ANALISI MATEMATICA B
LEZIONE 48 — 2.2.2024

Primitive di funzioni razionali

$$f(x) = \frac{P(x)}{Q(x)}$$ è razionale se P, Q sono polinomi

$$\int \frac{P(x)}{Q(x)}\,dx$$ c'è un metodo per trovare una primitiva

Procediamo per esempi.

(1) Se $\deg P \geq \deg Q$ faccio innanzitutto la divisione:

$$P = T \cdot Q + R \qquad \text{con } \deg R < \deg Q$$

$$\hookrightarrow \boxed{\frac{P}{Q} = T + \frac{R}{Q}}$$

Esempio $$\int \frac{x^3+1}{x-1}\,dx$$

$$\begin{array}{r|l} x^3+1 & x-1 \\ \hline x^3-x^2 & x^2+x+1 \\ \hline x^2+1 & \\ x^2-x & \\ \hline x+1 & \\ x-1 & \\ \hline 2 & \end{array}$$

Quoziente T

2 Resto R

$$x^3+1 = (x-1)(x^2+x+1) + 2$$

$$\int \frac{x^3+1}{x-1}\,dx = \int \left[x^2+x+1 + \frac{2}{x-1}\right] dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + \boxed{\int \frac{2}{x-1}\,dx}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + 2 \cdot \ln|x-1|$$

(ovvero: $$= \frac{x^3}{3} + \frac{x^2}{2} + x + \ln(x-1)^2 \;)$$

In generale per integrare $\frac{P(x)}{Q(x)}$ devo fattorizzare $Q(x)$.

Teorema fondamentale dell'algebra

Se $Q(z) \in \mathbb{C}[z]$ (polinomio a coefficienti complessi)

$\exists\, z : Q(z) = 0$.

Teorema di Ruffini Se $Q(\lambda) = 0$

allora Q è divisibile per $(z-\lambda)$

$$Q(z) = (z-\lambda)\cdot \tilde{Q}(z)$$

Iterando:

$$Q(z) = c\cdot(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$$

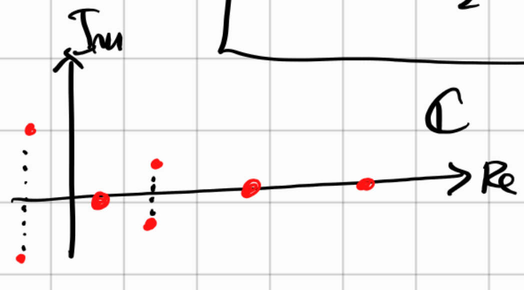
radici

In $\mathbb{R}[x]$ le cose sono più complicate:

$$Q(x) = c\,(x-\lambda_1)(x-\lambda_2)\cdots(x^2+\alpha x+\beta)\cdots$$

(x^2+x+1 non ha radici $\Delta<0$ ma ha due radici complesse: $z_{1,2} = \frac{-1\pm i\sqrt{3}}{2}$)

Osservazione Se Q ha coefficienti reali e se $Q(\lambda)=0$, $\lambda\in\mathbb{C}$ anche $Q(\bar{\lambda})=0$.



$$\underbrace{(z-\lambda)\cdot(z-\bar{\lambda})} = z^2-(\lambda+\bar{\lambda})z+\lambda\bar{\lambda}$$

$$\lambda+\bar{\lambda} = 2\,\mathrm{Re}\,\lambda \in\mathbb{R}$$

$$\lambda\bar{\lambda} = |\lambda|^2 \in\mathbb{R}$$

dim

$a_k\in\mathbb{R}$

$$Q(x) = \sum_{k=0}^{n} a_k x^k \qquad Q(z) = \sum_{k=0}^{n} a_k z^k$$

$$Q(\bar{z}) = \sum_{k=0}^{n} \bar{a}_k (\bar{z})^k = \overline{\sum_{k=0}^{n} a_k \cdot z^k}$$

