ANALISI MATEMATICA B
LEZIONE 56 - 21.2.2024

Esercizio $\displaystyle\int_0^{+\infty} \frac{(1+3x^2)^{2x} - (1+2x^2)^{3x}}{e^{x^2} - \sqrt[4]{1+4\sin^2 x}}\,dx$ converge?

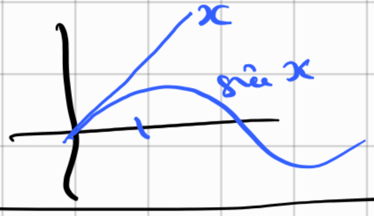
$e^{x^2} \overset{?}{>} \sqrt[4]{1+4\sin^2 x} \quad \forall x>0$

$f(x) = e^{4x^2} - 1 - 4\sin^2 x \overset{?}{>} 0$

$f(0)=0$

$f'(x) = \underbrace{8x}_{>0}\,\underline{e^{4x^2}} - 8\,\underbrace{\sin x}_{\text{se } x>0}\,\underline{\cos x}$

$\begin{cases} e^{4x^2} > 1 > \cos x \\ x > \sin x \end{cases}$



PUNTI CRITICI: 0^+, $+\infty$

$+\infty$

$a^b = e^{b\ln a}$

$(1+3x^2)^{2x} - (1+2x^2)^{3x} = e^{2x\ln(1+3x^2)} - e^{3x\ln(1+2x^2)}$

$\le e^{2x\ln(1+3x^2)} + e^{3x\ln(1+2x^2)}$

$\le 2\cdot e^{3x\ln(1+3x^2)}$

$3x\ln(1+3x^2) \sim 3x\,2\ln x = 6x\ln x$

e^{x^2} $\quad x^2 \gg x^{3/2}\cdot\sqrt{x}$ $\quad x^{3/2} \gg 6x\ln x$

$\dfrac{N(x)}{D(x)} \le \dfrac{2e^{3x\ln(1+3x^2)}}{e^{x^2} - \sqrt[4]{\dots}} \sim 2e^{3x\ln(1+3x^2) - x^2}$

$x^2 - 3x\ln(1+3x^2) \gg x\sqrt{x}$ $\qquad \displaystyle\int_1^{+\infty} e^{-x\sqrt{x}}\,dx < +\infty$

$$e^{-x\sqrt{x}} << \frac{1}{x^2} \qquad\qquad e^{x\sqrt{x}} >> x^2$$

0^+

per $x \to 0^+$

$$\frac{(1+3x^2)^{2x}-(1+2x^2)^{3x}}{e^{x^2}-\sqrt[4]{1+4\sin^2 x}} = \frac{e^{2x\ln(1+3x^2)}-e^{3x\ln(1+2x^2)}}{\%}$$

$$e^t = 1+t+\frac{t^2}{2}+o(t^2) \qquad (1+t)^{1/4} = 1+\frac{t}{4}-\frac{3}{32}t^2+o(t^2)$$

$$\% = \cancel{1}+x^2+\frac{x^4}{2}+o(x^4) - \Big(\cancel{1}+\sin^2 x - \frac{3}{32}16\sin^4 x + \underset{\overset{\|}{o(x^4)}}{o(\sin^4 x)}\Big)$$

$$= x^2+\frac{x^4}{2} - \Big(x-\frac{x^3}{6}+o(x^3)\Big)^2 + \frac{3}{2}x^4 + o(x^4)$$

$$= \cancel{x^2}+\frac{x^4}{2} - \Big(\cancel{x^2}-\frac{1}{3}x^4\Big) + \frac{3}{2}x^4 \qquad + o(x^4)$$

$$= \Big(\frac{1}{2}+\frac{1}{3}+\frac{3}{2}\Big)x^4 + o(x^4) = \frac{7}{3}x^4 + o(x^4).$$

$$N(x) = e^{2x\ln(1+3x^2)} - e^{3x\ln(1+2x^2)} \qquad\qquad \ln(1+t) = t-\frac{t^2}{2}+o(t^2)$$

$$= e^{2x\left(3x^2-\frac{9}{2}x^4+o(x^4)\right)} - e^{3x\left(2x^2-2x^4+o(x^4)\right)}$$

