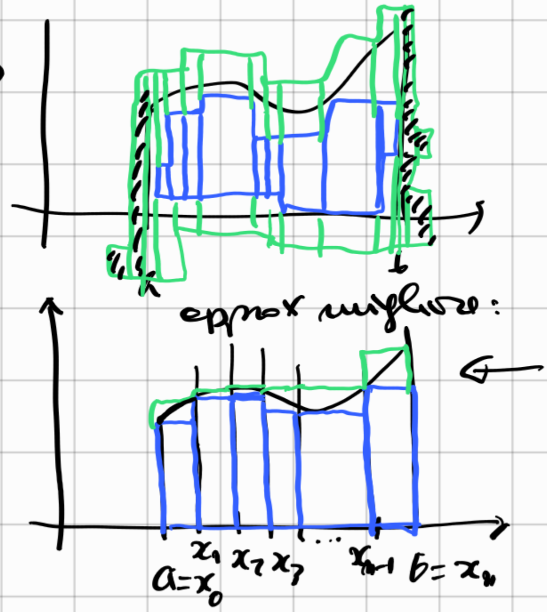
Integrale



$$f:[a,b]\to\mathbb{R}\ ,\ (f\geq 0 \text{ per ora}) \qquad (a\le b)$$

$$E=\{(x,y)\in\mathbb{R}^2:\ x\in[a,b],\ 0\le y\le f(x)\}$$

$m(E) \longrightarrow$



Def (integrale di Riemann) Sia $f:[a,b]\to\mathbb{R}$ una funzione limitata. Sia $P\subseteq[a,b]$ P finito con $a\in P$, $b\in P$ — diremo che P è una suddivisione di $[a,b]$.

$$P=\{x_0,\dots,x_n\} \qquad \#P=n+1.$$

$$a=x_0<x_1<\dots<x_{n-1}<x_n=b$$

Definiamo:

$$S^*(f,P)=\big(m(\ )\big)=\sum_{k=1}^{n}(x_k-x_{k-1})\cdot\sup_{x\in[x_{k-1},x_k]}f(x)$$

$$S_*(f,P)=\big(m(\ )\big)=\sum_{k=1}^{n}(x_k-x_{k-1})\cdot\inf_{x\in[x_{k-1},x_k]}f(x)$$

Sono finiti perché f è limitata

$$I^*(f) = \inf_P S^*(f,P)$$

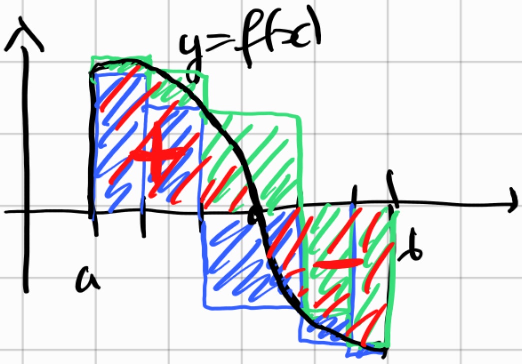
P suddivisione (*)

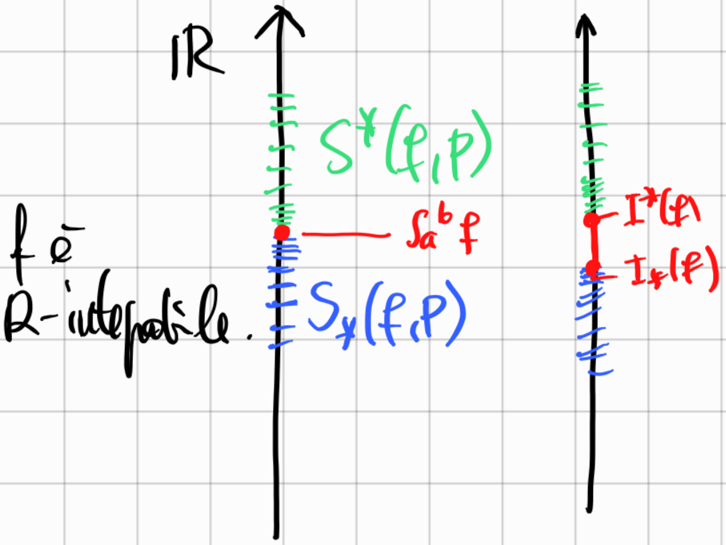
$$I_*(f) = \sup_P S_*(f,P)$$

Se $I^*(f) = I_*(f)$ diremo che f è Riemann-integrabile e porremo:

$$\int_a^b f = I_*(f) = I^*(f).$$

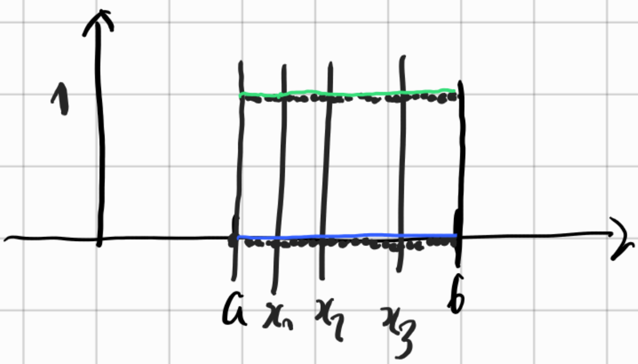
Oss Cosa succede se $f<0$?