$\bar{a}_k = a_k \Leftarrow a_k\in\mathbb{R}$

Se $Q(\lambda)=0$ $\quad Q(\bar{\lambda}) = \bar{0} = 0$ □

- $\underline{\text{Deg } Q = 2}$

Esempio $\int \frac{x+1}{x^2-x}\,dx$ $\qquad Q(x) = x^2 - x \quad (\Delta>0)$
$= x\cdot(x-1)$

$$\frac{x+1}{x^2-x} \overset{?}{=} \frac{A}{x} + \frac{B}{x-1}$$ ← decomposizione in fratti semplici.

$$= \frac{A(x-1)+Bx}{x\cdot(x-1)} = \frac{(A+B)x - A}{x(x-1)} \overset{!}{=} \frac{x+1}{x(x-1)}$$

$$\begin{cases} A+B=1 \\ -A=1 \end{cases} \qquad \begin{cases} A=-1 \\ B=2 \end{cases}$$

$$\int \frac{x+1}{x^2-x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{x-1}\,dx = -\ln|x| + 2\ln|x-1|$$

(ovvero: $= \ln \frac{(x-1)^2}{|x|}$

Esempio $(\Delta=0)$

$\int \frac{x-1}{x^2+2x+1}\,dx \qquad x^2+2x+1 = (x+1)^2$

$$\frac{x-1}{x^2+2x+1} \overset{?}{=} \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

$$= \frac{Ax+A+B}{(x+1)^2} \overset{!}{=} \frac{x-1}{(x+1)^2}$$

$$\begin{cases} A+B=-1 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-2 \end{cases}$$

$$\int \frac{x-1}{x^2+2x+1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x+1| + \frac{2}{x+1}$$

$$\left[\text{Oppure: } \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{Cx+D}{(x+1)^2}\right]$$

Esempio $\Delta < 0$ x^2+2x+3 $\Delta = 4-12 < 0$

$$\int \frac{x+2}{x^2+2x+3}\,dx$$

Idea: $\begin{cases} \bullet \int \frac{Q'(x)}{Q(x)}\,dx = \ln|Q(x)| \\ \bullet \int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x \end{cases}$

prima idea:

$$\frac{x+2}{x^2+2x+3} = \frac{\overbrace{\tfrac{1}{2}(2x+2)}\; + 1}{x^2+2x+3}$$

$$\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx = \frac{1}{2}\ln(\underbrace{x^2+2x+3}_{\text{è positivo}})$$

$$\int \frac{1}{\underbrace{x^2+2x+3}}\,dx \overset{\text{seconda idea}}{\underset{?}{=}} \int \frac{c}{1+(ax+b)^2}\,dx$$

completamento del quadrato:

$$x^2+2x+3 = (x+1)^2+2$$

$$\frac{1}{x^2+2x+3} = \frac{1}{2+(x+1)^2} = \frac{1}{2}\,\frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2} \xrightarrow{\int} \frac{\sqrt{2}}{2}\underbrace{\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right)}$$

$$y = \frac{x+1}{\sqrt{2}}$$

Risultato: $\frac{1}{2}\ln(x^2+2x+3) + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right).$

Metodo generale

$\dfrac{P(x)}{Q(x)}$ (1) svolgo la divisione: [mi riconduco a $\deg P < \deg Q$]

(2) Fattorizzo Q:

$$Q = \underbrace{Q_1^{p_1} \cdot Q_2^{p_2}}_{\text{grado } 1} \cdots \underbrace{\cdots Q_m^{p_m}}_{\text{grado } 2} \qquad \deg Q_k \le 2.$$

$$\frac{P(x)}{Q(x)} = \underbrace{\frac{c_{1,1}}{Q_1(x)} + \frac{c_{1,2}}{Q_1^2(x)} + \dots + \frac{c_{1,p_1}}{Q_1^{p_1}(x)} + \dots +}_{\to \int \frac{1}{y^p}}$$

$\forall$ Q_k di grado 1

grado 1

$$+ \frac{c_{m,1}x + d_{m,1}}{Q_m(x)} + \frac{c_{m,2}x + d_{m,2}}{Q_m^2(x)} + \dots + \frac{c_{m,p_m}x + d}{Q_m^{p_m}(x)}$$