$$= e^{6x^3-9x^5+o(x^5)} - e^{6x^3-6x^5+o(x^5)}$$

$$= \cancel{1}+\Big(\cancel{6x^3}-9x^5+o(x^5)\Big)+\frac{1}{2}\cancel{\Big(6x^3+o(x^3)\Big)^2}+o(x^6)$$

$$\boxed{e^t = 1+t+\frac{t^2}{2}+o(t^2) \\ = 1+t+o(t^2)}$$

$$-\cancel{1}-\Big(\cancel{6x^3}-6x^5+o(x^5)\Big)-\frac{1}{2}\cancel{\Big(6x^3+o(x^3)\Big)^2}+o(x^6)$$

$$= -9x^5+6x^5+o(x^6)$$

$$= -3x^5+o(x^5)$$

$$\boxed{⚠ \quad \Big(6x^3+o(x^3)\Big)^2-\Big(6x^3+o(x^3)\Big)^2 = o(x^6)}$$

NON E' ZERO!

$$\frac{N(x)}{D(x)} = \frac{-3x^5 + o(x^5)}{\frac{7}{3}x^4 + o(x^4)} = \frac{x^5}{x^4}\,\frac{-3+o(1)}{\frac{7}{3}+o(1)} = x\,\frac{-1+o(1)}{7+o(1)} \to 0$$

per $x \to 0$

è un finto integrale improprio.

$\int_0^1 \frac{N(x)}{D(x)}$ è convergente

$\Rightarrow \int_0^{+\infty} \frac{N(x)}{D(x)}$ converge.

Ripasso convergenza uniforme

$f_n : A \to \mathbb{R}$

$f : A \to \mathbb{R}$

$f_n \rightrightarrows f$ (converge uniformemente)

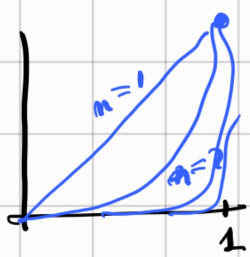
se (per definizione) $\lim_{n\to+\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$

è diversa dalla convergenza puntuale:
(più forte)

$\forall x \in A \quad \lim_{n\to+\infty} |f_n(x) - f(x)| = 0$

ovvero $f_n(x) \xrightarrow[n\to+\infty]{} f(x)$

$f_n : [0,1] \to \mathbb{R} \qquad f_n(x) = x^n$



$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{se } x < 1 \\ 1 & \text{se } x = 1. \end{cases}$$

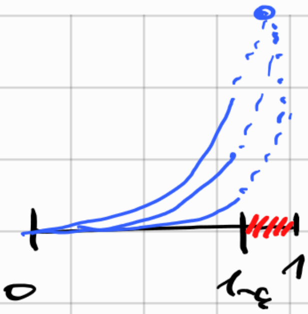
Se f_n converge uniformemente non può che convergere a f.

$$|f_n(x) - f(x)| = \begin{cases} x^n & \text{se } x < 1 \\ 0 & \text{se } x = 1 \end{cases}$$

$\sup_{x\in[0,1)} x^n = \lim_{x\to 1^-} x^n = 1.$

$\lim \sup |f_n - f| = 1 \neq 0$

non c'è conv. unif. su $[0,1]$.
neanche su $[0,1)$.
Ma c'è conv. uniforme su $[0,1-\varepsilon]$ $\forall \varepsilon>0$



$$\sup |f_n - f| = \sup_{x\in[0,1-\varepsilon]} |f_n(x)| = (1-\varepsilon)^n \to 0.$$

Continuità dell'integrale:

$$\lim_{\beta\to b} \int_a^\beta f(x)\,dx = \int_a^b f(x)\,dx.$$

(se f è limitata è un teorema facile altrimenti è una definizione).

Derivata

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

se f è continua — Teorema fondamentale.