Esempio di f non R-integrabile.

$f:[a,b] \to \mathbb{R}$ $\quad a<b$ $\quad f(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ 0 & \text{se } x \notin \mathbb{Q} \end{cases}$



$S^*(f,P) = b-a > 0$

$S_*(f,P) = 0$

Lemma Data f come sopra, P, Q suddivisioni

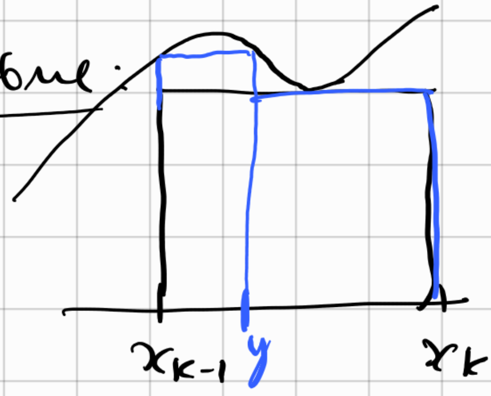
$$S_*(f,P) \overset{①}{\leq} S_*(f,P\cup Q) \overset{②}{\leq} S^*(f,P\cup Q) \overset{③}{\leq} S^*(f,Q)$$

dim ② è ovvio perché $\inf A \leq \sup A$
($\text{se } A \neq \emptyset$)

①

$P = \{ x_0 \quad x_1 \; \ldots \ldots \; x_n \}$
$Q = \{ y_0 \quad y_1 \; \cdot \quad y_m$

Per induzione:



Se $A \supseteq B$
$\inf A \leq \inf B$ $\Rightarrow$

$$\inf_{[x_{k-1},x_k]} f \overset{ⓐ}{\leq} \inf_{[x_{k-1},y]} f$$
$$\overset{ⓑ}{\leq} \inf_{[y,x_k]} f$$

$$(x_k - x_{k-1}) \inf_{[x_{k-1},x_k]} f = (y - x_{k-1}) \underset{ⓐ}{\inf_{[x_{k-1},x_k]} f} + (x_k - y) \cdot \underset{ⓑ}{\inf_{[x_{k-1},x_k]} f}$$

$$\leq (y - x_{k-1}) \inf_{[x_{k-1},y]} f + (x_k - y) \inf_{[y,x_k]} f$$

③ è analogo. [Se $A \supseteq B$ $\quad \sup A \geq \sup B$]

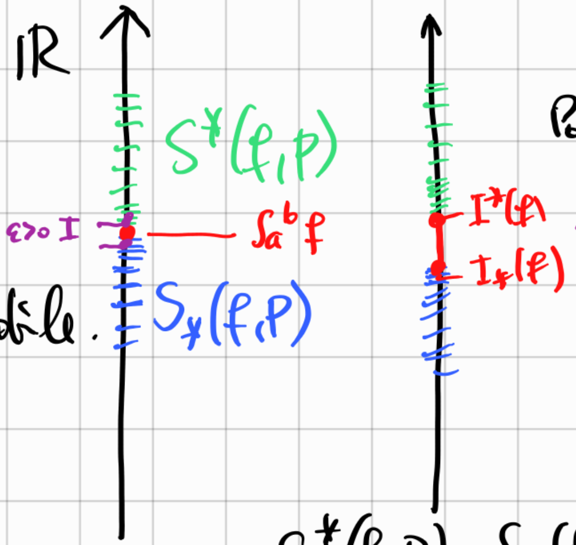
Teorema (criteri di integrabilità)

Sia $f:[a,b]\to\mathbb{R}$ limitata.

1. f è R-integrabile se e solo se:
$\forall \varepsilon>0$ $\exists$ P suddivisione di $[a,b]$ tale che

$$S^*(f,P) - S_*(f,P) < \varepsilon \qquad (*)$$

dim



So vale $(*)$

Posto $\varepsilon = \dfrac{I^*(f)-I_*(f)}{2}$

$$\left.\begin{aligned} &S^*(f,P) \geq \underset{\inf}{I^*(f)} \\ &S_*(f,P) \leq \underset{\sup}{I_*(f)} \end{aligned}\right\} S^*(f,P)-S_*(f,P)\geq 2\varepsilon \quad \text{contro } (*).$$

f integrabile.