$\forall$ Q_k di grado 2

$\hookrightarrow \displaystyle\int \frac{1}{(1+x^2)^2} = ?$

Oppure (decomposizione Hermite)

$$\frac{P(x)}{Q(x)} = \underbrace{\frac{c_1}{Q_1(x)} + \frac{c_2}{Q_2(x)} + \dots + \frac{c_m x + d_m}{Q_m(x)}}_{} + \left[\frac{\tilde{P}(x)}{\tilde{Q}(x)}\right]'$$

dove
$$\tilde{Q}(x) = \frac{Q(x)}{Q_1(x) \cdot Q_2(x) \cdots Q_m(x)}$$
$$= Q_1^{p_1 - 1}(x) \cdot Q_2^{p_2 - 1}(x) \cdots Q_m^{p_m - 1}(x)$$

e $\tilde{P}(x)$ ha grado inferiore a $\tilde{Q}(x)$

e va determinato.

$\tilde{P}(x)$ si trova espandendo la derivata $[\ \ ]'$

Esempio esageratamente complicato

$$\int \frac{x^{10}+x^8-x^6+1}{x^8-x^7+2x^6-2x^5+x^4-x^3}\,dx$$

(1) Faccio la divisione:

[.....]

$$\int \underbrace{x^2+x}_{\text{facile}} + \int \frac{x^4+1}{x^8-x^7+2x^6-2x^5+x^4-x^3}$$

$P(x)$

$Q(x)$

facile.

(2) Fattorizzo il denominatore:

$$\ldots = x^3(x-1)(x^2+1)^2$$

radici: $\lambda_1=0$ $\quad \lambda_2=1$ $\quad \lambda_3=i,\ \overline{\lambda_3}=-i$

$p_1=3$ $\quad p_2=1$ $\quad p_3=2$

$$3 + 1 + 2\cdot 2 = 8 = \deg Q$$

$$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \left[\frac{Ex^3+Fx^2+Gx+H}{x^2\cdot(x^2+1)}\right]'$$

Trovo $\underbrace{A, B, C, \ldots, H}_{8 \text{ coefficienti}}$ in modo che

Svolgo l'uguaglianza

(vedere sugli appunti)

$$\begin{cases} A=1,\ B=1/2 \\ C=-3/2 \quad D=1 \\ E=3/2 \quad F=1 \\ G=1 \quad H=1/2 \end{cases}$$

$$\int \frac{P(x)}{Q(x)}\,dx = \ln|x| + \frac{1}{2}\ln|x-1| + \operatorname{arctg} x - \frac{3}{4}\ln(x^2+1) + \frac{3/2\,x^3 + x^2 + x + \frac{1}{2}}{x^2(x^2+1)}$$

ES

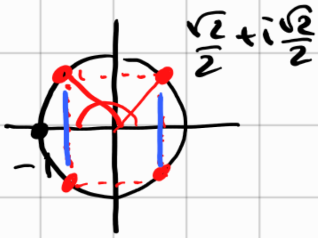
$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{Ax+B}{1+x^2} + \left[\frac{Cx+D}{1+x^2}\right]'$$

trovo A, B, C, D ....

$z^n = c$

Esempio $\int \frac{1}{x^4+1}\,dx$

$Q(x) = x^4+1 \qquad z^4 = -1$



$$z_{1234} = \frac{\sqrt{2}}{2}(\pm 1 \pm i)$$

$$z^4+1 = \underbrace{\left(z-\frac{\sqrt{2}}{2}(1+i)\right)\cdot\left(z-\frac{\sqrt{2}}{2}(1-i)\right)}\ \underbrace{\left(z-\frac{\sqrt{2}}{2}(-1+i)\right)\cdot\left(z-\frac{\sqrt{2}}{2}(-1-i)\right)}$$

$$= (z^2-\sqrt{2}z+1)\cdot(z^2+\sqrt{2}z+1)$$

$$\frac{1}{x^4+1} = \frac{Ax+B}{\underbrace{x^2-\sqrt{2}x+1}_{\Delta<0}} + \frac{Cx+D}{\underbrace{x^2+\sqrt{2}x+1}_{\Delta<0}} \quad \ldots\ldots$$