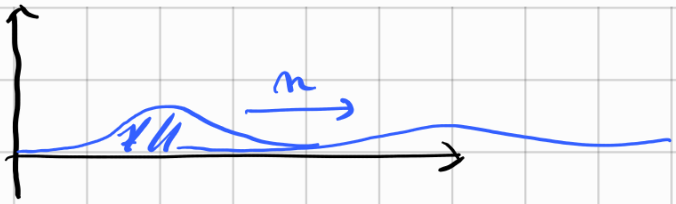
Esercizio: (già visto)

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt \left(= \frac{d}{dx}\left[F\right]_{a(x)}^{b(x)}\right) = f(b(x))\cdot b'(x) - f(a(x))\,a'(x)$$

Scambio della $\begin{cases}\text{derivata}\\ \text{limite}\end{cases}$ con l'integrale

$$\begin{cases} \displaystyle\lim_{x\to x_0} \int_a^b f(t,x)\,dt \overset{?}{=} \int_a^b \lim_{x\to x_0} f(t,x)\,dt \\ \displaystyle\frac{d}{dx}\int_a^b f(t,x)\,dt \overset{?}{=} \int_a^b \frac{\partial}{\partial x} f(t,x)\,dt \end{cases}$$

In generale NO!



$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt$$

Posso definire $f_n(t)$ in modo che $\int_{-\infty}^{+\infty} f_n(t)\,dt = 1$ $\forall n$.

$$f_1(t) = \frac{1}{1+t^2} \qquad \int_{-\infty}^{+\infty} \frac{1}{1+t^2}\,dt = [\operatorname{arctg} t]_{-\infty}^{+\infty} = \pi$$

$$f_n(t) = \frac{1}{1+(t-n)^2} \qquad \int_{-\infty}^{+\infty} f_n = \pi$$

ma $f_n(t) \to 0$

$f_n \to 0$ puntualmente

$$\lim \int f_n = \pi$$

$$\int \lim f_n = \int 0 = 0$$

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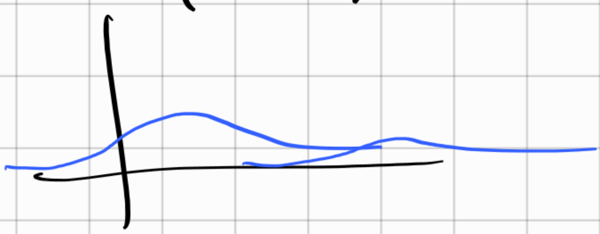
f_n non converge uniformemente.

Teo $\int_a^b f_n \to \int_a^b f$ se $f_n \rightrightarrows f$ $\quad a,b \in \mathbb{R}$, $a > -\infty$, $b < +\infty$.

Se $[a,b]$ non è limitato non vale sempre:

es: $g_n(t) = \dfrac{f_n\left(\frac{t}{n}\right)}{n}$ (credo)



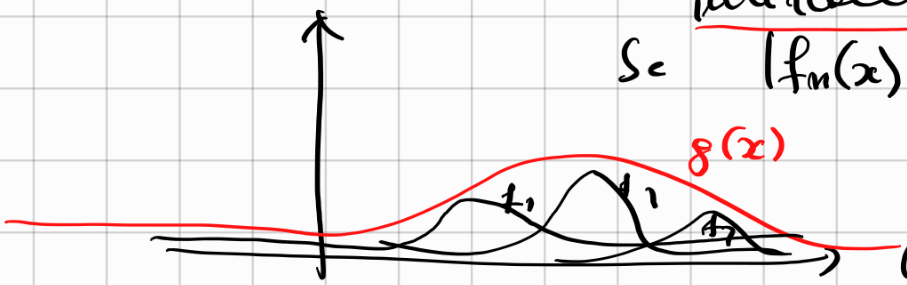
$g_n \underset{n\to\infty}{\rightrightarrows} 0$

ma $\int_{-\infty}^{+\infty} g_n = \pi$

Convergenza dominata

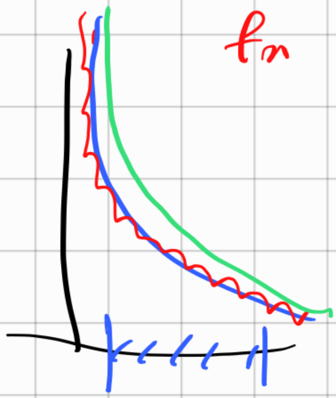
Se $f_n \to f$ su tutti gli intervalli chiusi e limitati

Se $|f_n(x)| \le g(x)$ e $\int_a^b g(x) < +\infty$



Allora $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

(Analogo per le derivate)



f_n $\quad f.$ $\quad \int g < +\infty$

$|f_n| \le g$