$\Downarrow$

$\exists$ P,Q con

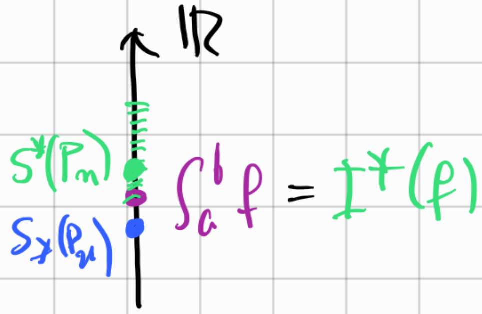
$$\begin{aligned} &S^*(f,P)-S_*(f,Q) < \varepsilon \\ &\quad\;\;\vee\!|\qquad\qquad\;\; \wedge\!| \\ &S^*(f,P\cup Q)-S_*(f,P\cup Q) \leq \end{aligned}$$

lemma.

2(b) Se f è R-integrabile su $[a,b]$ $\exists$ P_n successione di suddivisioni tali che:

$$2(a) \quad \lim_{n\to+\infty} S^*(f,P_n) = \lim_{n\to+\infty} S_*(f,P_n) = \int_a^b f.$$

dim



$$\begin{aligned} &S^*(f,P_n) - S_*(f,P_n) \leq \frac{1}{n} \\ &\qquad\qquad\qquad\quad \vee\!| \\ &S^*(f,P_n) - \int_a^b \end{aligned} \qquad \square$$

2(II) Viceversa se f è limitata, e se esiste una successione P_n di suddivisioni di $[a,b]$ tali che:

$$\lim_{n\to+\infty} S^*(f,P_n) - S_*(f,P_n) = 0$$

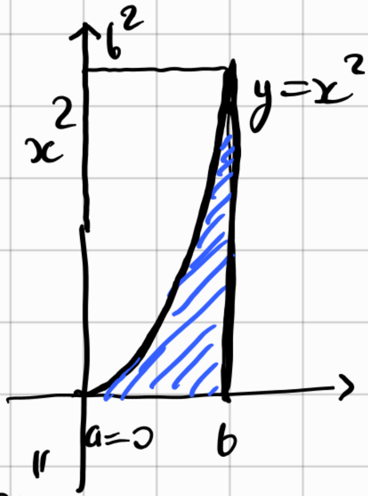
allora f è R-integrabile e vale 2(a)

dim $\forall \varepsilon>0 \ \exists n$ t.c. $S^*(f,P_n)-S_*(f,P_n)<\varepsilon$.

Uso punto I $\square$

Esempio $\boxed{!}$ Se $f(x)=mx+q$ si riesce a fare a trovare $\int_a^b f$ solo usando la definizione.

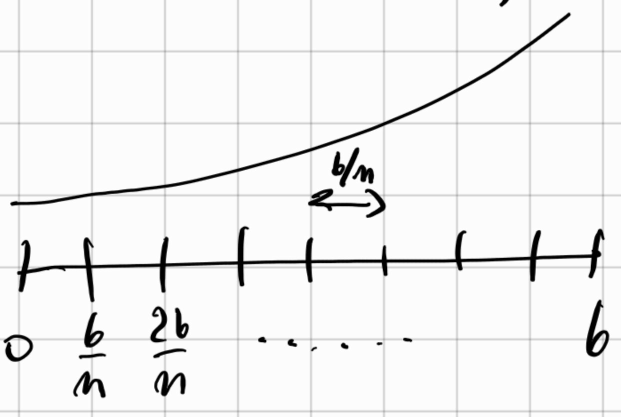
Esempio (uso dei criteri) Calcolo $\int_0^b x^2$



$f(x)=x^2 \qquad a=0,\ b>0$

P_n = "suddivisione in n parti uguali"

$$= \{x_k : k=0,\dots,n\} \text{ con } x_k = k\frac{b}{n}$$



$$x_k - x_{k-1} = \frac{b}{n}$$

$$S^{*}_{*}(f, P_n) = \sum_{k=1}^{n} \frac{b}{n} \cdot \overset{\inf}{\sup_{[x_{k-1}, x_k]}} f = \sum_{k=1}^{n} \frac{b}{n} f(\underset{x_{k-1}}{x_k}) =$$

$$= \sum_{k=1}^{n-1} \frac{b}{n} f\left(k\frac{b}{n}\right) = \sum_{k=1}^{n-1} \frac{b}{n} k^2 \frac{b^2}{n^2}$$

$$= \frac{b^3}{n^3} \cdot \sum_{k=1}^{n-1} k^2 = \frac{b^3}{n^3} \cdot \frac{\overset{(n-1)(n)(2n-1)}{n(n+1)(2n+1)}}{6}$$

$$= \frac{b^3}{6} \cdot \underset{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} \xrightarrow[n\to+\infty]{} \frac{b^3}{3} = \frac{b \cdot b^2}{3}$$

$\square$